

B-Focal Curves Of Biharmonic B-General Helices In Heisenberg Group

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ABSTRACT. In this paper, we study B -focal curves of biharmonic B -general

helices according to Bishop frame in the Heisenberg group Heis^3 . Finally, we characterize the B -focal curves of biharmonic B -general helices in terms of Bishop frame in the Heisenberg group Heis^3 .

Keywords: Biharmonic curve, Bishop frame, Heisenberg group, Parallel transport, Helix.

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1. INTRODUCTION

A smooth map $\phi : N \rightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 dv_h,$$

where $\mathcal{T}(\phi) := \text{tr} \nabla^\phi d\phi$ is the tension field of ϕ .

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The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \text{tr}R(\mathcal{T}(\phi), d\phi) d\phi, \quad (1.1)$$

and called the bitension field of ϕ . Non-harmonic biharmonic maps are called proper biharmonic maps, [7,8].

In this paper, we study \mathcal{B} –focal curves of biharmonic \mathfrak{B} –general helices according to Bishop frame in the Heisenberg group Heis^3 . Finally, we characterize the \mathcal{B} –focal curves of biharmonic \mathfrak{B} –general helices in terms of Bishop frame in the Heisenberg group Heis^3 .

2. THE HEISENBERG GROUP Heis^3

Heisenberg group Heis^3 can be seen as the space \mathbb{R}^3 endowed with the following multiplication:

$$(\bar{x}, \bar{y}, \bar{z})(x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \frac{1}{2}\bar{x}y + \frac{1}{2}x\bar{y}) \quad (2.1)$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of Heis^3 has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}, \quad (2.2)$$

for which we have the Lie products [11]

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

We obtain

$$\begin{aligned} \nabla_{\mathbf{e}_1} \mathbf{e}_1 &= \nabla_{\mathbf{e}_2} \mathbf{e}_2 = \nabla_{\mathbf{e}_3} \mathbf{e}_3 = 0, \\ \nabla_{\mathbf{e}_1} \mathbf{e}_2 &= -\nabla_{\mathbf{e}_2} \mathbf{e}_1 = \frac{1}{2} \mathbf{e}_3, \\ \nabla_{\mathbf{e}_1} \mathbf{e}_3 &= \nabla_{\mathbf{e}_3} \mathbf{e}_1 = -\frac{1}{2} \mathbf{e}_2, \\ \nabla_{\mathbf{e}_2} \mathbf{e}_3 &= \nabla_{\mathbf{e}_3} \mathbf{e}_2 = \frac{1}{2} \mathbf{e}_1. \end{aligned}$$

3. BIHARMONIC \mathfrak{B} -GENERAL HELICES WITH BISHOP FRAME IN THE HEISENBERG GROUP $Heis^3$

Let $\gamma : I \rightarrow Heis^3$ be a non geodesic curve on the Heisenberg group $Heis^3$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group $Heis^3$ along γ defined as follows:

\mathbf{T} is the unit vector field γ' tangent to γ , \mathbf{N} is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and \mathbf{B} is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N},\end{aligned}\tag{3.1}$$

where κ is the curvature of γ and τ is its torsion and

$$\begin{aligned}g(\mathbf{T}, \mathbf{T}) &= 1, \quad g(\mathbf{N}, \mathbf{N}) = 1, \quad g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0.\end{aligned}\tag{3.2}$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as [1]

$$\begin{aligned}\nabla_{\mathbf{T}}\mathbf{T} &= k_1\mathbf{M}_1 + k_2\mathbf{M}_2, \\ \nabla_{\mathbf{T}}\mathbf{M}_1 &= -k_1\mathbf{T}, \\ \nabla_{\mathbf{T}}\mathbf{M}_2 &= -k_2\mathbf{T},\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}g(\mathbf{T}, \mathbf{T}) &= 1, \quad g(\mathbf{M}_1, \mathbf{M}_1) = 1, \quad g(\mathbf{M}_2, \mathbf{M}_2) = 1, \\ g(\mathbf{T}, \mathbf{M}_1) &= g(\mathbf{T}, \mathbf{M}_2) = g(\mathbf{M}_1, \mathbf{M}_2) = 0.\end{aligned}\tag{3.4}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures. where $\theta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$. Thus, Bishop curvatures are defined by

$$\begin{aligned}k_1 &= \kappa(s) \cos \theta(s), \\ k_2 &= \kappa(s) \sin \theta(s).\end{aligned}\tag{3.5}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned}\mathbf{T} &= T^1\mathbf{e}_1 + T^2\mathbf{e}_2 + T^3\mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1\mathbf{e}_1 + M_1^2\mathbf{e}_2 + M_1^3\mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1\mathbf{e}_1 + M_2^2\mathbf{e}_2 + M_2^3\mathbf{e}_3.\end{aligned}\tag{3.6}$$

To separate a general helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as \mathfrak{B} -general helix.

Theorem 3.1. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic \mathfrak{B} -general helix. Then the parametric equation of $\gamma_{\mathfrak{B}}$ are*

$$\begin{aligned} x_{\mathfrak{B}}(s) &= \frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_2, \\ y_{\mathfrak{B}}(s) &= -\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_3, \quad 3.7(3.7) \\ z_{\mathfrak{B}}(s) &= (\cos \theta) s + \frac{\sin^2 \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \left(\frac{s}{2} - \frac{\sin 2\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right]}{4\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}}\right) \\ &\quad - \frac{\zeta_1 \sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_4, \end{aligned}$$

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration.

4. \mathcal{B} -FOCAL CURVE OF BIHARMONIC \mathfrak{B} -GENERAL HELICES WITH BISHOP FRAME IN THE HEISENBERG GROUP $Heis^3$

Denoting the focal curve by $\mathbf{focal}_{\gamma_{\mathfrak{B}}}^{\mathcal{B}}$ of $\gamma_{\mathfrak{B}}$, we can write

$$\mathbf{focal}_{\gamma_{\mathfrak{B}}}^{\mathcal{B}}(s) = (\gamma + \mathbf{f}_1$$

$\mathbf{BM}_1 + \mathbf{f}_2^{\mathcal{B}} \mathbf{M}_2)(s)$, (4.1) where the coefficients $\mathbf{f}_1^{\mathcal{B}}, \mathbf{f}_2^{\mathcal{B}}$ are smooth functions of the parameter of the curve γ , called the first and second focal curvatures of γ , respectively.

To separate a focal curve according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the focal curve defined above as \mathcal{B} -focal curve.

Theorem 4.1. *Let $\gamma_{\mathfrak{B}} : I \longrightarrow Heis^3$ be a unit speed biharmonic \mathfrak{B} -general helix with non-zero natural curvatures. Then, the position*

vector of $\text{focal}_{\gamma_{\text{B}}}^B$ is

$$\begin{aligned}
\text{focal}_{\gamma_{\text{B}}}^B(s) = & \left[\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \right. \\
& + \mathfrak{p} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \\
& + \frac{1 - \mathfrak{p}k_1}{k_2} \cos \theta \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_2\mathbf{e}_1 \\
& + \left[-\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \right. \\
& - \mathfrak{p} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \\
& + \frac{1 - \mathfrak{p}k_1}{k_2} \cos \theta \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_3\mathbf{e}_2 \\
& + \left[-\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_2\right] \\
& \left. \left[-\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_3\right] \right. \\
& + (\cos \theta) s + \frac{\sin^2 \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \left(\frac{s}{2} - \frac{\sin 2\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right]}{4\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}}\right) \\
& - \frac{\zeta_1 \sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \\
& \left. - \frac{1 - \mathfrak{p}k_1}{k_2} \sin \theta + \zeta_4\right] \mathbf{e}_3,
\end{aligned}$$

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration.

Proof. Assume that γ is a unit speed biharmonic curve and $\text{focal}_{\gamma_{\text{B}}}^B$ its \mathcal{B} -focal curve on Heis³.

By differentiating of the formula (4.1), we get

$$\mathfrak{F}_{\gamma}^{\mathcal{B}}(s)' = (1 - \mathfrak{f}_1^{\mathcal{B}}k_1 - \mathfrak{f}_2^{\mathcal{B}}k_2)\mathbf{T} + (\mathfrak{f}_1^{\mathcal{B}})'\mathbf{M}_1 + (\mathfrak{f}_2^{\mathcal{B}})'\mathbf{M}_2.$$

Using above equation, the first 2 components vanish, we get

$$\begin{aligned}
\mathfrak{f}_1^{\mathcal{B}}k_1 + \mathfrak{f}_2^{\mathcal{B}}k_2 &= 1, \\
(\mathfrak{f}_1^{\mathcal{B}})' &= 0.
\end{aligned}$$

Considering second equation above system, we chose

$$f_1^{\mathcal{B}} = \mathfrak{p} = \text{constant} \neq 0.$$

Then

$$f_2^{\mathcal{B}} = \frac{1 - \mathfrak{p}k_1}{k_2}.$$

On the other hand, we have

$$\begin{aligned} \mathbf{T} &= \sin \theta \cos\left[\left(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \mathbf{e}_1 \\ &+ \sin \theta \sin\left[\left(\frac{k_1^2 + k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \mathbf{e}_2 + \cos \theta \mathbf{e}_3. \end{aligned}$$

Thus, it is seen that

$$\text{focal}_{\gamma}^{\mathcal{B}}(s) = \left(\gamma + \mathfrak{p}\mathbf{M}_1 + \frac{1 - \mathfrak{p}k_1}{k_2}\mathbf{M}_2\right)(s),$$

By means of obtained equations, we express (4.2). This completes the proof.

In the light of Theorem 4.1, we express:

Theorem 4.2. *Let $\gamma_{\mathcal{B}} : I \rightarrow \text{Heis}^3$ be a unit speed biharmonic \mathcal{B} -general helix and $\text{focal}_{\gamma_{\mathcal{B}}}^{\mathcal{B}}$ its \mathcal{B} -focal curve on Heis^3 . Then, the*

parametric equations of $\mathbf{focal}_{\gamma_3}^B$ are given by

$$\begin{aligned}
x_{\mathbf{focal}_{\gamma}^B}(s) &= \left[\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \right. \\
&\quad + \mathfrak{p} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \\
&\quad \left. + \frac{1-\mathfrak{p}k_1}{k_2} \cos \theta \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_2\right], \\
z_{\mathbf{focal}_{\gamma}^B}(s) &= \left[-\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \right. \\
&\quad - \mathfrak{p} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \\
&\quad \left. + \frac{1-\mathfrak{p}k_1}{k_2} \cos \theta \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_3\right], \\
z_{\mathbf{focal}_{\gamma}^B}(s) &= \left[\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \right. \\
&\quad + \mathfrak{p} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \\
&\quad + \frac{1-\mathfrak{p}k_1}{k_2} \cos \theta \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_2 \\
&\quad \left[-\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \right. \\
&\quad - \mathfrak{p} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] \\
&\quad + \frac{1-\mathfrak{p}k_1}{k_2} \cos \theta \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_3 \\
&\quad + \left[-\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_2 \right] \\
&\quad \left. \left[-\frac{\sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] + \zeta_3 \right] \right. \\
&\quad \left. + (\cos \theta) s + \frac{\sin^2 \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \left(\frac{s}{2} - \frac{\sin 2\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right]}{4\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \right) \right. \\
&\quad \left. - \frac{\zeta_1 \sin \theta}{\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}}} \cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2 \theta} - \cos \theta\right)^{\frac{1}{2}} s + \zeta_0\right] - \frac{1-\mathfrak{p}k_1}{k_2} \sin \theta + \zeta_4 \right],
\end{aligned}$$

where $\zeta_0, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ are constants of integration.

Proof. Substituting (2.1) into (4.2), we obtain above system. This completes the proof.

If we use Mathematica both $\gamma_{\mathfrak{B}}$ and its focal curve, we have

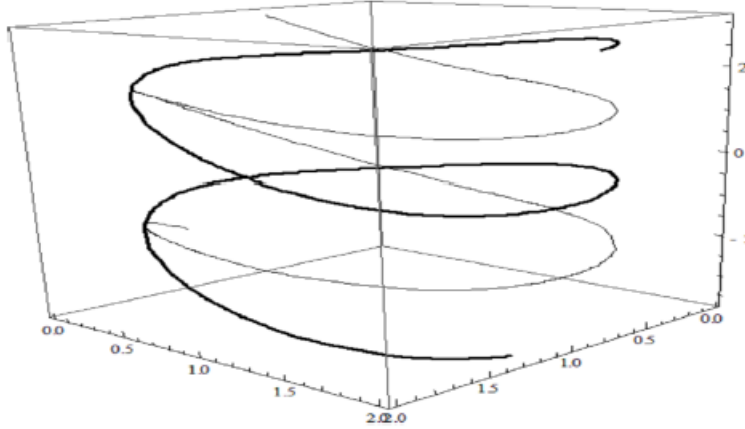


FIGURE 1.

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