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k-Tuple Domatic In Graphs

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Abstract. For every positive integer k, a set S of vertices in a graph $G = (V, E)$ is a k-tuple dominating set of G if every vertex of $V - S$ is adjacent to at least k vertices and every vertex of S is adjacent to at least $k-1$ vertices in S. The minimum cardinality of a k-tuple dominating set of G is the k-tuple domination number of G. When $k = 1$, a k-tuple domination number is the well-studied domination number. We define the k-tuple domatic number of G as the largest number of sets in a partition of V into k-tuple dominating sets. Recall that when $k = 1$, a k-tuple domatic number is the well-studied domatic number. In this study, basic properties and bounds for the k-tuple domatic number are derived.
Keywords: k-tuple dominating set; k-tuple domination number; k-tuple domatic number.

1. INTRODUCTION

The notation used in this study is as follows. Let G be a simple graph with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are

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denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If every vertex of G has degree k, then G is said to be *k-regular*. The *complement* of a graph G is denoted by $\overline{G}$ which is a graph with $V(\overline{G}) = V(G)$ and for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$. The subgraph *induced* by S in a graph G is denoted by $G[S]$. K_n for the *complete graph* of order n and $K_{n,m}$ for the *complete bipartite graph*are written..

For every positive integer k, the *k-join* $G \circ_k H$ of a graph G to a graph H, of order at least k, is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H.

A *dominating set* of a graph G is a subset S of the vertex set $V(G)$ such that every vertex of G is either in S or has a neighbor in S. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$ of G. It is common that the complement of a dominating set of minimum cardinality of a graph G without isolated vertices is also a dominating set. Hence one can partition the vertex set of G into at least two disjoint dominating sets. The maximum number of dominating sets which the vertex set of a graph G can be partitioned is called the *domatic number* of G, and denoted by $d(G)$. This graph invariant was introduced by Cockayne and Hedetniemi [2]. They also showed that

$$\gamma(G) \cdot d(G) \leq n. \tag{1.1}$$

To simplify matters of notation, a domatic partition of a graph G into ℓ dominating sets is given by a colouring $f : V(G) \to \{1, 2, ..., \ell\}$ of the vertex set $V(G)$ with ℓ colors. The dominating sets are recovered from f by taking the inverse, i.e. $D_i = f^{-l}(i)$, $i = 1, ..., \ell$. Clearly, a coloring f defines a domatic partition of G if and only if for every vertex $x \in V(G)$, $f(N[x]) = \{1, 2, ..., \ell\}$. Thus, any graph G satisfies $d(G) \leq \delta(G) + 1$. The word domatic, an amalgamation of the words domination and chromatic, refers to an analogy between the chromatic number (partitioning of the vertex set into independent sets) and the domatic number (partitioning into dominating sets). For a survey of results on the domatic number of graphs I refer the reader to [13]. It was first observed by Cockayne and Hedetniemi [2] that for every graph without isolated vertices $2 \leq d(G) \leq \delta(G)+1$. The upper bound $\delta(G)+1$ derived from interval graphs [9], for example.

Intuitively, it seems reasonable to expect that a graph with large minimum degree will have a large domatic number. Zelinka [14] showed that this is not necessarily the case. He gave examples for graphs of arbitrarily large minimum degree with domatic number 2. For more details about domatic number consider the following studies [1, 3, 10, 11].

The *total domatic number* $d_t(G)$ is similarly defined based on the concept of the *total domination number* $\gamma_t(G)$. Sheikholeslami and Volkmann, in a similar manner, generalized in [12] the concept of total domatic number to the k-tuple total domatic number $d_{\times k,t}(G)$ based on the concept of k-tuple total domination number $\gamma_{\times k,t}(G)$, which is defined by Henning and Kazemi in [8]. We recall that for every positive integer k, a *k-tuple total dominating set*, abbreviated kTDS, of a graph G is a subset S of the vertex set $V(G)$ such that every vertex of G is adjacent to at least k vertices of S. And the minimum cardinality of a kTDS of G is the *k-tuple total domination number* $\gamma_{\times k,t}(G)$ of G.

Harary and Haynes in [5] extend the concept of domatic number to k-tuple domatic number based on the concept of k-tuple domination number. For every positive integer k, a *k-tuple dominating set*, abbreviated kDS, of a graph G is a subset S of the vertex set $V(G)$ such that every vertex of G is either in S and is adjacent to at least $k-1$ vertices in S or is not in S and is adjacent to at least k vertices in S. The minimum cardinality of a kDS of G is the *k-tuple domination number* $\gamma_{\times k}(G)$ of G. For a graph to have a k-tuple dominating set, its minimum degree is at least $k-1$. The *k-tuple domatic number* $d_{\times k}(G)$ of G is the largest number of sets in a partition of $V(G)$ into k-tuple dominating sets. If $d = d_{\times k}(G)$ and $V(G) = V_1 \cup V_2 \cup ... \cup V_d$ is a partition of $V(G)$ into k-tuple dominating sets V_1, V_2, ... and V_d, we say that $\{V_1, V_2, ..., V_d\}$ is a *k-tuple domatic partition*, abbreviated kDP, of G. The k-tuple domatic number is well-defined and

$$d_{\times k}(G) \geq 1, \tag{1.2}$$

for all graphs G with $\delta(G) \geq k-1$, since the set consisting of $V(G)$ forms a k-tuple domatic partition of G.

Similar to the domatic partition of a graph, we simplify matters of notation as follows: a k-tuple domatic partition of a graph G into ℓ k-tuple dominating sets is given by a coloring $f : V(G) \to \{1, 2, ..., \ell\}$ of the vertex set $V(G)$ with ℓ colors. The k-tuple dominating sets are recovered from f by taking the inverse, i.e. $D_i = f^{-l}(i)$, $i = 1, ..., \ell$. Clearly, a coloring f defines a k-tuple domatic partition of G if and only if for every vertex $x \in V(G)$, $f(N(x)) = \{f(y) \mid y \in N(x)\}$ contains the mulitiset $\{t_1.1, t_2.2, ..., t_\ell.\ell\}$ such that for every i, $t_i \in \{k-1, k\}$ and for an index i, if $t_i = k-1$, then $f(x) = i$. Harary and Haynes in [5] proved that for each positive integer k and each graph G with minimum degree at least $k-1$,

$$d_{\times k}(G) \leq \frac{\delta(G)+1}{k}. \tag{1.3}$$

The next proposition is an immediate result of the inequalities (1.2) and (1.3).

Proposition 1.1. *Let $k \geq 1$ be an integer, and let G be a graph. If $k-1 \leq \delta(G) \leq 2k-2$, then $d_{\times k}(G) = 1$.*

Graphs for which $d_{\times k}(G)$ achieves this upper bound $\frac{\delta(G)+1}{k}$ we call *k-tuple domatically full.*

In this study basic properties and bounds for the k-tuple domatic number are deriven.

The following observations are useful.

observation

Let K_n be the complete graph of order $n \geq 1$. Then

$$d_{\times k}(K_n) = \lfloor \frac{n}{k} \rfloor.$$

Proof. The inequality (1.3) implies $d_{\times k}(K_n) \leq \frac{\delta(G)+1}{k} = \frac{n}{k}$, and hence $d_{\times k}(K_n) \leq \lfloor \frac{n}{k} \rfloor$. Now let $\lfloor \frac{n}{k} \rfloor = \ell$ and let $V(K_n) = \{1, 2, ..., n\}$. Let $V_i = \{(i-1)k + j | 1 \leq j \leq k\}$ for $1 \leq i \leq \ell - 1$ and $V_\ell = \{(\ell-1)k + 1, (\ell-1)k+2, ..., n\}$. Since $V(K_n) = V_1 \cup V_2 \cup ... \cup V_\ell$ is a partition of $V(K_n)$ into k-tuple dominating sets V_1, V_2, ... and V_ℓ, we obtain $d_{\times k}(K_n) = \ell = \lfloor \frac{n}{k} \rfloor$. □

observation

Let G be a bipartite graph with $\delta(G) \geq k-1 \geq 1$. If X and Y are the bipartite sets of G, then $\gamma_{\times k}(G) \geq 2k-2$ with equality if and only if $G = K_{k-1,k-1}$.

Proof. Let D be a $\gamma_{\times k}(G)$-set, and let $w \in X$ and $z \in Y$ be two arbitrary vertices. The definition implies that $\mid D \cap N(w) \mid \geq k-1$ and $\mid D \cap N(z) \mid \geq k-1$. Since $N(w) \cap N(z) = \emptyset$, we deduce that $\mid D \mid \geq 2k-2$ and thus $\gamma_{\times k}(G) \geq 2k-2$. Obviously, we can see that $\gamma_{\times k}(G) = 2k-2$ if and only if $G = K_{k-1,k-1}$. □

2. Properties of the k-Tuple Domatic Number

Here, we present basic properties of $d_{\times k}(G)$ and bounds on the k-tuple domatic number of a graph. I start my study with a theorem that characterizes graphs G with $\gamma_{\times k}(G) = m$, for some $m \geq k \geq 1$.

Theorem 2.1. *Let G be a graph with $\delta(G) \geq k-1 \geq 0$. Then for any integer $m \geq k$, $\gamma_{\times k}(G) = m$ if and only if $G = K'_m$ or $G = F \circ_k K'_m$, for some graph F and some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k-1$ such that m is minimum in the set*

$$\{t \mid G = F' \circ_k K'_t\}, \tag{2.1}$$

where F' is a graph and K'_t is a spanning subgraph of K_t with $\delta(K'_t) \geq k-1$.

Proof. Let S be a $\gamma_{\times k}(G)$-set and $\gamma_{\times k}(G) = m$, for some $m \geq k$. Then $\mid S \mid= m$ and every vertex in $V - S$ has at least k neighbors in S and otherwise $k-1$ neighbors. Then $G[S] = K'_m$, for some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k-1$. If $\mid V \mid= m$, then $G = K'_m$. If $\mid V \mid> m$, then let F be the induced subgraph $G[V-S]$. Then $G = F \circ_k K'_m$. Also by the definition of k-tuple domination number, m is minimum in the set given in (2.1).

Conversely, let $G = K'_m$ or $G = F \circ_k K'_m$, for some graph F and some spanning subgraph K'_m of K_m with $\delta(K'_m) \geq k-1$ such that m is minimum in the set given in (2.1). Then, since $V(K'_m)$ is a kDS of G of cardinality m, $\gamma_{\times k}(G) \leq m$. If $\gamma_{\times k}(G) = m' < m$, then the previous paragraph concludes that for some graph F' and some spanning subgraph $K'_{m'}$ of $K_{m'}$ with $\delta(K'_{m'}) \geq k-1$, $G = F' \circ_k K'_{m'}$, that is contradiction with the minimality of m. Therefore $\gamma_{\times k}(G) = m$. □

Corollary 2.2. *Let G be a graph with $\delta(G) \geq k-1 \geq 0$. Then $\gamma_{\times k}(G) = k$ if and only if $G = K_k$ or $G = F \circ_k K_k$, for some graph F.*

Theorem 2.3. *If G is a graph of order n and $\delta(G) \geq k-1$, then*

$$\gamma_{\times k}(G) \cdot d_{\times k}(G) \leq n.$$

Moreover, if $\gamma_{\times k}(G) \cdot d_{\times k}(G) = n$, then for each kDP $\{V_1, V_2, ..., V_d\}$ of G with $d = d_{\times k}(G)$, each set V_i is a $\gamma_{\times k}(G)$-set.

Proof. Let $\{V_1, V_2, ..., V_d\}$ be a kDP of G such that $d = d_{\times k}(G)$. Then

$$\begin{aligned} d \cdot \gamma_{\times k}(G) &= \sum_{i=1}^{d} \gamma_{\times k}(G) \\ &\leq \sum_{i=1}^{d} \mid V_i \mid \\ &= n. \end{aligned}$$

If $\gamma_{\times k}(G) \cdot d_{\times k}(G) = n$, then the inequality occurring in the proof becomes equality. Hence for the kDP $\{V_1, V_2, ..., V_d\}$ of G and for each i, $\mid V_i \mid= \gamma_{\times k}(G)$. Thus each set V_i is a $\gamma_{\times k}(G)$-set. □

The case $k = 1$ in Theorem 2.3 leads to the well-known inequality (1.1), given by Cockayne and Hedetniemi [2] in 1977.

Corollary 2.2 and Theorem 2.3 with this fact that for any graph G of order n with minimum degree at least $k-1$, $\gamma_{\times k}(G) \geq k$, imply the next result.

Corollary 2.4. *If G is a graph of order n with $\delta(G) \geq k-1 \geq 0$, then*

$$d_{\times k}(G) \leq \frac{n}{k},$$

with equality if and only if $G = K_k$ or $G = F \circ_k K_k$, for some graph F.

For bipartite graphs, we can improve the bound given in Corollary 2.4, by Observation 1.

Corollary 2.5. *Let G be a bipartite graph of order n with vertex partition $V(G) = X \cup Y$ and $\delta(G) \geq k-1 \geq 1$. Then*

$$d_{\times k}(G) \leq \frac{n}{2k-2},$$

with equality if and only if $G = K_{k-1,k-1}$.

Theorem 2.6. *If G is a graph of order n and $\delta(G) \geq k-1 \geq 2$, then*

$$\gamma_{\times k}(G) + d_{\times k}(G) \leq n+1.$$

Proof. Applying Theorem 2.3, we obtain

$$\gamma_{\times k}(G) + d_{\times k}(G) \leq \frac{n}{d_{\times k}(G)} + d_{\times k}(G).$$

Since $d_{\times k}(G) \geq 1$, by inequality (1.2), and $k \geq 3$, Corollary 2.4 implies that $d_{\times k}(G) \leq \frac{n}{2}$. Using these inequalities, and the fact that the function $g(x) = x + \frac{n}{x}$ is decreasing for $1 \leq x \leq n^{1/2}$ and increasing for $n^{1/2} \leq x \leq \frac{n}{2}$, we obtain

$$\gamma_{\times k}(G) + d_{\times k}(G) \leq \max\{n+1, \frac{n}{2}+2\} = n+1,$$

and this is the desired bound. □

If $G = \ell K_k$ for integers $\ell \geq 1$ and $k \geq 3$, then $\gamma_{\times k}(G) = n(G) = \ell k$ and $d_{\times k}(G) = 1$. Therefore $\gamma_{\times k}(G) + d_{\times k}(G) = n+1$, and so the upper bound $n+1$ in Theorem 2.6 is sharp.

By closer look at the proof of Theorem 2.6 we have:

Corollary 2.7. [5] *Let G be a graph of order n with $\delta(G) \geq k-1 \geq 2$. If $d_{\times k}(G) \geq 2$, then*

$$\gamma_{\times k}(G) + d_{\times k}(G) \leq \frac{n}{2} + 2.$$

If $G = K_{2k}$, then $\gamma_{\times k}(G) = k$ and $d_{\times k}(G) = 2$. Therefore $\gamma_{\times k}(G) + d_{\times k}(G) = n/2 + 2$, and so the upper bound $n/2+2$ in Corollary 2.7 is sharp. As a further application of the inequality (3), Harary and Haynes in [5] proved the next theorem.

Theorem 2.8. [5] *For every graph G of order n in which $min\{\delta(G), \delta(\overline{G})\} \geq k-1$,*

$$d_{\times k}(G) + d_{\times k}(\overline{G}) \leq \frac{n+1}{k}.$$

The upper bound in Theorem 2.8 is sharp. Since for $k \geq 2$, we have

$$d_{\times k}(K_{k,k}) + d_{\times k}(\overline{K_{k,k}}) = 1 + 1 = \lfloor \frac{2k+1}{k} \rfloor.$$

Finally, we compare the k-tuple domatic number of a graph with its k-tuple total domatic number.

Theorem 2.9. *Let G be a graph with $\delta(G) \geq k \geq 1$. Then*

$$d_{\times k,t}(G) \leq d_{\times k}(G) \leq 2d_{\times k,t}(G),$$

and this bounds are sharp.

Proof. Since every k-tuple total dominating set of G is a k-tuple dominating set and the union of at least two disjoint k-tuple dominating sets is a k-tuple total dominating set, then $d_{\times k,t}(G) \leq d_{\times k}(G) \leq 2d_{\times k,t}(G)$.

The lower bound is sharp for the complete bipartite graph $K_{mk,mk}$, where $k \geq 2$ and $m \geq 1$. Because $d_{\times k,t}(G) = d_{\times k}(G) = m$. Also for the cycle C_4, we have $d(C_4) = d_t(C_4) = 2$.

The upper bound is sharp for the graphs G which is obtained let H_1, H_2, H_3 and H_4 be four disjoint copies of the complete graph K_k, where $k \geq 1$. Let G be the union of the four graphs H_1, H_2, H_3 and H_4 such that for each $1 \leq i \leq 3$ every vertex of H_i is adjacent to all vertices of H_{i+1}. Obviously $V(H_2) \cup V(H_3)$ is the unique $\gamma_{\times k,t}(G)$-set, and so $d_{\times k,t}(G) = 1$. This follows that $d_{\times k}(G) \leq 2d_{\times k,t}(G) = 2$. Since the sets $V(H_2) \cup V(H_3)$ and $V(H_1) \cup V(H_4)$ are two disjoint $\gamma_{\times k}(G)$-sets, then $d_{\times k}(G) = 2 = 2d_{\times k,t}(G)$. □

Corollary 2.10. [15] *Let G be a graph with no isolated vertices. Then*

$$d_t(G) \leq d(G) \leq 2d_t(G).$$

Theorem 2.11. *Let $k \geq 1$ be integer. If one of the numbers $d_{\times k}(G)$ and $d_{\times k,t}(G)$ for a graph G is infinite, then*

$$d_{\times k}(G) = d_{\times k,t}(G).$$

Proof. Let $d_{\times k}(G) = \alpha$, where α is an infinite cardinal number. Then there exists a k-tuple domatic partition $\Re$ having α classes. The family $\Re$ can be partitioned into two subfamilies $\Re_1$ and $\Re_2$ which both have the cardinality of α. There exists a bijection $f : \Re_1 \to \Re_2$. Let $\Re_0 = \{D \cup f(D) \mid D \in \Re_1\}$. This is evidently a k-tuple total domatic partition of G having α classes and thus $d_{\times k,t}(G) \geq \alpha = d_{\times k}(G)$. Since $d_{\times k,t}(G) \leq d_{\times k}(G)$, we have $d_{\times k,t}(G) = d_{\times k}(G) = \alpha$. If $d_{\times k,t}(G)$ is infinite, then so is $d_{\times k}(G)$ and also $d_{\times k,t}(G) = d_{\times k}(G)$ will be, too. □

Corollary 2.12. [16] *If one of the numbers the $d(G)$ and $d_t(G)$ for a graph G is infinite, then*

$$d(G) = d_t(G) will be, too.$$

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