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Weak solvability via Lagrange multipliers for Frictional antiplane contact problems of p(x)-Kirchhoff type

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Abstract. This paper is concerned with the existence and uniqueness of solutions for a class of frictional antiplane contact problems of p(x)-Kirchhoff type on a bounded domain $\Omega \subseteq \mathbb{R}^2$. Using an abstract Lagrange multiplier technique and the Schauder fixed point theorem we establish the existence of weak solutions. Imposing some suitable monotonicity conditions on the datum f_1 the uniqueness of the solution is obtained.

Keywords: frictional antiplane contact problems; p(x)-Kirchhoff equation; Schauder fixed point theorem; uniqueness

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1. Introduction

In this work, we are concerned with the following Kirchhoff type problem

$$
\begin{aligned}
-M\,(L(u))\,\mathrm{div}(a(|\nabla u|^{p(x)})|\nabla u|^{p(x)}\nabla u) = f_1(x,u) \qquad & \text{in} \quad \Omega \\
u = 0 \qquad & \text{on } \Gamma_1 \\
M(L(u))a(|\nabla u|^{p(x)})|\nabla u|^{p(x)}\nabla u.\nu = f_2(x) \qquad & \text{on} \quad \Gamma_2 \\
|M(L(u))a(|\nabla u|^{p(x)})|\nabla u|^{p(x)}\nabla u.\nu| \leq g(x), \qquad & \\
M(L(u))a(|\nabla u|^{p(x)})|\nabla u|^{p(x)}\nabla u.\nu = -g\frac{u}{|u|}, \quad \text{if} \quad u \neq 0 \qquad & \text{on} \quad \Gamma_3
\end{aligned}
\tag{1.1}
$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain with smooth enough boundary Γ, partitioned in three parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that meas $(\Gamma_i) > 0$, $(i = 1,2,3)$; $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$, $f_2 : \Gamma_2 \to \mathbb{R}$, $g : \Gamma_3 \to \mathbb{R}$, $M : [0,+\infty[\to [m_0,+\infty[$ and $a : \mathbb{R}^+{}_0 \to \mathbb{R}^+{}_0$ are given functions, $p \in C(\overline{\Omega})$ and $L(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx$, with $A(t) = \int_0^t a(\tau)\,d\tau$.

The study of the $p(x)$- Kirchhoff type equations with nonlinear boundary conditions of different class have been a very interesting topic in the recent years. Let us just quote[1, 8, 16, 24] and references therein. One reason of such interest is due to their frequent appearance in applications such as the modeling of electrorheological fluids [20], image restoration [9], elastic mechanics [25] and continuum mechanics [3]. The other reason is that the nonlocal problems with variable exponent, in addition to their contributions to the modelization of many physical and biological phenomena, are very interesting from a purely mathematical point of view as well; we refer the reader to [2, 18, 22]. Cojocaru-Matei [6] studied the unique solvability of problem (1.1) in the case $M(s) = 1 = a(s), f_1(x,u) \equiv f_1(x), p =$ constant ≥ 2, which models the antiplane shear deformation of a nonlinearly elastic cylindrical body in frictional contact on Γ_3 with a rigid foundation; see, e.g. [21]. They used a technique involving dual Lagrange multipliers, which allows to write efficient algorithms to approximate the weak solutions; see [17]. For this situation, the behavior of the material is described by the Hencky-type constitutive law:

$$\boldsymbol{\sigma(x)} = ktr\boldsymbol{\varepsilon(u(x))I_3} + \mu(x)\|\boldsymbol{\varepsilon^D(u(x))}\|^{\frac{p(x)-2}{2}}\boldsymbol{\varepsilon^D(u(x))}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, tr is the trace of a Cartesian tensor of second order,$\boldsymbol{\sigma(x)}$ $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor, $\boldsymbol{u}$ is the displacement vector,$\boldsymbol{I_3}$ is the identity tensor, k, μ are material parameters, p is a given function;$\boldsymbol{\varepsilon^D}$ is the *desviator* of the tensor $\boldsymbol{\varepsilon}$ defined by $\boldsymbol{\varepsilon^D} = \boldsymbol{\varepsilon} - \frac{1}{3}(tr\boldsymbol{\varepsilon})\boldsymbol{I_3}$ where $tr\boldsymbol{\varepsilon} = \sum_{i=1}^{3} \varepsilon_{ii}$; see for instance [15].

Inspired by the above works, we study the existence of weak solutions for problem (1.1), under appropriate assumptions on M and f_1, via Lagrange multipliers and the Schauder fixed point theorem. In this sense, we extend and generalize the result the main result in [6]. Also, we state a simple uniqueness result under suitable monotonicity condition on f_1.

The paper is designed as follows. In Section 2, we introduce the mathematical preliminaries and give several important properties of $p(x)$-Laplacian-like operator. We deliver a weak variational formulation with Lagrange multipliers in a dual space. Section 3, is devoted to the proofs of main results.

2. Preliminaries

For the reader's convenience, we point out some basic results on the theory of Lebesgue-Sobolev spaces with variable exponent. In this context we refer the reader to [11, 20] for details. Firstly we state some basic properties of spaces $W^{1,p(x)}(\Omega)$ which will be used later. Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\, dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Proposition 2.1 ([14])**.** *The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*

Proposition 2.2 ([14]). *Set* $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$. *For any* $u \in L^{p(x)}(\Omega)$, *then*

(1) *for* $u \neq 0$, $|u|_{p(x)} = \lambda$ *if and only if* $\rho(\frac{u}{\lambda}) = 1$;
(2) $|u|_{p(x)} < 1$ $(= 1; > 1)$ *if and only if* $\rho(u) < 1$ $(= 1; > 1)$;
(3) *if* $|u|_{p(x)} > 1$, *then* $|u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}$;
(4) *if* $|u|_{p(x)} < 1$, *then* $|u|^{p^+}_{p(x)} \leq \rho(u) \leq |u|^{p^-}_{p(x)}$;
(5) $\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ *if and only if* $\lim_{k\to+\infty} \rho(u_k) = 0$;
(6) $\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ *if and only if* $\lim_{k\to+\infty} \rho(u_k) = +\infty$.

Proposition 2.3 ([12, 14]). *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) \leq p^*(x)$ $(q(x) < p^*(x))$ *for* $x \in \overline{\Omega}$, *then there is a continuous (compact) embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.4 ([14]). *The conjugate space of* $L^{p(x)}(\Omega)$ *is* $L^{q(x)}(\Omega)$, *where* $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ *holds a.e. in* Ω. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{q(x)}(\Omega)$, *we have the following Hölder-type inequality*

$$\Big|\int_\Omega uv\,dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

We introduce the following closed space of $W^{1,p(x)}(\Omega)$

$$X = \{v \in W^{1,p(x)}(\Omega) : \gamma u = 0 \quad \text{a. e. on} \quad \Gamma_1\} \tag{2.1}$$

where γ denotes the Sobolev trace operator and $\Gamma_1 \subseteq \Gamma$, meas $(\Gamma_1) > 0$, therefore X is a separable reflexive Banach space. Now, we denote

$$\|u\|_X = |\nabla u|_{p(x)}, \qquad u \in X.$$

This functional represents a norm on X.

Proposition 2.5 ([4]). *There exists* $c > 0$ *such that*

$$\|u\|_{1,p(x)} \leq C\|u\|_X \quad \text{for all } u \in X.$$

Then, the norms $\|.\|_X$ and $\|.\|_{1,p(x)}$ are equivalent on X.

We assume that $a(x,\xi) : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}^n$ is the continuous derivative with respect to ξ of the continuous mapping $\Phi : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}, \Phi = \Phi(x,\xi)$, i,e. $a(x,\xi) = \nabla_\xi \Phi(x,\xi)$. The mappings a and Φ verify the following assumptions:

(Φ_1) $\Phi(x,0) = 0$for a.e.$x \in \overline{\Omega}$.
(Φ_2) There exists $c > 0$ such that the function a satisfies the growth condition $|a(x,\xi)| \leq c(1 + |\xi|^{p(x)-1})$for a.e.$x \in \overline{\Omega}$and all$\xi \in \mathbb{R}^N$, where $|.|$ denotes the Euclidean norm.

(Φ_3) The monotonicity condition

$$C_3|\xi - \varsigma|^{p(x)} \leq (a(x,\xi) - a(x,\varsigma)).(\xi - \varsigma)$$

holds for a.e. $x \in \overline{\Omega}$ and $\forall \xi, \varsigma \in \mathbb{R}^N$. With equality if and only if $\xi = \varsigma$

(Φ_4) The inequalities $|\xi|^{p(x)} \leq a(x,\xi).\xi \leq p(x)A(x,\xi)$ hold for a.e. $x \in \overline{\Omega}$ and all $\xi \in \mathbb{R}^N$.

(Φ_5) There exists $C_5 > 0$ such that for all $\xi, \varsigma \in \mathbb{R}^N$ and almost every $x \in \overline{\Omega}$

$$|a(x,\xi) - a(x,\varsigma)| \leq C_5(1 + |\xi|^{p(x)-2} + |\varsigma|^{p(x)-2})|\xi - \varsigma|$$

The operator L is well defined and of class $C^1(W^{1,p(x)}(\Omega), \mathbb{R})$. The Fréchet derivative operator of L in weak sense $L' : X \to X'$ is

$$\langle L'u, v \rangle = \int_\Omega a(x, \nabla u).\nabla v \, dx, \ \forall u, v \in X. \tag{2.2}$$

Proposition 2.6. *The functional $L : X \to \mathbb{R}$ is convex. The mapping $L' : X \to X'$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \ \textit{and} \ \limsup_{n\to+\infty} L'(u_n)(u_n - u) \leq 0 \ \textit{implies} \ u_n \to u,$$

where X' is the dual space of X.

Proof. This result is obtained in a similar manner as the one given in [23], thus we omit the details. □

Now, we define the spaces

$$S = \Big\{ u \in W^{\frac{1}{p'(x)}, p(x)}(\Gamma) : \exists v \in X \quad \text{such that} \quad u = \gamma v \quad \text{a.e on} \quad \Gamma \Big\} \tag{2.3}$$

which is a real reflexive Banach space, $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ for all $x \in \Omega$, and

$$Y = S', \text{ the dual of the space S.} \tag{2.4}$$

Let us introduce a bilinear form

$$b : X \times Y \longrightarrow \mathbb{R} \quad : b(v, \mu) = \langle \ \mu, \gamma v \ \rangle_{Y \times S}\,, \tag{2.5}$$

a Lagrange multiplier $\lambda \in Y$,

$$\langle \ \lambda, z \ \rangle = -\int_{\Gamma_3} M\left(L(u)\right) a(x, \nabla u).\nu z \, d\Gamma \quad , \quad \forall z \in S$$

and the set of Lagrange multipliers

$$\Lambda = \Big\{ u \in Y : \langle \ \mu, z \ \rangle \leqslant \int_{\Gamma_3} g(x)|z(x)| \quad , \quad \forall z \in S \Big\}. \tag{2.6}$$

From $(1.1)_4$ we deduce that $\lambda \in \Lambda$.

Let u be a regular enough function satisfying problem (1.1). After some computations we get (by using density results)

$$M\left(L(u)\right)\int_\Omega a(x,\nabla u).\nabla v\,dx = \int_\Omega f_1(x,u)v\,dx$$

$$+\int_{\Gamma_2} f_2(x)\gamma v\,d\Gamma + M\left(L(u)\right)\int_{\Gamma_3} a(x,\nabla u)\gamma v\,d\Gamma \qquad (2.7)$$

for all $v \in X$, where u satisfies $(1.1)_5$ on Γ_3

Now, we write problem (2.7) as an abstract mixed variational problem (by means a Lagrange multipliers technique)

We define the following operators:

$i)$ $A : X \to X'$, given by

$$\langle\ Au, v\ \rangle = M\left(L(u)\right)\int_\Omega a(x,\nabla u).\nabla v\,dx,\ \ u,v \in X. \qquad (2.8)$$

$ii)$ $F : X \to X'$, given by

$$\langle\ F(u), v\ \rangle = \int_\Omega f_1(x,u)v dx + \int_{\Gamma_2} f_2(x)\gamma v\,dx \quad , \quad u,v \in X.$$

So, we are led to the following variational formulation of problem (1.1)

Problem 1. Find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} \langle\ Au, v\ \rangle + b(v,\lambda) &= \langle\ F(u), v\ \rangle \quad , \quad \forall v \in X \qquad (2.9)\\ b(u,\mu-\lambda) &\leq 0 \quad \forall \mu \in \Lambda \subseteq Y \end{aligned}$$

To solve this problem, we will apply the Schauder fixed point theorem.

Firstly, we "freeze" the state variable u on the function F, that is we fix $w \in X$ such that $f = F(w) \in X'$.

Hence, we arrive at the following abstract mixed variational problem.

Problem 2. Given $f \in X'$ find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} \langle\ Au, v\ \rangle + b(v,\lambda) &= \langle\ f, v\ \rangle \quad , \quad \forall v \in X\\ b(u,\mu-\lambda) &\leq 0 \quad \forall \mu \in \Lambda \subseteq Y. \qquad (2.10) \end{aligned}$$

The unique solvability of Problem 2 is given under the following generalized assumptions.

Let $(X, \|\|_X)$ and $(Y, \|\|_Y)$ be two real reflexive Banach space.

(B_1): $A : X \to X'$ is hemicontinuous;

(B_2): $\exists h : X \to \mathbb{R}$ such that

(a) $h(tw) = t^\gamma h(w)$ with $\gamma > 1$, $\forall t > 0, w \in X$;

(b) $\langle\ Au - Av, u - v\ \rangle_{X\times X} \geq h(v-u),\ \forall u,v \in X$;

(c) $\forall (x_\nu) \subseteq X : x_\nu \rightharpoonup x$ in$X \Longrightarrow h(x) \leq \lim_{\nu\to\infty} \sup h(x_\nu)$

(B_3): A is coercive.
(B_4): The form $b : X \times Y$ es bilinear, and
 (i) $\forall (u_\nu) \subseteq X : u_\nu \rightharpoonup u$ in $X \Longrightarrow b(u_\nu, \lambda_\nu) \to b(u, \lambda)$
 (ii) $\forall (\lambda_\nu) \subseteq Y : \lambda_\nu \rightharpoonup y$ in $Y \Longrightarrow b(v_\nu, \lambda_\nu) \to b(v, \lambda)$
 (iii) $\exists\, \widehat{\alpha} > 0 : \inf\limits_{\substack{\mu \in I \\ u \neq 0}} \sup\limits_{\substack{v \in X \\ v \neq 0}} \dfrac{b(v,\mu)}{|v|_X |\mu|_Y} \geq \widehat{\alpha}$
(B_5): Λ is a bounded closed convex subset of Y such that $0_Y \in \Lambda$.
(B_6): $\exists C_1 > 0, q > 0 : h(v) \geq C_1 \|v\|_X^q \quad , \quad \forall v \in X.$

Theorem 2.7. *Assume* (B_1) - (B_6)*. Then there exists a unique solution* $(u, \lambda) \in X \times \Lambda$ *of Problem* 2*.*

Proof. See [6].
To solve Problem 1, we start by stating the following assumptions on M , f_1 , f_2 and g

- (A_1) $M : [0, +\infty[\to [m_0, +\infty[$ is a locally Lipschitz-continuous and nondecreasing function; $m_0 > 0$.
- (A_2) $f_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying
$$|f_1(x,t)| \leq c_1 + c_2 |t|^{\alpha(x)-1} \ , \ \forall (x,t) \in \Omega \times \mathbb{R},$$
$\alpha \in C_+(\overline{\Omega})$with $\alpha(x) < p^*(x),\ \alpha^+ < p^-$.
- (A_3) $f_2 \in L^{p'(x)}(\Gamma_2),\ g \in L^{p'(x)}(\Gamma_3),\ g(x) \geq 0$ a.e on Γ_3.

We have the following properties about the operator A.

Proposition 2.8. *If (A_1) holds, then*

- (i) *A is locally Lipschitz continuous.*
- (ii) *A is bounded, strictly monotone. Furthermore*
$$\langle Au - Av, u - v \rangle \geq k_p \|u - v\|_X^{\hat{p}}$$
where
$$\hat{p} = \begin{cases} p^- & \text{if } \|u-v\|_X > 1, \\ p^+ & \text{if } \|u-v\|_X \leq 1. \end{cases}$$
So, we can take $h(v) = k_p \|v\|_X^{\hat{p}}$.
- (iii) $\frac{\langle Au, u \rangle}{\|u\|_X} \to +\infty \quad$ *as* $\|u\|_X \to +\infty$.

Proof. (i) Assume that M is Lipschitz in $[0, R_1]$ with Lipschitz constant L_M, $R_1 > 0$. We have, for $u, v, w \in B(0, R_1)$
$$\begin{aligned} \langle Au - Av, w \rangle = & [M(L(u)) - M(L(v))] \int_\Omega a(x, \nabla u).\nabla w \, dx \\ & + M(L(v)) \int_\Omega (a(x,\nabla u) - a(x, \nabla v)).\nabla w \, dx. \end{aligned}$$

Using the Lipschitz continuity of M, the Holder inequality and (Φ_5) we get

$$|\langle Au - Av, w\rangle| \leq C\|u - v\|_X \|w\|_X,$$

which implies $\|Au - Av\|_{X'} \leq C\|u - v\|_X$.

ii)The functional $S : X \to X'$ defined by

$$\langle Su, v\rangle = \int_\Omega a(x, \nabla u).\nabla v\, dx \qquad \forall u, v \in X, \tag{2.11}$$

is bounded (See [19]). Then

$$\langle Au, v\rangle = M(L(u))\langle Su, v\rangle \qquad \forall u, v \in X. \tag{2.12}$$

Hence, since M is continuous and L is bounded (see Proposition 2.6), A is bounded.

To obtain that A is strictly monotone, we observe that L' is strictly monotone.Hence, L is strictly convex. Moreover, since M is nondecreasing, $\hat{M}(t) = \int_0^t M(\tau)\, d\tau$ is convex in $[0, +\infty[$. Consequently, $\forall s, t \in\,]0, 1[$ with $s + t = 1$ one has

$$\hat{M}(L(su+tv)) < \hat{M}(sL(u)+tL(v)) \leq s\hat{M}(L(u))+t\hat{M}(L(v)), \forall u, v \in X, u \neq v.$$

This shows $\Psi(u) = \hat{M}(L(u))$ is strictly convex, then $\Psi'(u) = M(L(u))L'(u)$ is strictly monotone, which means that A is strictly monotone.

To establish the inequality in ii), we apply Lemma 3 in [5] to obtain

$$\begin{aligned}\langle Au - Av, u - v\rangle \geq &\int_\Omega \left[M(L(u))a(x, \nabla u) - M(L(v))a(x, \nabla u)\right].(\nabla v - \nabla u)\, dx\\ \geq &m_0 \int_\Omega \frac{1}{p(x)}(|\nabla u - \nabla u|^{p(x)})\, dx \geq \frac{m_0}{p^+}\int_\Omega |\nabla u - \nabla u|^{p(x)}\, dx\\ \geq &\frac{m_0}{p^+}\|u - v\|_X^{\hat{p}}.\end{aligned}$$

iii)For $u \in X$ with $\|u\|_X > 1$ we have

$$\begin{aligned}\frac{\langle Au, u\rangle}{\|u\|_X} = &\frac{M(L(u))\displaystyle\int_\Omega \left[\left(1 + \frac{|\nabla u|^{p(x)}}{\sqrt{1 + |\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)}\right] dx}{\|u\|}\\ \geq &m_0\|u\|_X^{p^- - 1} \to +\infty \text{ as } \|u\|_X \to +\infty.\end{aligned}$$

□

Proposition 2.9. *The form $b : X \times Y \to \mathbb{R}$ defined in* (2.5) *is bilinear and, it verifies i), ii) and iii) in assumption* (B_4). *Moreover*

$$b(u,\mu) \leq \int_{\Gamma_3} g(x)|u(x)|\,d\Gamma \text{ for all } \mu \in \Lambda; \tag{2.13}$$

$$b(u,\lambda) = \int_{\Gamma_3} g(x)|u(x)|\,d\Gamma \tag{2.14}$$

$$b(u,\mu-\lambda) \leq 0 \quad \text{for all } \mu \in \Lambda. \tag{2.15}$$

Moreover, Λ *is bounded.*

Proof. The assertions i), ii), iii) and Λ bounded are word for word as [6], Theorem 3, pags 138-139.

It is obvious to check (2.13). To justify (2.14), we have to show that, a.e. $x \in \Omega$

$$-M((L(u))\left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\frac{\partial u(x)}{\partial \nu}u(x) = g(x)|u(x)|$$

In fact, let $x \in \Omega$. If $|u(x)| = 0$, then

$$-M((L(u))\left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\frac{\partial u(x)}{\partial \nu}u(x) = 0 = g(x)|u(x)| \text{ on } \Gamma_3.$$

Otherwise, if $|u(x)| \neq 0$,then

$$\begin{aligned}-M((L(u))\left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\frac{\partial u(x)}{\partial \nu}u(x) &= g(x)\frac{(u(x))^2}{|u(x)|}\\ &= g(x)|u(x)| \text{ on } \Gamma_3\end{aligned}$$

Furthermore, for all $\mu \in \Lambda$:

$$b(u,\mu-\lambda) = b(u,\mu) - b(u,\lambda) = \langle\ \mu, \gamma u\ \rangle_{Y\times S} - \langle\ \lambda, \gamma u\ \rangle_{Y\times S}\,. \tag{2.16}$$

Hence, thanks to (2.13), (2.14) and (2.16), we obtain (2.15). □

3. Existence and uniqueness of solutions

We are ready to solve problem 1. For this, we consider the Banach spaces X and Y given in (2.1) and (2.4) respectively, and the set Λ in (2.6)

Theorem 3.1. *Suppose* $(B_1) - (B_6)$ *hold. Then problem* 1 *admits a solution* $(u,\lambda) \in X \times \Lambda$.

Proof. We apply the Schauder fixed point theorem.

As has been said before, we ”freeze” the state variable u on the function F , that is, we fix $w \in X$ and consider the problem:

Find $u \in X$ and $\lambda \in \Lambda$ such that

$$\langle\ Au, v\ \rangle + b(v, \lambda) \quad = \quad \langle\ f, v\ \rangle \quad , \quad \forall v \in X \tag{3.1}$$

$$b(u, \mu - \lambda) \quad \leq \quad 0 \quad \forall \mu \in \Lambda \subseteq Y. \tag{3.2}$$

with $f = F(w) \in X'$. Note that by the hypotheses on α and f_1, given in (A_2), we have $f_1(w) \in L^{\alpha'(x)}(\Omega) \hookrightarrow X'$.

By Theorem 2.7, problem (3.1)-(3.2) has a unique solution $(u_w, \lambda_w) \in X \times \Lambda$.

Here we drop the subscript w for simplicity. Setting $v = u$ in (3.1) and $\mu = 0_Y$ in (3.2), using proposition 2.8 ii), we get

$$k_p \|u\|_X^{\hat{p}} \leq (2C_1 C_\alpha \|w\|_X^\sigma + 2C_2 C_\alpha |\Omega| + c_p |f_2|_{p'(x),\Gamma_2}) \|u\|_X \tag{3.3}$$

where

$$\sigma = \begin{cases} \alpha^- & \text{if } \|w\|_X > 1, \\ \alpha^+ & \text{if } \|w\|_X \leq 1, \end{cases}$$

and C_χ is the embedding constant of $X \hookrightarrow L^{\chi(x)}(\Omega)$.

Then

$$\|u\|_X \leq [C(1 + \|w\|_X)]^{\frac{1}{\hat{p}-1}}\ .$$

Therefore, either $\|u\|_X \leq 1$ or

$$\|u\|_X \leq [C(1 + \|w\|_X)]^{\frac{1}{p^- - 1}}\ . \tag{3.4}$$

Since $p^- > \alpha^+ + 1$, we have

$$t^{p^- - 1} - Ct^\sigma - C \to +\infty \qquad \text{as } t \to +\infty$$

Hence, there is some $\bar{R}_1 > 0$ such that

$$\bar{R_1}^{p^- - 1} - C\bar{R_1}^\sigma - C \geq 0 \tag{3.5}$$

From (3.4) and (3.5) we infer that if $\|w\|_X \leq \bar{R}_1$ then $\|u\|_X \leq \bar{R}_1$.

Thus there exists $R_1 = \min\{1, \bar{R}_1\}$ such that

$$\|u\|_X \leq R_1 \quad \text{for all } u \in X. \tag{3.6}$$

For this constant, define K as

$$K = \{v : v \in L^{\alpha(x)}(\Omega), \|v\|_X \leq R_1\}$$

which is a nonempty, closed, convex subset of $L^{\alpha(x)}(\Omega)$. We can define the operator

$$T : K \to L^{\alpha(x)}(\Omega), \qquad Tw = u_w$$

where u_w is the first component of the unique pair solution of the problem (3.1)-(3.2), $(u_w, \lambda_w) \in X \times \Lambda$

From (3.6) $\|Tw\|_X \leq R_1$, for every $w \in K$, so that $T(K) \subseteq K$.

Moreover, if $(u_\nu)_{\nu\geq 1}$ $(u_{w_\nu} \equiv u_\nu)$ is a bounded sequence in K, then from (3.6) is also bounded in X. Consequently, from the compact embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, $(Tw_\nu)_{\nu\geq 1}$ is relatively compact in $L^{\alpha(x)}(\Omega)$ and hence, in K.

To prove the continuity of T , let $(w_\nu)_{\nu\geq 1}$ be a sequence in K such that

$$w_\nu \to w \quad \text{strongly in } L^{\alpha(x)}(\Omega) \tag{3.7}$$

and suppose $u_\nu = Tw_\nu$. The sequence $\{(u_\nu, \lambda_\nu)\}_{\nu\geq 1}$ satisfies

$$\begin{aligned} \langle\ Au_\nu, v\ \rangle + b(v, \lambda_\nu) &= \langle\ F(w_\nu), v\ \rangle \quad , \quad \forall v \in X \\ b(u_\nu, \mu - \lambda_\nu) &\leq 0 \quad \forall \mu \in \Lambda. \end{aligned}$$

Using (3.6)-(3.7) we can extract a subsequence (u_{ν_k}) of (u_ν) and a subsequence (w_{ν_k}) of (w_ν) such that

$$\begin{gathered} u_{\nu_k} \to u^* \text{weakly in} X, \\ u_{\nu_k} \to u^* \text{ strongly in } L^{\alpha(x)}(\Omega) \text{ and a.e. in } \ \Omega, \\ w_{\nu_k} \to w \quad \text{a.e. in } \Omega, \\ L(u_{\nu_k}) \to t_0, \quad \text{for some } t_0 \geq 0, \end{gathered} \tag{3.8}$$

and in view of continuity of M

$$M(L(u_{\nu_k})) \to M(t_0). \tag{3.9}$$

We shall show that $u^* = Tw$. To this end, by choosing $u_{\nu_k} - u^*$ as a test function, we have

$$\begin{gathered} \langle\ Au_{\nu_k}, u_{\nu_k} - u^*\ \rangle + b(u_{\nu_k} - u^*, \lambda_\nu) = \langle\ F(w_{\nu_k}), u_{\nu_k} - u^*\ \rangle \\ \langle\ Au^*, u_{\nu_k} - u^*\ \rangle + b(u_{\nu_k} - u^*, \lambda^*) = \langle\ F(w), u_{\nu_k} - u^*\ \rangle\,. \end{gathered} \tag{3.10}$$

Then

$$\begin{aligned} &[M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2} \nabla u^* . (\nabla u_{\nu_k} - \nabla u^*)\, dx + \\ &M(L(u_{\nu_k})) \int_\Omega \left[\left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2} \nabla u^* - \left(1 + \frac{|\nabla u_{\nu_k}|^{p(x)}}{\sqrt{1+|\nabla u_{\nu_k}|^{2p(x)}}}\right)\right. \\ &\left. |\nabla u_{\nu_k}|^{p(x)-2} \nabla u_{\nu_k} \right] . (\nabla u_{\nu_k} - \nabla u^*)\, dx + b(u_{\nu_k} - u^*, \lambda^* - \lambda_{\nu_k}) = \langle\ F(w) - F(w_{\nu_k}), u_{\nu_k} - u^*\ \rangle\,. \end{aligned} \tag{3.11}$$

Since $b(u_{\nu_k} - u^*, \lambda^* - \lambda_{\nu_k}) \geq 0$, again by the inequality of Lemma 3 in [5], $p \geq 2$, we obtain

$$m_0 C_p \int_\Omega |\nabla u_{\nu_k} - \nabla u^*|^{p(x)}\, dx + [M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2}\nabla u^*.(\nabla u_{\nu_k} - \nabla u^*)\, dx \leq |\langle\ F(w_{\nu_k}) - F(w), u_{\nu_k} - u^*\ \rangle| \tag{3.12}$$

But, using (3.8) we get

$$|[M(L(u^*) - M(L(u_{\nu_k})] \int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2}\nabla u^*.(\nabla u_{\nu_k} - \nabla u^*)\, dx|$$

$$\leq \frac{\vartheta_{\nu_k}}{p^-} |\int_\Omega \left(1 + \frac{|\nabla u^*|^{p(x)}}{\sqrt{1+|\nabla u^*|^{2p(x)}}}\right) |\nabla u^*|^{p(x)-2}\nabla u^*.(\nabla u_{\nu_k} - \nabla u^*)\, dx| \to 0 \text{ as } k \to \infty, \tag{3.13}$$

where $\vartheta_{\nu_k} = \max\{\|u_{\nu_k}\|_X^{p^-}, \|u_{\nu_k}\|_X^{p^+}\} + \max\{\|u^*\|_X^{p^-}, \|u^*\|_X^{p^+}\}$ is bounded.

Also, by (A_2), (3.8) and the compact embedding of $X \hookrightarrow L^{\alpha(x)}(\Omega)$ we deduce, thanks to the Krasnoselki theorem, the continuity of the Nemytskii operator

$$\begin{aligned} N_{f_1} : L^{\alpha(x)}(\Omega) &\to L^{\alpha'(x)}(\Omega) \\ w &\longmapsto N_{f_1}(w), \end{aligned} \tag{3.14}$$

given by $(N_{f_1}(w))(x) = f_1(x, w(x)), \quad x \in \Omega.$

Hence

$$\|f_1(w_{\nu_k}) - f_1(w)\|_{\alpha'(x)} \to 0$$

It follows from the definition of F and the above convergence that

$$|\langle\ F(w_{\nu_k}) - F(w), u_{\nu_k} - u^*\ \rangle| \to 0 \tag{3.15}$$

Thus, from (3.12)-(3.15) we conclude that

$$u_{\nu_k} \to u^* \quad \text{strongly in } X$$

Since the possible limit of the sequence $(u_\nu)_{\nu \geq 1}$ is uniquely determined, the whole sequence converges toward $u^* \in X$

Therefore, from (3.7) and the continuous embedding $X \hookrightarrow L^{\alpha(x)}(\Omega)$, we get $u^* = Tw \equiv u_w$.

On the other hand

$$\begin{aligned}\frac{b(v,\lambda)}{\|v\|_X} &= \frac{\langle F(w),v\rangle - \langle Au,v\rangle}{\|v\|_X} \le \frac{\langle F(w),v\rangle}{\|v\|_X} + \|Au\|_{X'} \\ &\le \frac{1}{\|v\|_X}\left[\int_\Omega f_1(x,w)v\,dx + \int_{\Gamma_2} f_2(x)\gamma v\,d\Gamma\right] + L_A\|u\|_X + \|A0\|_{X'} \\ &\le C(\|f_1(w)\|_{\alpha'(x)} + \|f_2\|_{p'(x),\Gamma_2} + \|A0\|_{X'} + 1)\end{aligned} \tag{3.16}$$

Next, using the boundedness of the operator N_{f_1} and the sequence $(u_\nu)_{\nu\ge1}$, and the inf-sup property of the form b, we get $\|\lambda_\nu\|_Y \le C$. It follows that up to a subsequence

$$\lambda_\nu \to \lambda_0 \quad \text{weakly in } Y$$

for some $\lambda_0 \in Y$.

So (u^*,λ^*) and (u^*,λ_0) are solutions of problem (3.1)-(3.2).Then, by the uniqueness $\lambda_0 = \lambda^* \equiv \lambda_w$. This shows the continuity of T.

To prove that T is compact, let $(w_\nu)_{\nu\ge1} \subseteq K$ be bounded in $L^{\alpha(x)}(\Omega)$ and $u_\nu = T(w_\nu)$. Since $(w_\nu)_{\nu\ge1} \subseteq K$, $\|w_\nu\|_X \le C$ and then, up to a subsequence again denoted by $(w_\nu)_{\nu\ge1}$ we have

$$w_\nu \to w \quad \text{weakly in } X$$

By the compact embedding Xinto $L^{\alpha(x)}(\Omega)$, it follows that

$$w_\nu \to w \text{ strongly in } L^{\alpha(x)}(\Omega).$$

Now, following the same arguments as in the proof of the continuity of T we obtain

$$u_\nu = T(w_\nu) \to T(w) = u \quad \text{strongly in } X$$

Thus

$$T(w_\nu) \to T(w) \;\text{ strongly in }\; L^{\alpha(x)}(\Omega).$$

Hence, we can apply the Schauder fixed point theorem to obtain that T possesses a fixed point. This gives us a solution of $(u,\lambda_0) \in X \times \Lambda$ of Problem 1, which concludes the proof. □

Next, we consider the uniqueness of solutions of (2.9). To this end, we also need the following hypothesis on the nonlinear term f_1.

(A4) There exists $b_0 \ge 0$ such that

$$(f(x,t) - f(x,s))(t-s) \le b_0|t-s|^{p(x)} \quad \text{a.e. } \; x \in \Omega, \forall t,s \in \mathbb{R}.$$

Our uniqueness result reads as follows.

Theorem 3.2. *Assume that* $(A1) - (A4)$ *hold. If, in addition* $2 \le p$ *for all* $x \in \bar{\Omega}$, *then* (2.9) *has a unique weak solution provided that*

$$\frac{k_p}{b_0\lambda_*^{-1}} < 1,$$

where

$$\lambda_* = \inf_{u\in X\setminus\{0\}} \frac{\int_\Omega |\nabla u|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx} > 0.$$

Proof. Theorem 3.1 gives a weak solution $(u,\lambda)\in X\times\Lambda$. Let $(u_1,\lambda_1),(u_2,\lambda_2)$ be two solutions of (2.9). Considering the weak formulation of u_1 and u_2 we have

$$\begin{aligned} \langle\, Au_i, v\,\rangle + b(v,\lambda_i) &= \langle\, F(u_i), v\,\rangle \quad , \quad \forall v\in X \qquad (3.17)\\ b(u_i,\mu-\lambda_i) &\le 0 \quad \forall \mu\in\Lambda\subseteq Y \quad i=1,2. \end{aligned}$$

By choosing $v=u_1-u_2$, $\mu=\lambda_2$ if $i=1$ and $\mu=\lambda_1$ if $i=2$, we have

$$\begin{aligned} \langle\, Au_1-Au_2, u_1-u_2\,\rangle + b(u_1-u_2,\lambda_1-\lambda_2) &= \langle\, F(u_1)-F(u_2), u_1-u_2\,\rangle \ ,\forall v\in X\\ b(u_1-u_2,\lambda_2-\lambda_1) &\le 0 \quad \forall\mu\in\Lambda\subseteq Y. \qquad (3.18) \end{aligned}$$

It gives

$$\langle\, Au_1-Au_2, u_1-u_2\,\rangle \le \langle\, F(u_1)-F(u_2), u_1-u_2\,\rangle\,.$$

Then, from (3.18) and repeating the argument used in the proof of Proposition 2.8, ii), we get

$$\begin{aligned} k_p\int_\Omega |\nabla u_1-\nabla u_2|^{p(x)}\,dx &\le |\,\langle\, f_1(u_1)-f_1(u_2), u_1-u_2\,\rangle\,|\\ &\le |\int_\Omega (f_1(x,u_1)-f_1(x,u_2))(u_1-u_2)\,dx|\\ &\le |\int_\Omega |u_1-u_2|^{p(x)}\,dx \le b_0\lambda_*^{-1}\int_\Omega |\nabla u_1-\nabla u_2|^{p(x)}\,dx \end{aligned}$$

Consequently when $\frac{k_p}{b_0\lambda_*^{-1}}<1$, it follows that $u_1=u_2$. This completes the proof. □

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