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The Hadamard-type k-step Pell sequences in Finite Groups

Ömür Deveci [1] Yeşim Aküzüm [2] and Muhammad Eshaq Rashedi [3]
[1,2,3] Department of Mathematics, Faculty of Science and Letters, Kafkas University 36100, Turkey

Abstract. In this work, we study the Hadamard-type k-step Pell sequence modulo m and then, we obtain the cyclic groups which are generated by the multiplicative orders of the Hadamard-type k-step Pell matrix when read modulo m. Then we extend the Hadamard-type k-step Pell sequence to groups and we redefine the Hadamard-type k-step Pell sequence by means of the elements of groups. Finally, we obtain the periods of the Hadamard-type 3-step Pell sequence in the semi-dihedral group SD_{2^m} and the quasi-dihedral group QD_{2^m}.

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1. Introduction

In [6], Deveci et al. defined the Hadamard-type k-step Pell sequence for $k \geq 3$ and $n \geq 0$ as follows:

$$HP_{n+k}^{k} = 2HP_{n+k-1}^{k} + HP_{n+k-2}^{k} + \cdots + HP_{n+2}^{k} - 2HP_{n+1}^{k} - HP_{n}^{k} \quad (1.1)$$

with initial constants $HP_0^k = HP_1^k = \cdots = HP_{k-2}^k = 0$ and $HP_{k-1}^k = 1$.

[1]odeveci36@hotmail.com
[2]Corresponding author: yesim_036@hotmail.com
[3]muhammadeshaqrashedi82@gmail.com
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Also in [6], they gave the generating matrix for the Hadamard-type k-step Pell sequence as shown:

$$H_k^p = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 & -2 & -1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix}_{k\times k.}$$

Then by an inductive argument, they derived that
(i) . For $k = 3$,

$$(H_3^p)^n = \begin{bmatrix} HP_{n+2}^3 & -2HP_{n+1}^3 - HP_n^3 & -HP_{n+1}^3 \\ HP_{n+1}^3 & -2HP_n^3 - HP_{n-1}^3 & -HP_n^3 \\ HP_n^3 & -2HP_{n-1}^3 - HP_{n-2}^3 & -HP_{n-1}^3 \end{bmatrix},$$

(ii) . For $k = 4$,

$$(H_4^p)^n = \begin{bmatrix} HP_{n+3}^4 & HP_{n+4}^4 - 2HP_{n+3}^4 & -2HP_{n+2}^4 - HP_{n+1}^4 & -HP_{n+2}^4 \\ HP_{n+2}^4 & HP_{n+3}^4 - 2HP_{n+2}^4 & -2HP_{n+1}^4 - HP_n^4 & -HP_{n+1}^4 \\ HP_{n+1}^4 & HP_{n+2}^4 - 2HP_{n+1}^4 & -2HP_n^4 - HP_{n-1}^4 & -HP_n^4 \\ HP_n^4 & HP_{n+1}^4 - 2HP_n^4 & -2HP_{n-1}^4 - HP_{n-2}^4 & -HP_{n-1}^4 \end{bmatrix},$$

(iii) . For $k \geq 5$,

$$\left(H_k^p\right)^n = \left[\begin{matrix} HP_{n+k-1}^k & HP_{n+k}^k - 2HP_{n+k-1}^k \\ HP_{n+k-2}^k & HP_{n+k-1}^k - 2HP_{n+k-2}^k \\ \vdots & \vdots \\ HP_{n+1}^k & HP_{n+2}^k - 2HP_{n+1}^k \\ HP_n^k & HP_{n+1}^k - 2HP_n^k \end{matrix} \right. \quad H_k^{p*}$$

$$\left. \begin{matrix} -2HP_{n+k-2}^k - HP_{n+k-3}^k & -HP_{n+k-2}^k \\ -2HP_{n+k-3}^k - HP_{n+k-4}^k & -HP_{n+k-3}^k \\ \vdots & \vdots \\ -2HP_n^k - HP_{n-1}^k & -HP_n^k \\ -2HP_{n-1}^k - HP_{n-2}^k & -HP_{n-1}^k \end{matrix} \right]$$

where H_k^{p*} is a $(k) \times (k-4)$ matrix as follows:

$$\begin{bmatrix} HP^k_{n+k-2}+HP^k_{n+k-3}+\cdots+HP^k_{n+3}-2HP^k_{n+2}-HP^k_{n+1} & HP^k_{n+k-2}+HP^k_{n+k-3}+\cdots+HP^k_{n+4}-2HP^k_{n+3}-HP^k_{n+2} & \cdots & HP^k_{n+k-2}-2HP^k_{n+k-3}-HP^k_{n+k-4} \\ HP^k_{n+k-3}+HP^k_{n+k-4}+\cdots+HP^k_{n+2}-2HP^k_{n+1}-HP^k_{n} & HP^k_{n+k-3}+HP^k_{n+k-4}+\cdots+HP^k_{n+3}-2HP^k_{n+2}-HP^k_{n+1} & \cdots & HP^k_{n+k-3}-2HP^k_{n+k-4}-HP^k_{n+k-5} \\ \vdots & \vdots & \vdots & \vdots \\ HP^k_{n}+HP^k_{n-1}+\cdots+HP^k_{n-k+5}-2HP^k_{n-k+4}-HP^k_{n-k+3} & HP^k_{n}+HP^k_{n-1}+\cdots+HP^k_{n-k+6}-2HP^k_{n-k+5}-HP^k_{n-k+4} & \cdots & HP^k_{n}-2HP^k_{n-1}-HP^k_{n-2} \\ HP^k_{n-1}+HP^k_{n-2}+\cdots+HP^k_{n-k+4}-2HP^k_{n-k+3}-HP^k_{n-k+2} & HP^k_{n-1}+HP^k_{n-2}+\cdots+HP^k_{n-k+5}-2HP^k_{n-k+4}-HP^k_{n-k+3} & \cdots & HP^k_{n-1}-2HP^k_{n-2}-HP^k_{n-3} \end{bmatrix}$$

for $n \geq k-3$. It is important to note that $\det \left(H_k^p\right)^n = (-1)^{kn}$.

Definition 1.1. A sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. In particular, if the first k elements in the sequence form a repeating subsequence, then the sequence is simply periodic and its period is k.

Definition 1.2. The semi-dihedral group SD_{2^m}, $(m \geq 4)$ is defined by the presentation

$$SD_{2^m} = \left\langle x, y | \ x^{2^{m-1}} = y^2 = e, \ yxy = x^{-1+2^{m-2}} \right\rangle.$$

Note that the orders x and y are 2^{m-1} and 2, respectively.

Definition 1.3. The quasi-dihedral group QD_{2^m}, $(m \geq 4)$ is defined by the presentation

$$QD_{2^m} = \left\langle x, y | \ x^{2^{m-1}} = y^2 = e, \ yxy = x^{1+2^{m-2}} \right\rangle.$$

Wall [18] proved that the lengths of the periods of the recurring sequences obtained by reducing Fibonacci sequences by a modulo m are equal to the lengths of the ordinary 2-step Fibonacci recurrences in cyclic groups. As a natural generalization of the problem, Wilcox [19] investigated the Fibonacci lengths to abelian groups. Recently, many authors have studied some special linear recurrence sequences in algebraic structures; see for example, [1, 2, 3, 4, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17]. In [4, 5, 12] several authors obtained the cyclic groups via some special matrices. Deveci et al. [6] defined the Hadamard-type k-step Pell sequences. In this work, we firstly study the Hadamard-type k-step Pell sequence modulo m and we consider the Hadamard-type k-step Pell matrix. Then, we obtain the cyclic groups which are generated by the multiplicative orders of the Hadamard-type k-step Pell matrix when read modulo m. Furthermore, we derive the relationship between the order the cyclic groups obtained and the periods of the Hadamard-type k-step Pell sequence modulo m. Secondly, we extend the Hadamard-type k-step Pell sequence to groups and we redefine the Hadamard-type k-step Pell sequence by means of the elements of groups. Finally, we obtain the periods of the Hadamard-type 3-step Pell sequence in the semi-dihedral group SD_{2^m} and the quasi-dihedral group QD_{2^m}.

2. The Hadamard-type k-step Pell sequences in Finite Groups

Reducing the Hadamard-type k-step Pell sequence $\left\{HP_n^k\right\}$ by a modulus m, then we get the repeating sequence, denoted by

$$\left\{HP_n^{k,m}\right\} = \left\{HP_0^{k,m}, HP_1^{k,m}, \ldots, HP_i^{k,m}, \ldots\right\}$$

where $HP_i^{k,m} = HP_i^k \,(mod\ m)$. It has the same recurrence relation as in (1.1).

Theorem 2.1. *The sequence $\left\{HP_n^{k,m}\right\}$ is simply periodic for $k \geq 3$.*

Proof. Suppose that $P = (p_0, p_1, \ldots, p_{k-1}) \mid {p_i}'$s are integers such that $0 \leq\ q_i \leq m-1\}$, then $|P| = m^k$. Since there are m^k distinct k-tuples of elements of Z_m, at least one of the k-tuples appears twice in the sequence $\left\{HP_n^{k,m}\right\}$. So, the subsequence following this k-tuples repeats; that is, the sequence $\left\{HP_n^{k,m}\right\}$ is periodic. So if

$$HP_i^{k,m} \equiv HP_j^{k,m}, HP_{i+1}^{k,m} \equiv HP_{j+1}^{k,m}, \ldots, HP_{i+k-1}^{k,m} \equiv HP_{j+k-1}^{k,m}$$

such that $i > j$, then $i \equiv j mod\,(k)$. By the definition of the Hadamard-type k-step Pell sequence, we can easily obtain

$$HP_{i-1}^{k,m} \equiv HP_{j-1}^{k,m}, HP_{i-2}^{k,m} \equiv HP_{j-2}^{k,m}, \ldots, HP_{i-j}^{k,m} \equiv HP_0^{k,m}.$$

Thus we get that the sequence $\left\{HP_n^{k,m}\right\}$ is a simply periodic. □

Given an integer matrix $Q = [q_{ij}]$, $Q\,(mod\ m)$ means that all entries of Q are reduced modulo m, that is, $Q\,(mod\ m) = (q_{ij}\,(mod\ m))$. Let us consider the set $\langle Q\rangle_m = \left\{Q^i\,(mod\ m) \mid i \geq 0\right\}$. If $\gcd(m, \det Q) = 1$, then $\langle Q\rangle_m$ is a cyclic group. We denote the order of the set $\langle Q\rangle_m$ by $|\langle Q\rangle_m|$. Since $\det H_k^p = (-1)^k$, it is clear that the set $\left\langle H_k^p\right\rangle_m$ is a cyclic group for every positive integer m.

Theorem 2.2. *Let s be a prime and let $\left\langle H_k^p\right\rangle_{s^m}$ be cyclic groups. If j is the largest positive integer such that $\left|\left\langle H_k^p\right\rangle_s\right| = \left|\left\langle H_k^p\right\rangle_{s^j}\right|$, then $\left|\left\langle H_k^p\right\rangle_{s^w}\right| = s^{w-j}.\left|\left\langle H_k^p\right\rangle_s\right|$ for every $w \geq j$. In particular, if $\left|\left\langle H_k^p\right\rangle_s\right| \neq \left|\left\langle H_k^p\right\rangle_{s^2}\right|$, then $\left|\left\langle H_k^p\right\rangle_{s^w}\right| = s^{w-1}.\left|\left\langle H_k^p\right\rangle_s\right|$ for every $w \geq 2$.*

Proof. Suppose that α is a positive integer and $\left|\left\langle H_k^p\right\rangle_{s^m}\right|$is denoted by $L^{k,p}(s^m)$. If $\left(H_k^p\right)^{L^{k,p}\left(s^{\alpha+1}\right)} = I \left(mod\ s^{\alpha+1}\right)$, then $\left(H_k^p\right)^{L^{k,p}\left(s^{\alpha+1}\right)} = I \ (mod\ s^{\alpha})$ where I is a $(k)\times(k)$ identity matrix. Then we show that $L^{k,p}\left(s^{\alpha+1}\right)$ is divided by $L^{k,p}(s^{\alpha})$. Also, writing $\left(H_k^p\right)^{L^{k,p}(s^{\alpha})} = I + \left(h_{ij}^{(\alpha)} s^{\alpha}\right)$, by the binomial theorem, we have

$$\left(H_k^p\right)^{L^{k,p}(s^{\alpha})\cdot s} = \left(I + \left(h_{ij}^{(\alpha)} s^{\alpha}\right)\right)^s = \sum_{i=0}^{s} \binom{s}{i} \left(h_{ij}^{(\alpha)} s^{\alpha}\right)^i \equiv I \left(mod\ s^{\alpha+1}\right).$$

So we get that $L^{k,p}\left(s^{\alpha+1}\right)$ divides $L^{k,p}(s^{\alpha})\cdot s$. Therefore, $L^{k,p}\left(s^{\alpha+1}\right) = L^{k,p}(s^{\alpha})$ or $L^{k,p}\left(s^{\alpha+1}\right) = L^{k,p}(s^{\alpha})\cdot s$. It is clear that $L^{k,p}\left(s^{\alpha+1}\right) = L^{k,p}(s^{\alpha})\cdot s$ holds if and only if there exists an integer $h_{ij}^{(\alpha)}$ which is not divisible by s. Since q is the largest positive integer such that $L^{k,p}(s) = L^{k,p}(s^q)$, we have $L^{k,p}(s^q) \neq L^{k,p}\left(s^{q+1}\right)$. Then, there exists an integer $h_{ij}^{(q+1)}$ which is not divisible by s. So we get that $L^{k,p}\left(s^{q+1}\right) \neq L^{k,p}\left(s^{q+2}\right)$. The proof is finished by induction on q. □

Let the notation $h^{k,p}(m)$ be the length of period of the sequence $\left\{HP_n^{k,m}\right\}$.

Theorem 2.3. *If $m = \prod_{i=1}^{u} (s_i)^{\beta_i}$, $(u \geqslant 1)$ where s_i's are distinct primes, then*

$$h^{k,p}(m) = lcm\left[h^{k,p}\left((s_1)^{\beta_1}\right), h^{k,p}\left((s_2)^{\beta_2}\right), \ldots, h^{k,p}\left((s_u)^{\beta_u}\right)\right].$$

Proof. Since $h^{k,p}\left((s_i)^{\beta_i}\right)$ is the length of the period of the sequence $\left\{HP_n^{k,(s_i)^{\beta_i}}\right\}$, the sequence $\left\{HP_n^{k,(s_i)^{\beta_i}}\right\}$ repeats only after blocks of length $\varepsilon.h^{k,p}\left((s_i)^{\beta_i}\right)$, $(\varepsilon \in N)$. Also, $h^{k,p}(m)$ is the length of the period $\left\{HP_n^{k,m}\right\}$, which implies that $\left\{HP_n^{k,(s_i)^{\beta_i}}\right\}$ repeats after $h^{k,p}(m)$ terms for all values i. So, $h^{k,p}(m)$ is the form $\varepsilon.h^{k,p}\left((s_i)^{\beta_i}\right)$ for all values of

i, and since any such number gives a period of $\left\{HP_n^{k,m}\right\}$. we conclude that

$$h^{k,p}(m) = lcm\left[h^{k,p}\left((s_1)^{\beta_1}\right), h^{k,p}\left((s_2)^{\beta_2}\right), \ldots, h^{k,p}\left((s_u)^{\beta_u}\right)\right].$$

□

Let G be a finite j-generator group and let X be the subset of $\underbrace{G \times G \times \cdots \times G}_{j \text{ times}}$ such that $(a_0, a_1, \ldots, a_{j-1}) \in X$ if and only if G is generated by $a_0, a_1, \ldots, a_{j-1}$. $(a_0, a_1, \ldots, a_{j-1})$ is said to be a generating j-tuple for G.

Definition 2.4. Let G be a j-generator group and let $(a_0, a_1, \ldots, a_{j-1})$ be a generating j-tuple for G. Then, we define the Hadamard-type k-step Pell sequence in G as follows:
if $k \leq j$, then

$$\begin{aligned} x_0 &= a_0,\ x_1 = a_1, \ldots, x_{j-1} = a_{j-1} \text{ and} \\ x_n &= (x_{n-k})^{-1}(x_{n-k+1})^{-2}(x_{n-k+2})\cdots(x_{n-2})(x_{n-1})^2 \text{ for } n \geq j; \end{aligned}$$

if $k = j+1$, then

$$\begin{aligned} x_0 &= a_0,\ x_1 = a_1, \ldots, x_{j-1} = a_{j-1}, x_j = (x_0)^{-2}(x_1)\cdots(x_{j-2})(x_{j-1})^2 \text{ and} \\ x_n &= (x_{n-k})^{-1}(x_{n-k+1})^{-2}(x_{n-k+2})\cdots(x_{n-2})(x_{n-1})^2 \text{ for } n \geq k; \end{aligned}$$

if $k \geq j+2$, then

$$\begin{aligned} x_0 &= a_0,\ x_1 = a_1, \ldots, x_{j-1} = a_{j-1} \text{ and} \\ x_n &= \begin{cases} (x_0)(x_1)\cdots(x_{n-2})(x_{n-1})^2 \text{ for } j \leq n \leq k-2, \\ (x_0)^{-2}(x_1)\cdots(x_{n-2})(x_{n-1})^2 \text{ for } n = k-1, \\ (x_{n-k})^{-1}(x_{n-k+1})^{-2}(x_{n-k+2})\cdots(x_{n-2})(x_{n-1})^2 \text{ for } n \geq k. \end{cases} \end{aligned}$$

The Hadamard-type k-step Pell sequence in the group G for the generating j-tuple $(a_0, a_1, \ldots, a_{j-1})$ is denoted by $HP^k_{(G;a_0,a_1,\ldots,a_{j-1})}$.

Theorem 2.5. *A Hadamard-type k-step Pell sequence in a finite group is periodic. In particular, for $k \geq j$, a Hadamard-type k-step Pell sequence in a j-generated finite group is simply periodic.*

Proof. Suppose that p is the order of the group G. Since there are p^k distinct k-tuples of elements of G, at least one of the k-tuples appears twice in the sequence $HP^k_{(G;a_0,a_1,\ldots,a_{j-1})}$. Thus, the subsequence following this k-tuples repeats; hence, the sequence $HP^k_{(G;a_0,a_1,\ldots,a_{j-1})}$ is periodic. Since the sequence $HP^k_{(G;a_0,a_1,\ldots,a_{j-1})}$ is periodic, there exist natural numbers g and h with $g > h$, such that

$$x_{g+1} = x_{h+1}, x_{g+2} = x_{h+2}, \ldots, x_{g+k} = x_{h+k}.$$

By the definition of the sequence $HP^{k}_{(G;a_0,a_1,\ldots,a_{j-1})}$, we know that

$$x_n = (x_{n+1})^{-2}(x_{n+2})\cdots(x_{n+k-2})(x_{n+k-1})^{2}(x_{n+k})^{-1}.$$

If $k \geq j$, then we may write $x_g = x_h$, and hence

$$x_{g-1} = x_{h-1}, x_{g-2} = x_{h-2}, \ldots, x_{g-g} = x_0 = x_{h-g}$$

which implies that the sequence $HP^{k}_{(G;a_0,a_1,\ldots,a_{j-1})}$ is simply periodic. □

We denote the period of the sequence $HP^{k}_{(G;a_0,a_1,\ldots,a_{j-1})}$ by $LHP^{k}_{(G;a_0,a_1,\ldots,a_{j-1})}$.

We consider the periods of the Hadamard-type 3-step Pell sequence in the semi-dihedral group SD_{2^m} and the quasi-dihedral group QD_{2^m}.

Theorem 2.6. *The period of the Hadamard-type* 3*-step Pell sequence in the semi-dihedral group* SD_{2^m} *is* $3 \cdot 2^{m-2}$.

Proof. We prove the result by direct calculation. The sequence $HP^{3}_{(SD_{2^m};x,y)}$ is

$$\begin{array}{l} x, y, x^{-2}, x^{-5}, yx^{-6}, x^{12}, x^{29}, yx^{28}, x^{-70}, \\ x^{-169}, yx^{-170}, x^{408}, x^{985}, yx^{984}, x^{-2378}, \ldots. \end{array}$$

Using the above, it possible to show that the sequence has the following form

$$\begin{array}{rcl} x_0 & = & x, x_1 = y, x_2 = x^{-2}, \ldots \\ x_{6i} & = & x^{4i\beta_1+1}, x_{6i+1} = yx^{4i\beta_1}, x_{6i+2} = x^{-4i\beta_2-2}, \ldots, \end{array}$$

where β_1 and β_2 are positive integers such that $\gcd(\beta_1, \beta_2) = 1$. So we need the smallest $i \in \mathbb{N}$ such that $4i = 2^{m-1}k$ $(k \in \mathbb{N})$. If we choose $i = 2^{m-3}$, we obtain

$$x_{6\cdot 2^{m-3}} = x, x_{6\cdot 2^{m-3}+1} = y, x_{6\cdot 2^{m-3}+2} = x^{-2}.$$

Since the elements succeeding $x_{6\cdot 2^{m-3}}$, $x_{6\cdot 2^{m-3}+1}, x_{6\cdot 2^{m-3}+2}$ depend on x and y for their values, the cycle begins again with the (6.2^{m-3}) *nd* element. Thus it is verified that

$$LHP^{3}_{(SD_{2^m};x,y)} = 3 \cdot 2^{m-2}.$$

□

Theorem 2.7. *Consider the quasi-dihedral group* QD_{2^m} *with generating pair* (x, y)*. Then the period of the sequence* $HP^{3}_{(QD_{2^m};x,y)}$ *is* $3 \cdot 2^{m-3}$.

Proof. The sequence $HP^{3}_{(QD_{2^m};x,y)}$ is

$$\begin{array}{l} x, y, x^{-2}, x^{-5}, yx^{-6}, e, x^{17}, yx^{40}, x^{46}, \\ x^{-5}, yx^{-142}, x^{-320}, x^{-351}, yx^{80}, x^{1182}, \ldots. \end{array}$$

Thus, we also have

$$\begin{aligned} x_0 &= x, x_1 = y, x_2 = x^{-2}, \ldots \\ x_{6i} &= x^{8i\beta_1+1}, x_{6i+1} = yx^{8i\beta_2}, x_{6i+2} = x^{8i\beta_3-2}, \ldots, \end{aligned}$$

where β_1 , β_2 and β_3 are positive integers such that $\gcd(\beta_1, \beta_2, \beta_3) = 1$. So we need the smallest $i \in \mathbb{N}$ such that $8i = 2^{m-1}k$ $(k \in \mathbb{N})$. If we choose $i = 2^{m-4}$, we obtain

$$x_{6\cdot 2^{m-4}} = x, x_{6\cdot 2^{m-4}+1} = y, x_{6\cdot 2^{m-4}+2} = x^{-2}.$$

So we get

$$LHP^3_{(QD_{2^m};x,y)} = 3 \cdot 2^{m-3}.$$

□

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