Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
`http://cjms.journals.umz.ac.ir`
ISSN: 1735-0611

CJMS. **1**(2)(2012), 53-60

Fan-KKM Theorem in Minimal Vector Spaces and its Applications

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Abstract. In this paper, after reviewing some results in minimal space, some new results in this setting are given. We prove a generalized form of the Fan-KKM type theorem in minimal vector spaces. As some applications, the open type of matching theorem and generalized form of the classical KKM theorem in minimal vector spaces are given.

Keywords: Minimal vector space, Fan-KKM theorem, Matching theorem.

1. Introduction

Many problems in nonlinear analysis can be solved by the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point, an optimal point, a solution point for complementarity problem, a solution point for variational problem, or others of the corresponding problem under consideration. The first result on the nonempty intersection was the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in [5],

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Received: 27 Dec 2011
Revised: 4 Feb 2012
Accepted: 4 Feb 2012

which is concerned with certain types of multimaps called the KKM maps. The KKM theory, first called by Park [10], is the study of KKM maps and their applications. Generalized form of KKM theorem namely Fan-KKM principle provides a foundation for many of the modern essential results in diverse areas of mathematical sciences (see [8, 9]).

At the peresent paper, after reviewing some results in minimal space, some new results in this setting are given. We prove a generalized form of the Fan-KKM type theorem in minimal vector spaces. As some applications, the open type of matching theorem and generalized form of the classical KKM theorem in minimal vector spaces are given.

2. Preliminaries

The concepts of minimal structures and minimal spaces, as generalization of topology and topological spaces were introduced in [7]. Further results about minimal spaces can be found in [1, 2, 3, 6] and [11]. For easy understanding of the material incorporated in this paper we recall some basic definitions and results. Also some new concepts are introduced in minimal spaces.

A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be a *minimal structure* on X if $\emptyset, X \in \mathcal{M}$. In this case $(X, \mathcal{M})$ is called a *minimal space*. For example, let (X, τ) be a topological space, then τ, $SO(X)$, $PO(X)$, $\alpha O(X)$ and $\beta O(X)$ are minimal structures on X [6].
In a minimal space $(X, \mathcal{M})$, $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. For any set $A \subseteq X$, set $m\text{-Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and $m\text{-Cl}(A) = \bigcap\{F : A \subseteq F,\ F^c \in \mathcal{M}\}$. Note that for $A \subseteq X$, m-Cl(A) (m-Int(A)) is not necessarily m-closed (m-open) respectively.

Definition 2.1. [11] Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be two minimal spaces. A function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is called *minimal continuous* (briefly *m-continuous*) if $f^{-1}(U) \in \mathcal{M}$ for any $U \in \mathcal{N}$.

Definition 2.2. [1] For two minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ we define *minimal product structure* for $X \times Y$ as follows :

$$\mathcal{M}\times\mathcal{N} = \{A \subseteq X\times Y : \forall\,(x, y) \in A,\ \exists U \in \mathcal{M}, \exists V \in \mathcal{N}; (x, y) \in U\times V \subseteq A\}.$$

Definition 2.3. [1] A *linear minimal structure* on a vector space X over the complex field $\mathbb{F}$ is a minimal structure $\mathcal{M}$ on X such that the two mappings

$$\begin{aligned} + &: X \times X \to X, (x, y) \mapsto x + y \\ . &: \mathbb{F} \times X \to X, (t, x) \mapsto tx \end{aligned}$$

are m-continuous, where $\mathbb{F}$ has the usual topology and both $\mathbb{F} \times X$ and $X \times X$ have the corresponding product minimal structures. A *linear minimal space* (or *minimal vector space*) is a vector space together with a linear minimal structure. The convex hull or convex span of $E \subseteq X$ is denoted by $co(E)$.

It is not hard to see that there are many minimal spaces which are not topological space and obviously any topological vector space is a minimal vector space. In the following, it is shown that there is a linear minimal space which is not a topological vector space.

Example 2.4. Consider the real field $\mathbb{R}$. Clearly $\mathcal{M} = \{(a, b) : a, b \in \mathbb{R} \cup \{\pm\infty\}\}$ is a minimal structure on $\mathbb{R}$. We claim that $\mathcal{M}$ is a linear minimal structure on $\mathbb{R}$. For this, we must prove that, two operations $+$ and $\cdot$ are m-continuous. Suppose $(x_0, y_0) \in +^{-1}(a, b)$,

so $x_0 + y_0 \in (a, b)$. Put $\epsilon = \min\{x_0 + y_0 - a, b - (x_0 + y_0)\}$ and hence $x_0 \in (x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2})$ and $y_0 \in (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})$. Therefore,

$$x_0 + y_0 \in ((x_0 - \frac{\epsilon}{2}, x_0 + \frac{\epsilon}{2}) + (y_0 - \frac{\epsilon}{2}, y_0 + \frac{\epsilon}{2})) \subseteq (a, b);$$

which implies that $+^{-1}(a, b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$; that is $+$ is m-continuous. Also, suppose $(\alpha_0, x_0) \in \cdot^{-1}(a, b)$. Since $\alpha_0 x_0 \in (a, b)$ and $\lim_{s,t \to 0}(\alpha_0 - s)(x_0 - t) = \alpha_0$

x$_0$, so one can find some $\delta > 0$ for which $\mid \alpha_0 - s \mid < \delta$ and $\mid x_0 - t \mid < \delta$ imply that $a < (\alpha_0 - s)(x_0 - t) < b$. Therefore, $(\alpha_0, x_0) \in (\alpha_0 - \delta$

$, \alpha_0 + \delta) \cdot (x_0 - \delta, x_0 + \delta) \subseteq (a, b)$; i.e., $\cdot^{-1}(a, b)$ is m-open in the minimal product space $\mathbb{R} \times \mathbb{R}$, which implies that the operation $\cdot$ is m-continuous.

Definition 2.5. [11] For a minimal space $(X, \mathcal{M})$,

(a) a family of m-open sets $\mathcal{A} = \{A_j : j \in J\}$ in X is called an *m-open cover* of K if $K \subseteq \bigcup_j A_j$. Any subfamily of $\mathcal{A}$ which is also an m-open cover of K is called a *subcover* of $\mathcal{A}$ for K;

(b) a subset K of X is *m-compact* whenever given any m-open cover of K has a finite subcover.

Lemma 2.6. [3] Suppose $(X, \mathcal{M})$ is an m-compact minimal space, $\{A_i : i \in I\}$ is a family of subsets of X. If $\{m\text{-Cl}(A_i) : i \in I\}$ has the finite intersection property, then

$$\bigcap_{i \in I} m\text{-Cl}(A_i) \neq \emptyset.$$

Definition 2.7. Suppose (X, τ) is a topological space and also suppose $(Y, \mathcal{N})$ is a minimal space. A function $f : (X, \tau) \to (Y, \mathcal{N})$ is called *(τ, m)-continuous* if $f^{-1}(U) \in \tau$ for any $U \in \mathcal{N}$.

Lemma 2.8. *Suppose $(X, \mathcal{M})$ is a minimal vector space. Consider the multimap $\Gamma : \langle X \rangle \multimap X$ defined by $\Gamma(\{a_0, a_1, \cdots, a_n\}) = \{\Sigma_{i=0}^{n} \lambda_i a_i : 0 \leq \lambda_i \leq 1, \Sigma_{i=0}^{n} \lambda_i = 1\}$. For $A \in \langle X \rangle$ with $| A |= n+1$ define $\psi : \mathbb{R}^{n+1} \longrightarrow X$ by $\psi(\lambda_0, \lambda_1, \cdots, \lambda_n) = \sum\limits_{i=0}^{n} \lambda_i a_i$. Then ψ is (τ, m)-continuous.*

Proof. Suppose U is an m-open set, we must show that $\psi^{-1}(U)$ is open in $\mathbb{R}^{n+1}$. If $(\lambda_0, \lambda_1, \cdots, \lambda_n) \in \psi^{-1}(U)$, then $\lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_n a_n \in U$. Since $+$ and $\cdot$ are m-continuous, so there are open sets $D_0, D_1, \cdots, D_n \subseteq \mathbb{R}$ and m-open sets $V_0, V_1, \cdots, V_n$ in X with $\lambda_i \in D_i$ and $a_i \in V_i$ for $i = 0, 1, \cdots, n$ in which

$$D_0 \cdot V_0 + D_1 \cdot V_1 + \cdots + D_n \cdot V_n \subseteq U.$$

Therefore, $(\lambda_0, \lambda_1, \cdots, \lambda_n) \in D_0 \times D_1 \times D_2 \times \cdots \times D_n \subseteq \psi^{-1}(U)$ which implies that ψ is (τ, m)-continuous.

A *multimap* $F : X \multimap Y$ is a function from a set X into the power set of Y. Given $A \subseteq X$, set $F(A) = \bigcup_{x \in A} F(x)$. The multimap $F : X \multimap Y$ is said to be *surjective* if for any $y \in Y$, there is an $x \in X$ such that $y \in F(x)$. Note that $F : X \multimap Y$ is surjective if and only if $Y = \bigcup_{x \in X} F(x)$.

Definition 2.9. Suppose that D is a convex subset of minimal vector space X. A multimap $F : D \multimap X$ is called a *KKM map* if $co(A) \subseteq F(A)$ for any $A \in \langle D \rangle$.

3. Main Result

Theorem 3.1. (**Fan-KKM Principle**) *Suppose D is the set of vertices of an n-simplex Δ_n and also suppose that the multimap $F : D \multimap \Delta_n$ is a closed valued KKM map. Then $\bigcap\limits_{z \in D} F(z) \neq \emptyset$.*

Theorem 3.2. *Suppose that D is a nonempty subset of minimal vector space X and $F : D \multimap X$ is a multimap satisfying*

(a) *F has m-closed values,*

(b) *F is a* KKM *map.*

Then $\{F(z) : z \in D\}$ has the finite intersection property.
Further, if

(c) *$\bigcap_{z \in M} F(z)$ is m-compact for some $M \in \langle D \rangle$,*

then $\bigcap\limits_{z \in D} F(z) \neq \emptyset$.

Proof. Assume $N = \{a_0, a_1, ..., a_n\} \in \langle D \rangle$. Define the map $\phi_N : \Delta_n \longrightarrow co(N)$ such that $\phi_N\big(\sum_{i=0}^{n} \lambda_i e_i\big) = \sum_{i=0}^{n} \lambda_i a_i$ where $0 \leq \lambda_i \leq 1$, for $i = 0, \cdots, n$ and $\sum_{i=0}^{n} \lambda_i = 1$. Now, suppose that $\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\} \subseteq$

$\{e_0, e_1, \cdots, e_n\}$ for any choice $0 \leq i_0 < \cdots < i_k \leq n$. So

$$\phi_N\Big(\sum_{j=0}^{k} \lambda_j e_{i_j}\Big) = \sum_{j=0}^{k} \lambda_j a_{i_j} \text{ where } 0 \leq \lambda_j \leq 1 \text{ and } \sum_{j=0}^{k} \lambda_j = 1. \ (1)$$

Since F is a KKM map, $\sum_{j=0}^{k} \lambda_j a_{i_j} \in co(\{a_{i_0}, a_{i_1}, \cdots, a_{i_k}\}) \subseteq \bigcup_{j=0}^{k} F(a_{i_j})$ and via (1) we have

$$co(\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\}) \subseteq \bigcup_{j=0}^{k} \phi_N^{-1} F(a_{i_j}). \ (2)$$

Therefore from (2) the multimap $\phi : \Delta_n \multimap \Delta_n$ defined by $\phi(e_i) = \phi_N^{-1}(F(a_i))$ is a KKM map on $\{e_0, e_1, ..., e_n\}$. It follows from Lemma 2.8, Definition 2.7 and (a) that $\phi_N^{-1}(F(a_{i_j}))$ is closed in Δ_n. Now, Theorem 3.1 implies that

$$\bigcap_{i=0}^{n} \phi_N^{-1}(F(a_i)) \neq \emptyset,$$

and clearly $\bigcap_{i=0}^{n} F(a_i) \neq \emptyset$. For the second part, on the contrary suppose that

$$\emptyset = \bigcap_{z \in D} F(z) = \bigcap_{z \in M} F(z) \cap \bigcap_{z \in D \setminus M} F(z), \text{ and so } \bigcap_{z \in M} F(z) \subseteq \Big(\bigcap_{z \in D \setminus M} F(z)\Big)^c = \bigcup_{z \in D \setminus M} F(z)^c.$$

According to (c) there is $N \in \langle D \setminus M \rangle$ for which $\bigcap_{z \in M} F(z) \subseteq \bigcup_{z \in N} F(z)^c$, and hence

$$\bigcap_{z \in M \cup N} F(z) = \emptyset.$$

This contradicts with the fact that F has the finite intersection property on D.

Remark 3.3. Condition (c) in Theorem 3.2 is satisfied when X is an m-compact minimal space.

Lemma 3.4. *Suppose that Y is a minimal vector space and $X \subseteq Y$ is an m-compact convex set. Consider a surjective multimap $F : X \multimap Y$ such that $F(x)$ is m-open for any $x \in X$. Then there exists $\{x_1, x_2, \ldots, x_n\} \subseteq X$ such that $Co\{x_1, x_2, \ldots, x_n\} \cap \bigcap_{i=1}^{n} F(x_i) \neq \emptyset$.*

Proof. On the contrary, suppose for any $\{x_1, x_2, \ldots, x_n\} \subseteq X$ we have $co\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} (F(x_i))^c$. Now, define the multimap $G : X \multimap X$ by $G(x) = X \cap (F(x))^c$ for any $x \in X$. So the multimap G is m-closed valued in X. Since X is convex, $co\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} (F(x_i))^c \cap X = \bigcup_{i=1}^{n} G(x_i)$. So G is a KKM map and according to the Theorem 3.2, the family $\{G(x) : x \in X\}$ has the finite intersection property. Lemma 2.6 implies that $\bigcap_{x \in X} G(x) \neq \emptyset$. Hence

$\bigcap_{x\in X}(F(x))^c \neq \emptyset$ or $Y \neq \bigcup_{x\in X} F(x)$ which contradicts the fact that F is surjective.

In some works Lemma 3.4 is called open type of matching theorem. However it has a generalized form which is our next aim. The two following theorems are applied to prove Lemma 3.7 which has an essential role in the proof of Theorem 3.8

Theorem 3.5. [4] The product minimal space $\left(\prod_{\alpha\in I} X_\alpha, \prod_{\alpha\in I} \mathcal{M}_\alpha\right)$ is m-compact if and only if $(X_\alpha, \mathcal{M}_\alpha)$ is an m-compact minimal space, for any $\alpha \in I$.

Theorem 3.6. [11] Suppose that X and Y are two minimal spaces and $f : X \to Y$ is an m-continuous function. For any m-compact subset $K \subseteq X$, $f(K)$ is m-compact in Y.

Lemma 3.7. *Let X be a minimal vector space and $E \subseteq X$ be m-compact and convex. Let $A = \{a_1, a_2, ..., a_n\}$ be a finite subset of X. Then $co(A \cup E)$ is m-compact.*

Proof. Let $\Delta_n = \{\sum_{i=1}^n \lambda_i e_i : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1\}$, where $\{e_i\}$ is the standard base of $\mathbb{R}^n$. Define $\varphi : \Delta_{n+1} \times \{a_1\} \times \{a_2\} \times \cdots \times \{a_n\} \times E \longrightarrow co(A \cup E)$ by $\varphi(\lambda_1, ..., \lambda_{n+1}, a_1, ..., a_n, e) = \sum_{i=1}^n \lambda_i a_i + \lambda_{n+1} e$. The surjective function φ is m-continuous and $\Delta_{n+1} \times \{a_1\} \times \{a_2\} \times \cdots \times \{a_n\} \times E$ is m-compact by Theorem 3.5. It follows from Theorem 3.6 that $\varphi(\Delta_{n+1} \times \{a_1\} \times \{a_2\} \times \cdots \times \{a_n\} \times E) = co(A \cup E)$ is m-compact.

Theorem 3.8. (***Open Type Matching Theorem***) *Suppose that Y is a minimal vector space and $X \subseteq Y$ is a convex set. Consider the surjective multimap $F : X \multimap Y$ such that $F(x)$ is m-open for any $x \in X$. Suppose that $X_0 \subseteq X$ is m-compact and convex in Y such that $Y \setminus \bigcup_{x\in X_0} F(x)$ is m-compact or empty. Then there exists $\{x_1, x_2, \ldots, x_n\} \subseteq X$ such that $co\{x_1, x_2, \ldots, x_n\} \cap \bigcap_{i=1}^n F(x_i) \neq \emptyset$.*

Proof. If $Y = \bigcup_{x\in X_0} F(x)$, then conclusion follows from Lemma 3.4. Otherwise suppose that $H : X \multimap Y$ is a multimap defined as $H(x) = Y \setminus F(x)$ for any $x \in X$ and suppose that $D = Y \setminus \bigcup_{x\in X_0} F(x) = \bigcap_{x\in X_0} H(x)$, which is nonempty and m-compact subset of Y. On the contrary, suppose that $co\{x_1, x_2, \ldots, x_n\} \subseteq Y \setminus \bigcap_{i=1}^n F(x_i) = \bigcup_{i=1}^n H(x_i)$. Now, choose fixed $\{x_1, x_2, \ldots, x_n\} \subseteq X$. Set $X_1 = X_0 \cup \{x_1, x_2, \ldots, x_n\}$ and $K = co(X_0 \cup \{x_1, x_2, \ldots, x_n\})$. Since X_0 is m-compact, Lemma 3.7 implies that $K \subseteq Y$ is m-compact. Define the multimap $G : X_1 \multimap K$ such that $G(y) = K \cap H(y)$ for any $y \in X_1$. For each $y \in Y$, $F(y)$ is m-closed and for any $\{y_1, y_2, \ldots, y_m\} \subseteq X_1$ we have

$$co(\{y_1, y_2, \ldots, y_m\}) \subseteq K \cap \bigcup_{j=1}^m H(y_j) = \bigcup_{j=1}^m G(y_j).$$

According to Theorem 3.2 via Remark 3.3, we can deduce that $\bigcap_{y\in X_1} G(y) \neq \emptyset$. Since

$$\begin{aligned}\bigcap_{y\in X_1} G(y) &= \bigcap_{x\in X_0} \cap \bigcap_{i=1}^{n} H(x_i) \cap K \\ &= D \cap \bigcap_{i=1}^{n} H(x_i) \cap K \\ &\subseteq D \cap \bigcap_{i=1}^{n} H(x_i),\end{aligned}$$

So for any $\{x_1, x_2, \ldots, x_n\} \subseteq X$, $D \cap \bigcap_{i=1}^{n} H(x_i) \neq \emptyset$. Hence the family $\{D\cap H(x) : x \in X\}$ has the finite intersection property in m-compact set D. Then Lemma 2.6 implies that $\bigcap_{x\in X} D\cap H(x) \neq \emptyset$ or $\bigcap_{x\in X} H(x) \neq \emptyset$. This means that $Y \neq \bigcup_{x\in X} F(x)$, which contradicts the fact that F is surjective.

As an application of Theorem 3.8, we prove the following theorem which is a generalization of the classical KKM theorem [5]. This is the minimal version of the KKM theorem.

Theorem 3.9. *Suppose that Y is a minimal vector space, $X \subseteq Y$ is a convex set and $F : X \multimap Y$ is a* KKM *multimap with m-closed values. Let $X_0 \subseteq X$ be an m-compact convex set such that $\bigcap_{x\in X_0} F(x)$ is m-compact. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Proof. Suppose that $\bigcap_{x\in X} F(x) = \emptyset$. Define the multimap $H : X \multimap Y$ such that $H(x) = Y \setminus F(x)$ for all $x \in X$. So H is surjective and open valued. Since $\bigcap_{x\in X_0} F(x) = \bigcap_{x\in X_0} Y \setminus H(x) = Y \setminus \bigcup_{x\in X_0} H(x)$, hence $Y \setminus \bigcup_{x\in X_0} H(x)$ is m-compact or empty. All conditions of Theorem 3.8 are satisfied for H and so there exists $\{x_1, x_2, \ldots, x_n\} \subseteq X$ such that $co\{x_1, x_2, \ldots, x_n\} \cap \bigcap_{i=1}^{n} H(x_i) \neq \emptyset$. Then it is easy to see that $co\{x_1, x_2, \ldots, x_n\} \nsubseteq \bigcup_{i=1}^{n} F(x_i)$ which is a contradiction, because F is a KKM map.

Corollary 3.10. *Suppose that Y is a topological vector space, $X \subseteq Y$ is a convex set and $F : X \multimap Y$ is a* KKM *multimap with closed values. Let $F(x)$ is compact for at least one $x \in X$. Then $\bigcap_{x\in X} F(x) \neq \emptyset$.*

Remark 3.11. Note that

(1) Corollary 3.10 is a generalized type of classical KKM theorem in topological vector spaces which is restated here as a corollary of Theorem 3.9.

(2) Lemma 3.4, Theorem 3.8 and Theorem 3.9 originally go back to Ky Fan.

References

[1] M. Alimohammady, M. Roohi, Linear minimal space, Chaos, Solitons & Fractals. 33(2007) 1348–1354.

[2] M. Alimohammady, M. Roohi. M. R. Delavar, Knaster-Kuratowski-Mazurkiewicz theorem in minimal generalized convex spaces, Nonlinear Funct. Anal. Appl. 13(2008) 483–492.

[3] M. Alimohammady, M. Roohi, M. R. Delavar, Transfer closed multimaps and Fan-KKM principle, Nonlinear Funct. Anal. Appl. 13(2008) 597–611.

[4] M. R. Delavar, S. A. Mohammadzadeh, M. Roohi, Fan's minimax inequality in MG-convex spaces, Nonlinear Funct. Anal. Appl. (to apear).

[5] B. Knaster, K. Kuratowski, S. Mazurkiewicz, Ein Beweis des fixpunktsatzes fur n-dimensionale simplexe, Fund. Math. 14(1929) 132–137.

[6] H. Maki, On generalizing semi-open sets and preopen sets, Meeting on Topological Spaces Theory and its Application, Augoust 1996, 13–18.

[7] H. Maki, J. Umehara, T. Noiri, Every topological space is pre $T_{\frac{1}{2}}$, Mem. Fac. Sci. Kochi Univ. Ser A. Math. 17(1996) 33–42.

[8] S. Park, Eighty years of the Brouwer fixed point theorem, Antipodal points and Fixed Points(by J. Jaworoeski, W. A. Kirk ans S. Park), Lect. Notes Ser. 28, RIM-GARC, Seoul Nat. Univ. 1995, 55–97.

[9] S. Park, Ninety years of the Brouwer fixed point theorem, Vietnam J. Math. 27(1999) 193–232.

[10] S. Park, Some coincidence theorem on acyclic multifunctions and applications to KKM theory, Fixed Point Theory and Applications(K.-K. Tan, ed.), World Sci. River Edge, NJ, 1992, 248–277.

[11] V. Popa, T. Noiri, On M-continuous functions, Anal. Univ. Dunarea Jos-Galati, Ser. Mat. Fiz. Mec. Teor. Fasc. II 18(2000) 31–41.