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On the solution of conformable fractional heat conduction equation

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Abstract. In this article, we study a conformable fractional heat conduction equation. Applying the method of separation variables to this problem, we get a conformable fractional Sturm–Liouville eigenvalue problem. Later, we prove the existence of a countably infinite set of eigenvalues and eigenfunctions. Finally, we establish uniformly convergent expansions in the eigenfunctions.

Keywords: Conformable Fractional Sturm-Liouville operator, eigenfunction expansion, Green's function, completely continuous operator.

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1. Introduction

Fractional calculus is a non-integer order derivative and integral calculus. The concept of non-integer order derivative and integration, proposed by Leibniz and L'Hospital, is now available in many studies on the subject of fractional analysis ([1]-[8]). When the literature is examined, we see that there have been many studies about fractional derivatives and the studies are continuing intensively today. Recently, a new definition of the fractional derivative has been proposed which is called a conformable fractional derivative and based on the classical derivative definition ([9]). Khalil et al. were defined conformable fractional derivative ([9]). In ([10]), the authors defined the right and left conformable fractional derivatives.

Conformable fractional derivative aims to extend the definition of derivative in the known sense by providing the natural properties of the classical derivative and to give new perspectives to the theory of differential equations with the help of conformable fractional differential equations obtained using this derivative definition. In [11, 12], the authors studied the conformable fractional Sturm-Liouville problems. In their study, the authors discussed a conformable fractional Sturm-Liouville boundary-value problem. In [13], they used sine-Gordon expansion (SGE) approach and generalized Kudryashov (GK) scheme to generate broad spectral solutions containing unknown parameters. The dynamic behavior of the waves drawn for the individual values of the parameter was analyzed in the 3D and contour graphics of the results they obtained. In [23], the authors studied the conformable fractional heat equation. Later, in [24, 25], the authors introduced the conformable fractional Fourier series and gave its applications to solve some conformable fractional equations. In [26], the authors introduced the concept of a mild solution of conformable fractional abstract initial value problem. They established the existence and uniqueness theorem using the contraction principle. In [27], the authors adopted the Adomian decomposition method and the Padé approximation technique to derive the approximate solutions of a conformable heat transfer equation by considering the new definition of the Adomian polynomials. Many researchers have done studies in this area [14, 15, 16, 17].

In the present article, we consider a conformable fractional heat conduction equation in the following form:

$$\frac{\partial^\alpha u(t,x)}{\partial t^\alpha}+q(x)u(t,x)=\frac{\partial^\alpha}{\partial x^\alpha}(r(x)\frac{\partial^\alpha u(t,x)}{\partial x^\alpha}),\ x\in(0,b),\ 0<b<\infty,\ t>0, \tag{1.1}$$

where

$$\frac{\partial^{\alpha}u\left(t,x\right)}{\partial t^{\alpha}}:=\lim_{\varepsilon\to 0}\frac{u\left(t+\varepsilon t^{1-\alpha},x\right)-u\left(t,x\right)}{\varepsilon},\ \alpha\in(0,1],$$

$r(.)$ and $q(.)$ are real-valued functions defined on $J:=[0,b]$ and satisfy the conditions $\frac{1}{r(.)},q\left(.\right)\in L_{\alpha}^{1}(J)$, and $r\left(x\right)>0$, $q\left(x\right)\geq 0$, $x\in J$. We shall assume that the system (1.1) satisfies the homogeneous boundary conditions

$$u\left(t,0\right)=0,\ u\left(t,b\right)=0, \tag{1.2}$$

and the initial condition

$$u\left(0,x\right)=f\left(x\right),\ x\in J. \tag{1.3}$$

We solve the problem (1.1) by the method of separation of variables. Let

$$u\left(t,x\right)=e^{-\lambda\frac{t^{\alpha}}{\alpha}}y\left(x\right),\ x\in J, \tag{1.4}$$

where λ is a complex constant. If we substitute (1.4) into (1.1)-(1.2), we arrive at

$$\Gamma y:=-T_{\alpha}\left(r\left(x\right)T_{\alpha}y(x)\right)+q\left(x\right)y(x)=\lambda y(x), \tag{1.5}$$

and

$$y(0)=y(b)=0, \tag{1.6}$$

where $x\in(0,b)$.

If λ is an eigenvalue and $y(x)$ is a corresponding eigenfunction of the problem (1.5)-(1.6) if and only if the function $u\left(t,x\right)$ in (1.4) is a nontrivial solution of the problem (1.1)-(1.2). Since the problem (1.1)-(1.2) is linear, the function defined as

$$u\left(t,x\right)=\sum_{k=1}^{\infty}c_{k}e^{-\lambda\frac{t^{\alpha}}{\alpha}}\varphi_{k}\left(x\right), \tag{1.7}$$

where $c_1,c_2,...$ are arbitrary constants, is a formal solution of the problem (1.1)-(1.2). The initial condition (1.3) gives

$$f\left(x\right)=\sum_{k=1}^{\infty}c_{k}\varphi_{k}\left(x\right),\ x\in J.$$

In this process, the natural question now is: is it possible to expand a given function $f\left(x\right)$ in eigenfunctions $\varphi_1\left(x\right),\varphi_2\left(x\right),...$? Our purpose of this paper is to answer this question.

This paper is organized as follows. In Section 2, we give some definitions and theorems related to conformable fractional calculus. In Section 3, we obtain an eigenfunctions expansion. This expansion is $\alpha-$square convergent (that is, in an L_{α}^{2}-metric). The existence of a countably infinite set of eigenvalues and eigenfunctions is proved. Finally, in Section

4, we obtain uniformly convergent expansions in the eigenfunctions. In the analysis that follows, we will largely follow the development of the theory in [18, 22, 21, 19].

2. Preliminaries

Definition 2.1 (see [10])**.** Assume α be a positive number with $0 < \alpha < 1$. A function $f : (0, b) \to \mathbb{R} = (-\infty, \infty)$ the conformable fractional derivative of order α of f at $x > 0$ was defined by

$$T_\alpha f(x) = \lim_{\varepsilon \to 0} \frac{f\left(x + \varepsilon x^{1-\alpha}\right) - f(x)}{\varepsilon}, \tag{2.1}$$

and the fractional derivative at 0 is defined

$$\left(T_\alpha f\right)(0) = \lim_{x \to 0} T_\alpha f(x).$$

Definition 2.2 (see [10])**.** The conformable fractional integral starting from 0 of a function f of order $0 < \alpha \leq 1$ is defined by

$$\left(I_\alpha f\right)(x) = \int\limits_0^x s^{\alpha-1} f(s) ds = \int\limits_0^x f(s) d_\alpha s.$$

Lemma 2.3 (see [10])**.** *Assume that f is a continuous function on $(0, b)$ and $0 < \alpha < 1$. Then, we have*

$$T_\alpha I_\alpha f(x) = f(x),$$

for all $x \in (0, b)$.

Theorem 2.4. [10] *Let $z, y : [0, b] \to R$ be two functions such that z and y are conformable fractional differentiable. Then, we have*

$$\int_0^b y(x) T_\alpha(z)(x) d_\alpha x + \int_0^b z(x) T_\alpha(y)(x) d_\alpha x = z(b) y(b) - z(0) y(0). \tag{2.2}$$

3. $\alpha-$Square Convergent Expansions

Let us denote by $L_\alpha^2(J)$, the space of all real-valued functions defined on J such that

$$\|y\| := \sqrt{\int_0^b y^2(x) d_\alpha x} < \infty,$$

where $0 < b < \infty$. The space $L_\alpha^2(J)$ is a Hilbert space with the inner product

$$\langle z, y \rangle = \int_0^b z(x) y(x) d_\alpha x, \text{ where } z, y \in L_\alpha^2(J).$$

Let us consider the linear set $D_{\max}$ consisting of all vectors $y \in L_{\alpha}^{2}(J)$ such that y and $T_{\alpha}y$ are absolutely continuous on J and $\Gamma y \in L_{\alpha}^{2}(J)$. We define the *maximal operator* $S_{\max}$ on $D_{\max}$ by the equality $S_{\max}y = \Gamma y$.

Let $D_{\min}$ be the linear set of all vectors $y \in D_{\max}$ satisfying the conditions

$$y\left(0\right) = T_{\alpha}y\left(0\right) = y\left(b\right) = T_{\alpha}y\left(b\right) = 0. \tag{3.1}$$

The operator $S_{\min}$, that is the restriction of the operator $S_{\max}$ to $D_{\min}$ is called the *minimal operator.*

For $y_1, y_2 \in D_{\max}$, we have the following $\alpha-$Green's formula

$$\int_0^b \left[\Gamma[y_1](x)y_2(x) - y_1(x)\Gamma[y_2](x)\right] d_{\alpha}x = \left[y_1, y_2\right](b) - \left[y_1, y_2\right](0), \tag{3.2}$$

where

$$[y_1, y_2](x) = r\left(x\right)\left\{y_1(x)T_{\alpha}y_2(x) - T_{\alpha}y_1(x)y_2(x)\right\}$$

(see [12]).

Theorem 3.1. *The operator $S_{\min}$is Hermitian.*

Proof. By the formula (3.2), for $y, z \in D_{\min}$, we have

$$\int_0^b (\Gamma y)(x)z(x)d_{\alpha}x - \int_0^b y(x)(\Gamma z)(x)d_{\alpha}x = 0.$$

□

Theorem 3.2. *Let $\xi \in L_{\alpha}^{2}(J)$. Then, the equation*

$$\Gamma\left(y\right) = \xi \tag{3.3}$$

has a solution $y\left(x\right)$ satisfying the conditions

$$y\left(0\right) = T_{\alpha}y\left(0\right) = y\left(b\right) = T_{\alpha}y\left(b\right) = 0, \tag{3.4}$$

if and only if the function ξ is orthogonal to all solutions of the equation

$$\Gamma\left(y\right) = 0.$$

Proof. Let $y\left(x\right)$ be the solution of the equation $\Gamma\left(y\right) = \xi$ satisfying the conditions

$$y\left(0\right) = T_{\alpha}y\left(0\right) = 0. \tag{3.5}$$

There exists one such solution (see [12]). Let us denote by z_1 and z_2, a fundamental system of solutions of the equation $\Gamma\left(z\right) = 0$ satisfying the conditions

$$\begin{aligned} z_1\left(b\right) = 1,\ T_{\alpha}z_1\left(b\right) = 0, \\ z_2\left(b\right) = 0,\ T_{\alpha}z_2\left(b\right) = 1. \end{aligned} \tag{3.6}$$

Applying $\alpha-$Green's formula (3.2) to the functions $y\left(x\right)$ and $z_i\left(x\right)$ $\left(i = 1, 2\right)$, we conclude that

$$\left(\xi, z_i\right) = \left(\Gamma\left(y\right), z_i\right) = \left[y, z_i\right](b) - \left[y, z_i\right](0) + \left(y, \Gamma\left(z_i\right)\right). \tag{3.7}$$

By the condition (3.5), we deduce that $[y, z_i](0) = 0$. It follows from $\Gamma(z_i) = 0$ that

$$(\xi, z_i) = [y, z_i](b) = \begin{cases} -T_\alpha y(b) & for \quad i = 1 \\ y(b) & for \quad i = 2, \end{cases} \tag{3.8}$$

and this is precisely the assertion of the theorem. □

Now, let us denote by Ω the set of all solutions of the equation $\Gamma(z) = 0$. Further, we denote by M the range of the operator $S_{\min}$. It follows from Theorem 3.2 that

$$L^2_\alpha(J) = \Omega \oplus M. \tag{3.9}$$

Theorem 3.3. *For arbitrary real numbers $\alpha_1, \alpha_2, \alpha_3$ and α_4, there exists a function $y \in D_{\max}$ satisfying the conditions*

$$\begin{aligned} y(0) = \alpha_1,\ T_\alpha y(0) = \alpha_2, \\ y(b) = \alpha_3,\ T_\alpha y(b) = \alpha_4. \end{aligned} \tag{3.10}$$

Proof. Firstly, we will prove the theorem for the special case when α_1 and α_2 are zero. Let ξ be an arbitrary vector in $L^2_\alpha(J)$ satisfying the conditions

$$(\xi, z_i) = \begin{cases} -\alpha_4 & for \quad i = 1 \\ \alpha_3 & for \quad i = 2. \end{cases} \tag{3.11}$$

Here z_1 and z_2 are a fundamental system of solutions of the equation $\Gamma(z) = 0$. There exists such a vector ξ. If we put

$$\xi = c_1 z_1 + c_2 z_2,$$

then the conditions (3.11) provide a system of equations in the constants c_i $(i = 1, 2)$ whose determinant is the same as the Gram determinant for the linearly independent functions z_1, z_2, and does not vanish.

Let z denote the solution of the equation

$$\Gamma(z) = \xi$$

satisfying the conditions

$$z(0) = T_\alpha z(0) = 0. \tag{3.12}$$

Then we have

$$z(b) = \alpha_3,\ T_\alpha z(b) = \alpha_4.$$

Applying the formula (3.2) to the functions $z(x)$ and $z_i(x)$ $(i = 1, 2)$, we get

$$(\xi, z_i) = (\Gamma(z), z_i) = [z, z_i](a) - [z, z_i](0) + (z, \Gamma(z_i)). \tag{3.13}$$

It follows from $\Gamma(z_i) = 0$ and (3.12) that $[z, z_i](b) = 0$. From the conditions (3.6) and (3.8), we get

$$[z, z_i](b) = \begin{cases} -T_\alpha z(b) & for \quad i = 1 \\ z(b) & for \quad i = 2. \end{cases}$$

From (3.11) and (3.13), we conclude that

$$z(b) = \alpha_3, \ T_\alpha z(b) = \alpha_4.$$

Thus, we have constructed a function $z \in D_{\max}$ such that

$$z(0) = T_\alpha z(0) = 0, \ z(b) = \alpha_3, \ T_\alpha z(b) = \alpha_4.$$

Similarly, we can construct a function $k \in D_{\max}$ such that

$$k(0) = \alpha_1, \ T_\alpha k(0) = \alpha_2, \ k(b) = 0, \ T_\alpha k(b) = 0.$$

Then the function $y = z + k \in D_{\max}$ satisfies the conditions (3.10). □

Theorem 3.4. *$D_{\min}$ is dense in $L^2_\alpha(J)$.*

Proof. We will show that every vector ζ orthogonal to $D_{\min}$ is zero. Let ζ be such a vector, i.e.,

$$(\zeta, y) = 0, \text{ for all } y \in D_{\min}.$$

Let ν be any particular solution of the equation $\Gamma(\nu) = \zeta$. For an arbitrary vector $y \in D_{\min}$, we have

$$(\nu, S_{\min} y) = (S_{\max}\nu, y) = (\Gamma(\nu), y) = (\zeta, y) = 0.$$

An application of Theorem 3 yields $\zeta = 0$. □

It follows from Theorem 3.1 and Theorem 3.4 that $S_{\min}$ is a symmetric operator.

Theorem 3.5. *The equality $S_{\max} = S^*_{\min}$ holds.*

Proof. For arbitrary vectors $y \in D_{\min}$ and $z \in D_{\max}$, we have

$$(S_{\min} y, z) = (y, S_{\max} z),$$

i.e., $S_{\max} \subset S^*_{\min}$. Hence, we have to prove the converse. Let ζ be an arbitrary vector in the domain of definition $D^*_{\min}$ of the operator $S^*_{\min}$ and $S^*_{\min}\zeta = \nu$. Further, we denote by $\xi(z)$ any particular solution of the equation $\Gamma(\xi) = \nu$. Then we have

$$(\nu, y) = (\Gamma(\xi), y) = (S_{\max}\xi, y) = (\xi, S_{\min} y) \text{ for every } y \in D_{\min}. \tag{3.14}$$

By definition of the adjoint operator, we get

$$(\nu, y) = (S^*_{\min}\zeta, y) = (\zeta, S_{\min} y). \tag{3.15}$$

Subtracting (3.15) from (3.14), we have

$$(\xi - \zeta, S_{\min} y) = 0,$$

i.e., $\xi - \zeta \in M^\perp$. By virtue of (3.9), we deduce that $\xi - \zeta \in \Omega$. Thus, $\Gamma(\xi - \zeta) = 0$, i.e., $\Gamma\zeta = \Gamma\xi = \nu = S^*_{\min}\zeta$. □

Theorem 3.6. *The equality $S^*_{\max} = S_{\min}$ holds.*

Proof. From Theorem 3.5, we have

$$S^*_{\max} = S^{**}_{\min} \supset S_{\min}.$$

Thus, we have to show the opposite inclusion. Since $S_{\min} \subset S_{\max}$, we arrive at

$$S^*_{\max} \subset S^*_{\min} = S_{\max}. \tag{3.16}$$

Let ξ be a vector in the domain of definition $D^*_{\max}$ of the operator $S^*_{\max}$. From (3.16), we have $\xi \in D_{\max}$ and $S^*_{\max}\xi = S_{\max}\xi$. Then we get

$$\left(S^*_{\max}\xi, y\right) = \left(\xi, S_{\max}y\right),$$

$$\left(S_{\max}\xi, y\right) = \left(\xi, S_{\max}y\right) \text{ for all } y \in D_{\max}.$$

Using $\alpha-$Green's formula (3.2), we conclude that

$$[\xi, y](b) - [\xi, y](0) = 0 \text{ for all } y \in D_{\max}. \tag{3.17}$$

It follows from Theorem 3.3 that the equation (3.17) is possible if

$$\xi(0) = T_\alpha \xi(0) = \xi(b) = T_\alpha \xi(b) = 0,$$

i.e., $\xi \in D_{\min}$. □

It follows from Theorem 3.6 that $S_{\min}$ is a closed symmetric operator. Furthermore, the deficiency indices of the operator $S_{\min}$ is $(2,2)$.

Now, we will give the self-adjoint extension of the operator $S_{\min}$. Let D be the linear set of all vectors $y \in D_{\max}$ satisfying the conditions

$$y(0) = y(b) = 0. \tag{3.18}$$

Then we have the following theorem.

Theorem 3.7. *Let S be the restriction of the operator $S_{\max}$ to the set D. Then the operator S is a self-adjoint extension of the symmetric operator $S_{\min}$.*

Theorem 3.8. *The operator S is positive, i.e., for all $y \in D$ $(y \neq 0)$, we have*

$$(Sy, y) = \int_0^b \left\{ r(x) \left[T_\alpha y(x)\right]^2 + q(x) \left[y(x)\right]^2 \right\} d_\alpha x > 0 \,. \tag{3.19}$$

Proof. From the formula (2.2), we have for all $y \in D$

$$\begin{aligned}(Sy, y) &= r(x)y(x)T_\alpha y(x)|_0^b + \int_0^b \left\{ r(x)[T_\alpha y(x)]^2 + q(x)[y(x)]^2 \right\} d_\alpha x \\ &= \int_0^b \left\{ r(x)[T_\alpha y(x)]^2 + q(x)[y(x)]^2 \right\} d_\alpha x > 0.\end{aligned}$$

□

Since S is self-adjoint, we have the following properties:

i) All eigenvalues are real and positive

ii) Any two eigenfunctions corresponding to distinct eigenvalues are orthogonal.

iii) The eigenvalues are simple.

In the next result, we use the notation

$$\ker S = \{y \in D : Sy = 0\} .$$

Then we have the following proposition

Proposition 3.9.

$$\ker S = \{0\} .$$

Proof. Let $y \in D$ and $Sy = 0$. It follows from (3.19) that, for $x \in (0, b)$, $T_\alpha y(x) = 0$. Thus $y(x)$ is constant on $(0, b)$. From (1.6), we conclude that $y(x) \equiv 0$. $\square$

It follows from Proposition 3.9 that the inverse operator S^{-1} exists.

Now, we shall define Green's function for the problem (1.5)-(1.6). Let $\varphi\,(x, \lambda)$ and $\psi\,(x, \lambda)$ two linearly independent solutions in $L^2_\alpha(J)$ of the equation (1.5) and satisfy the following conditions

$$\begin{aligned} &\varphi\,(0, \lambda) = 0,\ r(0)T_\alpha\varphi\,(0, \lambda) = 1, \\ &\psi\,(b, \lambda) = 0,\ r(b)T_\alpha\psi\,(b, \lambda) = 1. \end{aligned} \tag{3.20}$$

Let

$$G(x, s) = \begin{cases} \varphi(x)\psi(s), & x \le s \\ \psi(x)\varphi(s), & x \ge s \end{cases} \tag{3.21}$$

(see [12]).

Then, for any $v \in L^2_\alpha(J)$, we get

$$(S^{-1}v)(x) = \int_a^b G(x, s)v(s)d_\alpha s. \tag{3.22}$$

It is clear that the operator S^{-1} is a compact symmetric in $L^2_\alpha(J)$ (see [12]). Since $\lambda = 0$ is not an eigenvalue of the operator S, the eigenvalue problem $Sv = \lambda v$, $v \in L^2_\alpha(J)$, ($v \ne 0$) is equivalent to the eigenvalue problem $Bv = \eta v, v \in L^2_\alpha(J)$, $(v \ne 0)$, where $B = S^{-1}$ and $\eta = \frac{1}{\lambda}$. It is clear that $\eta = 0$ cannot be an eigenvalue for B.

Now, we present below for the convenience of the reader.

Theorem 3.10 (see [20])**.** *Let A be a compact symmetric operator mapping a Hilbert space H into itself. Then there is an orthonormal system ϕ_1, ϕ_2, ... of eigenvectors of A, with corresponding nonzero* *eigenvalues* $\eta_1, \eta_2, ...$ *such that every element* $f \in H$ *has a unique representation of the form*

$$f = \sum_k c_k \phi_k + h,$$

where h *satisfies the condition* $Ah = 0$. *Moreover,*

$$Af = \sum_k \eta_k c_k \phi_k,$$

and

$$\lim_{k \to \infty} \eta_k = 0$$

in the case where there are infinitely many nonzero eigenvalues.

Corollary 3.11. *Let* A *be a compact symmetric operator mapping a Hilbert space* H *into itself. If* $\ker A = 0$, *then the eigenvectors of* A *form an orthonormal basis of* H.

From Corollary 3.11, we have the following theorem.

Theorem 3.12. *For the problem (1.5)-(1.6), there exists an orthonormal system* $\{\phi_k\}$ *of eigenfunctions corresponding to eigenvalues* $\{\lambda_k\}$ $(k \in \mathbb{N} := \{1, 2, 3, ...\})$. *Each eigenvalue* λ_k *is positive and simple. The system* $\{\phi_k\}$ *forms an orthonormal basis for the Hilbert space* $L^2_\alpha(J)$. *Therefore the number of the eigenvalues is equal to* $\dim L^2_\alpha(J) = \infty$. *Any function* $f \in L^2_\alpha(J)$ *can be expanded in eigenfunctions* ϕ_k *in the form*

$$f(x) = \sum_{k=1}^{\infty} c_k \phi_k(x), \tag{3.23}$$

where c_k *are the Fourier coefficients of* f *defined by*

$$c_k = \int_0^b f(x) \phi_k(x) d_\alpha x, \ k \in \mathbb{N}. \tag{3.24}$$

The sum in (3.23) converges to the function f *in metric of the space* $L^2_\alpha(J)$, *i.e.,*

$$\lim_{n \to \infty} \int_0^b \left(f(x) - \sum_{k=1}^{n} c_k \phi_k(x) \right)^2 d_\alpha x = 0. \tag{3.25}$$

By (3.25), we get the Parseval equality, i.e.,

$$\int_0^b f^2(x) d_\alpha x = \sum_{k=1}^{\infty} c_k^2, \tag{3.26}$$

since

$$\int_0^b \left(f(x) - \sum_{k=1}^n c_k \phi_k(x)\right)^2 d_\alpha x = \int_0^b f^2(x) d_\alpha x - \sum_{k=1}^n c_k^2.$$

4. Uniformly Convergent Expansions

In what follows, we present our main result.

Theorem 4.1. *Let $f(x)$ be a continuous real-valued function satisfying the boundary conditions (1.6) and such that it has a continuous $\alpha-$derivative in the interval J. Then the series*

$$\sum_{k=1}^\infty c_k \phi_k(x), \tag{4.1}$$

where

$$c_k = \int_0^b f(x) \phi_k d_\alpha x, \ \ k \in \mathbb{N}, \tag{4.2}$$

converges uniformly on J to the function f.

Proof. We follow the ideas of [19, 21, 22]. Define the functional $Q(y)$ by

$$Q(y) = \int_0^b \left\{ r(x) \left[T_\alpha y(x)\right]^2 + q(x) \left[y(x)\right]^2 \right\} d_\alpha x \geq 0.$$

If we substitute

$$y = f(x) - \sum_{k=1}^n c_k \phi_k(x),$$

into $Q(y)$, we conclude that

$$Q\left(f - \sum_{k=1}^n c_k \phi_k\right) = \int_0^b r(x) \left\{ T_\alpha f(x) - \sum_{k=1}^n c_k T_\alpha \phi_k(x) \right\}^2 d_\alpha x$$

$$+ \int_0^b q(x) \left[f - \sum_{k=1}^n c_k \phi_k \right]^2 d_\alpha x$$

$$= \int_0^b \left\{ r(x) \left[T_\alpha f(x)\right]^2 + q(x) \left[f(x)\right]^2 \right\} d_\alpha x$$

$$- 2 \sum_{k=1}^{n} c_k \left[\int_0^b \left\{ r(x)\, T_\alpha f(x) T_\alpha \phi_k(x) + q(x)\, f(x) \phi_k(x) \right\} d_\alpha x \right]$$

$$+ \sum_{k,m=1}^{n} c_k c_m \left[\int_0^b \left\{ r(x)\, T_\alpha \varphi_k(x) T_\alpha \phi_m(x) + q(x)\, \varphi_k(x) \phi_m(x) \right\} d_\alpha x \right] \tag{4.3}$$

Applications of (2.2) and (1.6) yield

$$\int_0^b \left\{ r(x)\, T_\alpha f(x) T_\alpha \phi_k(x) + q(x)\, f(x) \phi_k(x) \right\} d_\alpha x$$

$$= \int_0^b f(x) \left\{ -T_\alpha \left[r(x)\, T_\alpha \phi_k(x) \right] + q(x)\, \phi_k(x) \right\} d_\alpha x$$

$$+ r(b)\, f(b) T_\alpha \phi_k(b) - r(0)\, f(0) T_\alpha \phi_k(0)$$

$$= \lambda_k \int_0^b f(x) \phi_k(x) d_\alpha x = \lambda_k c_k, \;\; k \in \mathbb{N},$$

and

$$\int_0^b \left\{ r(x)\, T_\alpha \phi_k(x) T_\alpha \phi_m(x) + q(x)\, \varphi_k(x) \phi_m(x) \right\} d_\alpha x$$

$$= \int_0^b \phi_k(x) \left\{ -T_\alpha \left[r(x)\, T_\alpha \phi_m(x) \right] + q(x)\, \phi_m(x) \right\} d_\alpha x$$

$$+ r(b)\, \phi_k(b) T_\alpha \phi_m(b) - r(0)\, \phi_k(0) T_\alpha \phi_m(0)$$

$$= \lambda_m \int_0^b \phi_k(x) \phi_m(x) d_\alpha x = \lambda_k \delta_{km}, \;\; k, m \in \mathbb{N},$$

where δ_{km} is the Kronecker symbol. Therefore from (4.3), we conclude that

$$Q\left(f-\sum_{k=1}^{n}c_k\phi_k(x)\right)$$

$$=\int_0^b\left\{r\left(x\right)\left[T_\alpha y(x)\right]^2+q\left(x\right)\left[y(x)\right]^2\right\}-\sum_{k=1}^{n}\lambda_k c_k^2. \tag{4.4}$$

Since the left-hand side in (4.4) is nonnegative we arrive at

$$\sum_{k=1}^{\infty}\lambda_k c_k^2\leq\int_0^b\left\{r\left(x\right)\left[T_\alpha y(x)\right]^2+q\left(x\right)\left[y(x)\right]^2\right\}d_\alpha x \tag{4.5}$$

analogous to Bessel's inequality, and the convergence of the series in (4.5) follows. Since $\lambda_k>0$, all the terms of this series are nonnegative.

We now show that the series

$$\sum_{k=1}^{\infty}\left|c_k\phi_k(x)\right| \tag{4.6}$$

is uniformly convergent in J. It follows from (3.22) that

$$\phi_k(x)=\lambda_k\int_0^b G(x,s)\phi_k(s)d_\alpha s.$$

Then we can rewrite (4.6) as

$$\sum_{k=1}^{\infty}\lambda_k\left|c_k\tau_k(x)\right|, \tag{4.7}$$

where

$$\tau_k(x)=\int_0^b G(x,s)\phi_k(s)d_\alpha s$$

can be regarded as the Fourier coefficient of $G(x,s)$ as a function of s. It follows from (4.5) that

$$\sum_{k=1}^{\infty}\lambda_k\left[\tau_k(x)\right]^2\leq\int_0^b\left\{r\left(x\right)\left[\frac{\partial^\alpha}{\partial s^\alpha}G(x,s)\right]^2+q\left(x\right)\left[G(x,s)\right]^2\right\}d_\alpha s. \tag{4.8}$$

Since the function appearing under the integral sign in (4.8) is bounded, we get

$$\sum_{k=1}^{\infty} \lambda_k [\tau_k(x)]^2 \leq K,$$

where K is a constant. Applying the Cauchy–Schwarz inequality in (4.7), we conclude that

$$\sum_{k=s}^{s+p} \lambda_k |c_k \tau_k(x)| \leq \left(\sum_{k=s}^{s+p} \lambda_k c_k^2\right)^{1/2} \left(\sum_{k=s}^{s+p} \lambda_k [\tau_k(x)]^2\right)^{1/2} \leq K^{1/2} \left(\sum_{k=s}^{s+p} \lambda_k c_k^2\right)^{1/2}.$$

It follows from (4.5) and the convergence of the series with terms $\lambda_k c_k^2$ that the series in (4.7) is uniformly convergent on the interval J. Hence the series in (4.6) is uniformly convergent in this interval. Let

$$f_1(x) = \sum_{k=1}^{\infty} c_k \phi_k(x), \;\; k \in \mathbb{N}. \tag{4.9}$$

Since the series in (4.9) is uniformly convergent on J, we can multiply both sides of (4.9) by $\phi_m(x)$ and then α-integrate it term-by-term to get

$$\int_0^b f_1(x)\phi_m(x) d_\alpha x = c_m, \;\; k \in \mathbb{N}.$$

Since the Fourier coefficients of f_1 and f are the same, the Fourier coefficients of the difference $f_1 - f$ are zero. Applying (3.26) to the function $f_1 - f$, we deduce that $f_1 - f = 0$. Hence the sum of series in (4.1) is equal to $f(x)$, which proves the theorem. □

5. Conclusion

As a result, in our study, by examining a compatible fractional heat conduction equation, The method of separating the variables was suitable for this problem and we obtained the conformable fractional Sturm-Liouville eigenvalue problem with congruent fractions. Next, by proving the existence of a countably infinite set of eigenvalues and eigenfunctions, we construct uniformly convergent expansions of the eigenfunctions.

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