Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **11**(1)(2022), 52-68 (Research paper)

A special type of IF operations, IF modules and IF homomorphisms

Behnam Talaee [1], Mehrnoosh Sobhani Oskooie [2] and Mohammad Javad Nematollahi [3]
[1], [2] Department of Mathematics, Faculty of Basic Sciences,
Babol Noshirvani University of Technology, Shariati Ave., Babol, Iran
Post Code:47148-71167
[3] Department of Mathematics, Islamic Azad University, Arsanjan Branch, Arsanjan, Iran

Abstract. In this paper we study about IF binary operations on some IF sets, at first. Then we introduce IF groups, modules and IF homomorphisms under IF binary operation. We get some properties of IF groups rings and modules under binary operation. IF modules and IF homomorphisms over this kind of IF rings are introduced and investigated.

Keywords: IF operation, IF rings, IF modules, IF homomorphisms.

2020 Mathematics subject classification: 08A72, 16D60, 94D05.

1. Introduction

The concept of fuzzy subgroup of a group was first introduced by Rosenfeld [?] in 1971. The concept of fuzzy subset of a non-empty set was introduced by Zadeh [?] who introduced the notion of a fuzzy set as a method of representing uncertainty in real physical world. Negoita and Ralescu [?] introduced fuzzy module. By theuse of Yuan and Lee's [?] definition of fuzzy group based on fuzzy binary operation, Aktas and

[1]Corresponding author: behnamtalaee@nit.ac.ir
Received: 12 July 2020
Revised: 16 March 2021
Accepted: 31 March 2021

Cagman [?] defined a new kind of fuzzy ring.
In this study, we introduce a new kind of intuitionistic fuzzy module by using Yuan and Lee's definition of the intuitionistic fuzzy group and Aktas and Cagman's definition of fuzzy ring.
Let X be a non-empty set. A mapping $\mu : X \longrightarrow [0,1]$ is called a fuzzy subset of X. Rosenfeld [?] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group.

Definition 1.1. Let M be an $R-$module. Then the fuzzy set μ of M is called a *fuzzy submodule* (FSM) of M if
(1) $\mu(0) = 1$;
(2) $\mu(x+y) \geq min\{\mu(x), \nu(y)\}, \forall x, y \in M$;
(3) $\mu(rx) \geq \mu(x), \forall x \in M, r \in R$.

Definition 1.2. *Intersection* (logical and): the membership function of the intersection of two fuzzy sets A and B is defined as:
$\mu_{A\cap B}(x) = Min(\mu_A(x), \mu_B(x)),\ \forall x \in X$

Definition 1.3. *Union* (exclusive or): the membership function of the union is defined as:
$\mu_{A\cup B}(x) = Max(\mu_A(x), \mu_B(x)),\ \forall x \in X$

Definition 1.4. For two fuzzy $R-$modules μ_A and μ_B; a function $\tilde{f}$: $\mu_A \longrightarrow \nu_B$ is called *fuzzy $R-$homomorphism*, if f is an $R-$homomorphism and $\nu(f(a)) \geq \mu(a)(\forall a \in A)$. For simplicity, denote by $Hom(\mu_A,\ \nu_B)$ the set of fuzzy $R-$homomorphisms from μ_A to ν_B.

Definition 1.5. Let G be a nonempty set and R be a fuzzy subset of $G \times G \times G$. R is called a *fuzzy binary operation* on G if
(1) for all $a, b \in G$, $\exists c \in G$ such that $R(a, b, c) > \theta$;
(2) for all $a, b, c_1, c_2 \in G$, $R(a, b, c_1)$ 0 and $R(a, b, c_2)$ 0 implies $C_1 = C_2$.

Definition 1.6. Let G be a nonempty set and R be a fuzzy binary operation on G. (G, R) is called a *fuzzy group* if the following conditions are true:
(1) $\forall a, b, c, z_1, z_2 \in G$, $((aob)oc)(z_1) > 0$ and $(ao(boc))(z_2) > 0$ implies $z_1 = z_2$;
(2) $\exists e \in G$ such that $(eoa)(a) > 0$ and $(aoe)(a) > 0$ for any $a \in G$ (e is called an identity element of G);
(3) $\forall a \in G, \exists b \in G$ such that $(aoB)(a) > 0$ and $(boa)(e) > 0$ (b is called an inverse dement of a and is denoted as a^{-1}).

Definition 1.7. A fuzzy set μ of a ring R is called a *fuzzy ideal*, if it satisfies the following properties:
(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in R$.
(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$, for all $x, y \in R$.

Definition 1.8. An *intuitionistic fuzzy set* (briefly an IFS) A of a non-void set X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$, where the maps $\mu_A : X \longrightarrow [0,1]$ and $\nu_A : X \longrightarrow [0,1]$, are fuzzy subsets of X, denote respectively the degree of membership (namely $\mu_A(x)$) and the degree of
non-membership (namely $\nu_A(x)$) of each element $x \in X$, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.
For the sake of simplicity, we denote an IFS, $A = \{(x, \mu_A(x), \nu_A(x)); x \in X\}$ of the set X by $A = (\mu_A, \nu_A)$ or briefly A, and the set of all IFS of X by $IFS(X)$. If X is a non-empty set and $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$ are two IFS of X, then
$A \subseteq B$, if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$;
$A = B$ if and only if $\mu_A(x) = \mu_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$;
$A^c = (\nu_A, \mu_A)$;
$A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)); x \in X\}$;
$A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)); x \in X\}$.

Let $\{A_i = (\mu_{A_i}, \nu_{A_i})\}_{i \in I}$ be a family of IFS of X. Then
$\bigcap_{i \in I} A_i = (\mu_{(\cap_{i \in I} A_i)}, \nu_{(\cap_{i \in I} A_i)}) = \{(x, \bigwedge_{i \in I} \mu_{A_i}(x), \bigvee_{i \in I} \nu_{A_i}(x)); x \in X\}$
and
$\bigcup_{i \in I} A_i = (\mu_{(\cup_{i \in I} A_i)}, \nu_{(\cup_{i \in I} A_i)}) = \{(x, \bigvee_{i \in I} \mu_{A_i}(x), \bigwedge_{i \in I} \nu_{A_i}(x)); x \in X\}$

Definition 1.9. Let M be an $R-$module and $A = (\mu_A, \nu_A)$ an IFS of M. Then A is called an *intuitionistic fuzzy submodule of* M if A satisfies the
following:

(1) $\mu_A(0) = 1$, $\nu_A(0) = 0$
(2) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$, for all $x, y \in M$
$\nu_A(x + y) \leq \nu_A(x) \vee \nu_A(y)$, for all $x, y \in M$
(3) $\mu_A(rx) \geq \mu_A(x)$, for all $x \in M$ and $r \in R$
$\nu_A(rx) \leq \nu_A(x)$, for all $x \in M$ and $r \in R$

2. IF binary operations , IF fuzzy groups, basic properties and preliminaries

In this section we give some important definitions of IF sets and operations. Then we formulate some properties and results of them.

Definition 2.1. Let $\theta \in [0, 1)$, R and S be nonempty sets and let $f = (\mu_f, \nu_f)$ be an intuitionistic fuzzy subset of $R \times S$; then A is called a *(θ) intuitionistic fuzzy (IF) function* from R into S if

$$(1) \begin{cases} \forall x \in R,\ \exists y \in s \text{ such that } \mu_f(x, y) > \theta \\ (\forall x \in R,\ \exists y \in s \text{ such that } \nu_f(x, y) < 1 - \theta) \end{cases}$$

$$(2)\begin{cases}\forall x\in R \text{ for all } y_1,y_2\in S,\ \mu_f(x,y_1)>\theta \text{ and } \mu_f(x,y_2)>\theta \text{ imply } y_1=y_2\\ \forall x\in R \text{ for all } y_1,y_2\in S,\ \nu_f(x,y_1)<1-\theta \text{ and } \nu_f(x,y_2)<1-\theta \text{ imply } y_1=y_2\end{cases}$$

Definition 2.2. Let G be a nonempty set and let $R=(\mu_R,\nu_R)$ be an IF subset of $G\times G\times G$. Then $R=(\mu_R,\nu_R)$ with $\begin{cases}\mu_R: G\times G\times G\longrightarrow[0,1]\\ \nu_R: G\times G\times G\longrightarrow[0,1]\end{cases}$ is called an *intuitionistic fuzzy binary operation* on G if

$$(1)\begin{cases}\forall a,b\in G,\ \exists c\in G \text{ such that } \mu_R(a,b,c)>\theta\\ (\forall a,b\in G,\ \exists c\in G \text{ such that } \nu_R(a,b,c)<1-\theta)\end{cases}$$

$$(2)\begin{cases}\forall a,b,c_1,c_2\in G, \mu_R(a,b,c_1)>\theta \text{ and } \mu_R(a,b,c_2)>\theta \text{ imply } c_1=c_2\\ \forall a,b,c_1,c_2\in G, \nu_R(a,b,c_1)<1-\theta \text{ and } \nu_R(a,b,c_2)<1-\theta \text{ imply } c_1=c_2\end{cases}$$

Let R be an intuitionistic fuzzy binary operation on G; then we have a mapping

$$R: IF(G)\times IF(G)\longrightarrow IF(G),$$
$$(A,B)\longmapsto \alpha_R(A,B),$$

where $IF(G)$ is the set of all IF subsets of G, such that $\alpha_R(A,B)=(\mu_{\alpha_R},\nu_{\alpha_R})$ where

$$\begin{cases}\mu_{\alpha_R}(A,B)(c)=\bigvee_{a,b\in G}(\mu_A(a)\wedge\mu_B(b)\wedge\mu_R(a,b,c))\\ \nu_{\alpha_R}(A,B)(c)=\bigwedge_{a,b\in G}(\nu_A(a)\vee\nu_B(b)\vee\nu_R(a,b,c))\end{cases}$$

Let $A=\chi^{IF}_{\{a\}}=(\chi_{\{a\}},\chi^c_{\{a\}})$ and $B=\chi^{IF}_{\{b\}}=(\chi_{\{b\}},\chi^c_{\{b\}})$ and let $R(A,B)$ be denoted as $(aob)^{IF}=(\mu_{(aob)},\nu_{(aob)})$ and $(boa)^{IF}=(\mu_{(boa)},\nu_{(boa)})$; then

$$\begin{cases}\forall c\in G,\ (\mu_{(aob)})(c)=\mu_R(a,b,c),\\ \forall c\in G,\ (\nu_{(aob)})(c)=\nu_R(a,b,c),\end{cases}\quad \begin{cases}\forall c\in G,\ (\mu_{(boa)})(c)=\mu_R(b,a,c),\\ \forall c\in G,\ (\nu_{(boa)})(c)=\nu_R(b,a,c),\end{cases}$$

Now define $((aob)oc)^{IF}=(\mu_{((aob)oc)},\nu_{((aob)oc)})$ and $(ao(boc))^{IF}=(\mu_{(ao(boc))},\nu_{(ao(boc))})$ then

$$\begin{cases}\forall c\in G,\ \mu_{((aob)oc)}(z)=\bigvee_{d\in G}(\mu_R(a,b,d)\wedge\mu_R(d,c,z))\\ \forall c\in G,\ \nu_{((aob)oc)}(z)=\bigwedge_{d\in G}(\nu_R(a,b,d)\vee\nu_R(d,c,z))\end{cases}$$

$$\begin{cases}\forall c\in G,\ \mu_{(ao(boc))}(z)=\bigvee_{d\in G}(\mu_R(b,c,d)\wedge\mu_R(a,d,z))\\ \forall c\in G,\ \nu_{(ao(boc))}(z)=\bigwedge_{d\in G}(\nu_R(b,c,d)\vee\nu_R(a,d,z))\end{cases}$$

Definition 2.3. Let G be nonempty set and let $R=(\mu_R,\nu_R)$ be an IF binary operation on G. (G,R) is called an *IF group*, if the following conditions are true

$$(1)\begin{cases}\forall a,b,c,z_1,z_2\in G,\ \mu_{((aob)oc)}(z_1)>\theta \text{ and } \mu_{(ao(boc))}(z_2)>\theta\\ \qquad \text{imply } z_1=z_2;\\ \forall a,b,c,z_1,z_2\in G,\ \nu_{((aob)oc)}(z_1)<1-\theta \text{ and } \nu_{(ao(boc))})(z_2)<1-\theta\\ \qquad \text{imply } z_1=z_2;\end{cases}$$

(2) there exists $e_0\in G$, $(e_0oa)=(\mu_{(e_0oa)},\ \nu_{(e_0oa)})$, $(aoe_0)=(\mu_{(aoe_0)},\nu_{(aoe_0)})$ such that $\mu_{(e_0oa)}(a)>\theta$ and $\mu_{(aoe_0)}(a)>\theta$ (Consequently $\nu_{(e_0oa)}(a)<1-\theta$ and $\nu_{(aoe_0)}(a)<1-\theta$) for every $a\in G$ (e_0 is called an *identity element* of G).

(3) For every $a \in G$, there exists $b \in G$ such that $\mu_{(aob)}(e_0) > \theta$ and $\mu_{(boa)}(e_0) > \theta$ (Consequently $\nu_{(aob)}(e_0) < 1-\theta$ and $\nu_{(boa)}(e_0) < 1-\theta$ in this case b is called an *inverse element* of a and denoted by a^{-1}.

Proposition 2.4. $\begin{cases} \mu_{((aob)oc)}(d) > \theta \Longleftrightarrow (\mu_{(ao(boc))})(d) > \theta; \\ \nu_{((aob)oc)}(d) < 1-\theta \Longleftrightarrow \nu_{((aob)oc)}(d) < 1-\theta \end{cases}$.

Proof. Let $\begin{cases} \mu_{((aob)oc)}(d) > \theta; \\ \nu_{((aob)oc)}(d) < 1-\theta \end{cases}$ and let $z, w \in G$ such that $\begin{cases} \mu_R(b,c,z) > \theta; \\ \nu_R(b,c,z) < 1-\theta \end{cases}$ and $\begin{cases} \mu_R(a,z,w) > \theta; \\ \nu_R(a,z,w) < 1-\theta \end{cases}$. Then

$$\begin{cases} \mu_{(ao(boc))}(w) \geq \mu_R(b,c,z) \wedge \mu_R(a,z,w) > \theta; \\ \nu_{(ao(boc))}(w) \leq \nu_R(b,c,z) \vee \nu_R(a,z,w) < 1-\theta \end{cases}$$

Thus, $d = w$ and $\begin{cases} \mu_{(ao(boc))}(d) > \theta; \\ \nu_{(ao(boc))}(d) < 1-\theta \end{cases}$.

Similarly by $\begin{cases} \mu_{(ao(boc))}(d) > \theta; \\ \nu_{(ao(boc))}(d) < 1-\theta \end{cases}$. we have $\begin{cases} \mu_{((aob)oc)}(d) > \theta; \\ \nu_{((aob)oc)}(d) < 1-\theta \end{cases}$.

□

Proposition 2.5. *H is an IF subgroup of G if and only if*

(1) $\begin{cases} \forall a,b \in H, \forall c \in G, \mu_{(aob)}(c) > \theta \text{ imlies } c \in H; \\ \forall a,b \in H, \forall c \in G, \nu_{(aob)}(c) < 1-\theta \text{ imlies } c \in H \end{cases}$

(2) $a \in H$ implies $a^{-1} \in H$.

Definition 2.6. Let $H = (\mu_H, \nu_H)$ be an IF subgroup of G. Let $aH = \begin{cases} (a\mu_H)(z) = \bigvee_{x\in G} \mu_R(a,x,z); \\ (a\nu_H)(z) = \bigwedge_{x\in G} \nu_R(a,x,z). \end{cases}$ Ha=$\begin{cases} (\mu_H a)(z) = \bigvee_{x\in G} \mu_R(x,a,z); \\ (\nu_H a)(z) = \bigwedge_{x\in G} \nu_R(x,a,z). \end{cases}$
Then aH (Ha) is called a *left(right)coset* of H.

Definition 2.7. Let $H = (\mu_H, \nu_H)$ be an IF subgroup of G. If for $(ao(hoa^{-1})) = (\mu_{(ao(hoa^{-1}))},\ \nu_{(ao(hoa^{-1}))})$

$$\begin{cases} \forall a,b \in G,\ \forall h \in H,\ \mu_{(ao(hoa^{-1}))}(b) > \theta; \\ \forall a,b \in G,\ \forall h \in H,\ \nu_{(ao(hoa^{-1}))}(b) < 1-\theta. \end{cases}$$

then H is called a *normal IF subgroup* of G.

Definition 2.8. Let (G,R) be an IF subgroup. If
$\mu_{(aob)}(c) > \theta \Longleftrightarrow \mu_{(boa)}(c) > \theta,\ \forall a,b,c \in G$
$\nu_{(aob)}(c) < 1-\theta \Longleftrightarrow \nu_{(boa)}(c) < 1-\theta,\ \forall a,b,c \in G,$
then $(G,R)^{IF}$ is called an *abelian IF group*.

Theorem 2.9. *Let* $[aH]=\{a'H \mid a'H\sim aH\}$,
$\bar{a}=\{a' \mid a'\in G \text{ and } a'H\sim aH\},\ G/H=\{[aH] \mid a\in G\}$, *and*

$$\bar{R}=(\mu_{\bar{R}},\nu_{\bar{R}})=\begin{cases}\mu_{\bar{R}}:\frac{G}{H}\times\frac{G}{H}\times\frac{G}{H}\longrightarrow[0,1],\\ \nu_{\bar{R}}:\frac{G}{H}\times\frac{G}{H}\times\frac{G}{H}\longrightarrow[0,1].\end{cases}$$

$$([aH],[bH],[cH])\longmapsto\bar{R}([aH],[bH],[cH])=\begin{cases}\bigvee_{(a',b',c')\in\bar{a}\times\bar{b}\times\bar{c}}\mu_R(a',b',c'),\\ \bigwedge_{(a',b',c')\in\bar{a}\times\bar{b}\times\bar{c}}\nu_R(a',b',c').\end{cases}$$

Then $\bar{R}$ *is an* IF *binary relation on* $\frac{G}{H}$.

Proof.
(1) $\forall a,b\in G,\ \exists c\in G$ such that $\mu_R(a,b,c)>\theta$, then

$$\begin{cases}\mu_{\bar{R}}([aH],[bH],[cH])\geq\mu_R(a,b,c)>\theta,\\ \nu_{\bar{R}}([aH],[bH],[cH])\leq\mu_R(a,b,c)<1-\theta.\end{cases}$$

(2) Let

$$M=\begin{cases}\mu_{\bar{R}}([aH],[bH],[cH])>\theta,\\ \nu_{\bar{R}}([aH],[bH],[cH])<1-\theta.\end{cases}$$

and

$$N=\begin{cases}\mu_{\bar{R}}([aH],[bH],[dH])>\theta,\\ \nu_{\bar{R}}([aH],[bH],[dH])<1-\theta.\end{cases}$$

We need to prove $[cH]=[dH]$.
There exist $a_1\in\bar{a},\ b_1\in\bar{b},\ c_1\in\bar{c},\ a_1'\in\bar{a},\ b_1'\in\bar{b},\ d_1\in d$ such that

$$\begin{array}{c}\mu_R(a_1,b_1,c_1)>\theta\ \mu_R(a_1',b_1',c_1')>\theta,\\ \nu_R(a_1,b_1,c_1)<1-\theta\ \nu_R(a_1',b_1',c_1')<1-\theta.\end{array}$$

Since $a_1'H\sim a_1H,\ b_1'H\sim b_1H$, so there exist $h_1\in H,h_2\in H$ such that

$$\begin{array}{c}\mu_R(a_1',h_1,a_1)>\theta\ \mu_R(b_1',h_2,b_1)>\theta,\\ \nu_R(a_1',h_1,a_1)<1-\theta\ \nu_R(b_1',h_2,b_1)<1-\theta.\end{array}$$

Let $z\in G$ such that $\begin{cases}\mu_R(h_1,b_1',z)>\theta,\\ \nu_R(h_1,b_1',z)<1-\theta.\end{cases}$, then $\begin{cases}\mu_R(z,{b_1'}^{-1},h_1)>\theta,\\ \nu_R(z,{b_1'}^{-1},h_1)<1-\theta.\end{cases}$,
So ${b_1'}^{-1}H\sim zH$ and there exists $h_1'\in H$ such that $\begin{cases}\mu_R({b_1'}^{-1},z,h')>\theta,\\ \nu_R({b_1'}^{-1},z,h')<1-\theta.\end{cases}$.
Let $y\in G$ such that $\begin{cases}\mu_R(b_1',h_1',y)>\theta,\\ \nu_R(b_1',h_1',y)<1-\theta.\end{cases}$, then for
$(b_1'o({b_1'}^{-1}oz))^{IF}=(\mu_{(b_1'o({b_1'}^{-1}oz))},\nu_{(b_1'o({b_1'}^{-1}oz))})$,

$$\begin{cases}\mu_{(b_1'o({b_1'}^{-1}oz))}(y)\geq\mu_R({b_1'}^{-1},z,h_1')\wedge\mu_R(b_1',h_1',y)>\theta,\\ \nu_{(b_1'o({b_1'}^{-1}oz))}(y)\leq\nu_R({b_1'}^{-1},z,h_1')\vee\nu_R(b_1',h_1',y)<1-\theta\end{cases}$$

and $((b'_1 o b'^{-1}_1) o z)^{IF} = (\mu_{((b'_1 o b'^{-1}_1) o z)}, \nu_{((b'_1 o b'^{-1}_1) o z)})$,

$$\begin{cases} \mu_{((b'_1 o b'^{-1}_1) o z)}(z) \geq \mu_R(b'_1, b'^{-1}_1, e) \wedge \mu_R(e, z, z) > \theta, \\ \nu_{((b'_1 o b'^{-1}_1) o z)}(z) \leq \nu_R(b'_1, b'^{-1}_1, e) \vee \nu_R(e, z, z) < 1-\theta \end{cases}$$

Thus, $y = z$. Let $z_1, y_1 \in G$ such that $\begin{matrix} \mu_R(h_1, b_1, z_1) > \theta, \\ \nu_R(h_1, b_1, z_1) < 1-\theta. \end{matrix}$, $\begin{matrix} \mu_R(a'_1, z_1, y_1) > \theta, \\ \nu_R(a'_1, z_1, y_1) < 1-\theta. \end{matrix}$,
then $(a'_1 o (h_1 o b_1))^{IF} = (\mu_{(a'_1 o (h_1 o b_1))}, \nu_{(a'_1 o (h_1 o b_1))})$

$$\begin{cases} \mu_{(a'_1 o (h_1 o b_1))}(y_1) \geq \mu_R(h_1, b_1, z_1) \wedge \mu_R(a'_1, z_1, y_1) > \theta, \\ \nu_{(a'_1 o (h_1 o b_1))}(y_1) \leq \nu_R(h_1, b_1, z_1) \vee \nu_R(a'_1, z_1, y_1) < 1-\theta \end{cases}$$

$((a'_1 o h_1) o b_1)^{IF} = (\mu_{((a'_1 o h_1) o b_1)}, \nu_{((a'_1 o h_1) o b_1)})$

$$\begin{cases} \mu_{((a'_1 o h_1) o b_1)} \geq \mu_R(h_1, b_1, z_1) \wedge \mu_R(a'_1, z_1, y_1) > \theta, \\ \nu_{((a'_1 o h_1) o b_1)} \leq \nu_R(h_1, b_1, z_1) \vee \nu_R(a'_1, z_1, y_1) < 1-\theta \end{cases}$$

Thus, $y_1 = c_1$ and $\begin{cases} \mu_R(a'_1, z_1, c_1) > \theta, \\ \nu_R(a'_1, z_1, c_1) < 1-\theta \end{cases}$

Let $p_1 \in G$ such that $\begin{cases} \mu_R(z, h_2, p_1) > \theta, \\ \nu_R(z, h_2, p_1) < 1-\theta \end{cases}$ then for
$((h_1 o b'_1) o h_2)^{IF} = (\mu_{((h_1 o b'_1) o h_2)}, \nu_{((h_1 o b'_1) o h_2)})$

$$\begin{cases} \mu((h_1 o b'_1) o h_2)(p_1) \geq \mu_R(h_1, b'_1, z) \wedge \mu_R(z, h_2, p_1) > \theta, \\ \nu((h_1 o b'_1) o h_2)(p_1) \leq \nu_R(h_1, b'_1, z) \vee \nu_R(z, h_2, p_1) < 1-\theta \end{cases}$$

and for $(h_1 o (b'_1 o h_2))^{IF} = (\mu_{(h_1 o (b'_1 o h_2))}, \nu_{(h_1 o (b'_1 o h_2))}) \Rightarrow$

$$\begin{cases} \mu_{(h_1 o (b'_1 o h_2))}(z_1) \geq \mu_R(b'_1, h_2, b_1) \wedge \mu_R(h_1, b_1, z) > \theta, \\ \nu_{(h_1 o (b'_1 o h_2)}(z_1) \leq \nu_R(b'_1, h_2, b_1) \vee \nu_R(h_1, b_1, z) < 1-\theta \end{cases}$$

Thus, $p_1 = z_1$ and $\begin{cases} \mu_R(z, h_2, z_1) > \theta, \\ \nu_R(z, h_2, z_1) < 1-\theta \end{cases}$, $\begin{cases} \mu_R(y, h_2, z_1) > \theta, \\ \nu_R(y, h_2, z_1) < 1-\theta \end{cases}$.

Let $h \in G$, $w_1 \in G$ such that $\begin{cases} \mu_R(h'_1, h_2, h) > \theta, \\ \nu_R(h'_1, h_2, h) < 1-\theta \end{cases}$, $\begin{cases} \mu_R(b'_1, h, w_1) > \theta, \\ \nu_R(b'_1, h, w_1) < 1-\theta \end{cases}$,
then $h \in H$ and $(b'_1 o (h'_1 o h_2)^{IF} = (\mu_{(b'_1 o (h'_1 o h_2)}, \nu_{(b'_1 o (h'_1 o h_2)}) \Rightarrow$

$$\begin{cases} \mu_{(b'_1 o (h'_1 o h_2)}(w_1) \geq \mu_R(h'_1, h_2, h) \wedge \mu_R(b'_1, h, w_1) > \theta, \\ \nu_{(b'_1 o (h'_1 o h_2)}(w_1) \leq \nu_R(h'_1, h_2, h) \vee \nu_R(b'_1, h, w_1) < 1-\theta \end{cases}$$

$((b'_1 o h'_1) o h_2)^{IF} = (\mu_{((b'_1 o h'_1) o h_2)}, \nu_{((b'_1 o h'_1) o h_2)}) \Rightarrow$

$$\begin{cases} \mu_{((b'_1 o h'_1) o h_2)}(z_1) \geq \mu_R(b'_1, h'_1, y) \wedge \mu_R(y, h_2, z_1) > \theta, \\ \nu_{((b'_1 o h'_1) o h_2)}(z_1) \leq \nu_R(b'_1, h'_1, y) \vee \nu_R(y, h_2, z_1) < 1-\theta \end{cases}$$

Thus, $w_1 = z_1$ and $\begin{cases} \mu_R(b'_1, h, z_1) > \theta, \\ \nu_R(b'_1, h, z_1) < 1-\theta \end{cases}$.

Let $w \in G$ such that $\begin{cases} \mu_R(d_1, h, w) > \theta, \\ \nu_R(d_1, h, w) < 1-\theta \end{cases}$, then
$((a'_1 o b'_1) o h)^{IF} = (\mu_{((a'_1 o b'_1) o h)}, \nu_{((a'_1 o b'_1) o h)}) \Rightarrow$

$$\begin{cases} \mu_{((a'_1ob'_1)oh)}(w) \geq \mu_R(a'_1, b'_1, d_1) \wedge \mu_R(d_1, h, w) > \theta, \\ \nu_{((a'_1ob'_1)oh)}(w) \leq \nu_R(a'_1, b'_1, d_1) \vee \nu_R(d_1, h, w) < 1-\theta \end{cases}$$

$(a'_1o(b'_1oh))^{IF} = (\mu_{(a'_1o(b'_1oh))}, \nu_{(a'_1o(b'_1oh))}) \Rightarrow$

$$\begin{cases} \mu_{(a'_1o(b'_1oh))}(c_1) \geq \mu_R(b'_1, h, z_1) \wedge \mu_R(a'_1, z_1, c_1) > \theta, \\ \nu_{(a'_1o(b'_1oh))}(c_1) \leq \nu_R(b'_1, h, z_1) \vee \nu_R(a'_1, z_1, c_1) < 1-\theta \end{cases}$$

Thus, $w = c_1$ and $\begin{cases} \mu_R(d_1, h, c_1) > \theta, \\ \nu_R(d_1, h, c_1) < 1-\theta \end{cases}$. It follows that $cH \sim dH$ and consequently $[cH] = [dH]$.

Hence, $\bar{R}$ is an IF binary operation on $\frac{G}{H}$. Since $\bar{R}$ is an IF binary operation on $\frac{G}{H}$, so we have $([aH]o[bH])^{IF} = (\mu_{([aH]o[bH])}, \nu_{([aH]o[bH])})$

$$\begin{cases} \mu_{([aH]o[bH])}([cH]) = \mu_{\bar{R}}([aH], [bH], [cH]), \\ \nu_{([aH]o[bH])}([cH]) = \nu_{\bar{R}}([aH], [bH], [cH]) \end{cases}$$

$(([aH]o[bH])o[cH])^{IF} = (\mu_{(([aH]o[bH])o[cH])}, \nu_{(([aH]o[bH])o[cH])}) \Rightarrow$

$$\begin{cases} \mu_{(([aH]o[bH])o[cH])}([dh]) = \bigvee \mu_{\bar{R}}([aH], [bH], [xH]) \wedge \mu_{\bar{R}}([xH], [cH], [dH]), \\ \nu_{(([aH]o[bH])o[cH])}([dh]) = \bigwedge \nu_{\bar{R}}([aH], [bH], [xH]) \vee \nu_{\bar{R}}([xH], [cH], [dH]) \end{cases}$$

$([aH]o([bH]o[cH])^{IF} = (\mu_{([aH]o([bH]o[cH])}, \nu_{([aH]o([bH]o[cH])}) \Rightarrow$

$$\begin{cases} \mu_{([aH]o([bH]o[cH])}([wh]) = \bigvee \mu_{\bar{R}}([aH], [cH], [xH]) \wedge \mu_{\bar{R}}([aH], [xH], [wH]), \\ \nu_{([aH]o([bH]o[cH])}([wh]) = \bigwedge \nu_{\bar{R}}([aH], [cH], [xH]) \vee \nu_{\bar{R}}([aH], [xH], [wH]). \end{cases}$$

□

Theorem 2.10. $(\frac{G}{H}, \alpha_{\bar{R}})$ *is an IF group.*

Proof. Let
$(([aH]o[bH])o[cH])^{IF} = (\mu_{(([aH]o[bH])o[cH])}, \nu_{(([aH]o[bH])o[cH])})$ and $([aH]o([bH]o[cH]))^{IF} = (\mu_{([aH]o([bH]o[cH]))}, \nu_{([aH]o([bH]o[cH]))}) \Rightarrow$

$$\begin{cases} \mu_{(([aH]o[bH])o[cH])}([dh]) > \theta, \ \mu_{([aH]o([bH]o[cH]))}([wH]) > \theta, \\ \nu_{(([aH]o[bH])o[cH])}([dh]) < 1-\theta, \ \nu_{([aH]o([bH]o[cH]))}([wH]) < 1-\theta \end{cases}$$

Then, we have $a_1, a'_1, b_1, b'_1, c_1, c'_1, w_1 \in G$ such that $c_1H \sim c'_1H \sim cH$, $a'_1H \sim a_1H$, $b'_1H \sim b_1H \sim bH$, $d_1H \sim dH$, $w_1H \sim wH$ and there exist elements $h_1, h_2, h_3 \in H$, $x'_1, x'_2 \in G$ such that

$$\begin{cases} \mu_R(a_1, b_1, x'_1) \wedge \mu_R(x'_1, c_1, d_1) > \theta, \\ \nu_R(a_1, b_1, x'_1) \vee \nu_R(x'_1, c_1, d_1) < 1-\theta \end{cases}$$

$$\begin{cases} \mu_R(b'_1, c'_1, x'_2) \wedge \mu_R(a'_1, x'_2, w_1) > \theta, \\ \nu_R(b'_1, c'_1, x'_2) \vee \nu_R(a'_1, x'_2, w_1) < 1-\theta \end{cases}$$

$$\begin{cases} \mu_R(a'_1, h_1, a_1) > \theta \ \mu_R(b'_1, h_2, b_1) > \theta \ \mu_R(c'_1, h_3, c_1) > \theta, \\ \nu_R(a'_1, h_1, a_1) < 1-\theta \ \nu_R(b'_1, h_2, b_1) < 1-\theta \ \nu_R(c'_1, h_3, c_1) < 1-\theta \end{cases}$$

Let $z_1 \in G$ such that $\begin{cases} \mu_R(a'_1, b'_1, z_1) > \theta, \\ \nu_R(a'_1, b'_1, z_1) < 1-\theta \end{cases}$, then by $\begin{cases} \mu_R(a_1, b_1, x'_1) > \theta, \\ \nu_R(a_1, b_1, x'_1) < 1-\theta \end{cases}$, $\begin{cases} \mu_R(a'_1, h_1, a_1) > \theta, \\ \nu_R(a'_1, h_1, a_1) < 1-\theta \end{cases}$, $\begin{cases} \mu_R(a'_1, b'_1, z_1) > \theta, \\ \nu_R(a'_1, b'_1, z_1) < 1-\theta \end{cases}$, $\begin{cases} \mu_R(b'_1, h_2, b_1) > \theta, \\ \nu_R((b'_1, h_2, b_1) < 1-\theta \end{cases}$, and the proof of Theorem **??**, there exists $h_4 \in H$ such that $\begin{cases} \mu_R(z_2, h_4, d_1) > \theta, \\ \nu_R(z_2, h_4, d_1) < 1-\theta \end{cases}$.

$(a'_1 o(b'_1 o c'_1))^{IF} = (\mu_{(a'_1 o(b'_1 o c'_1))},\ \nu_{(a'_1 o(b'_1 o c'_1))})$ then

$$\begin{cases} \mu_{(a'_1 o(b'_1 o c'_1))}(w_1) > \mu_R(b'_1, c'_1, x'_2) \wedge \mu_R(a'_1, x'_2, w_1) > \theta, \\ \nu_{(a'_1 o(b'_1 o c'_1))}(w_1) < \nu_R(b'_1, c'_1, x'_2) \vee \nu_R(a'_1, x'_2, w_1) < 1-\theta \end{cases}$$

$((a'_1 o b'_1) o c'_1)^{IF} = (\mu_{((a'_1 o b'_1) o c'_1)}, \nu_{((a'_1 o b'_1) o c'_1)})$ then

$$\begin{cases} \mu_{((a'_1 o b'_1) o c'_1)}(w_1) > \mu_R(a_1, b'_1, z_1) \wedge \mu_R(z_1, c'_1, z_2) > \theta, \\ \nu_{((a'_1 o b'_1) o c'_1)}(w_1) < \nu_R(a_1, b'_1, z_1) \vee \nu_R(z_1, c'_1, z_2) < 1-\theta \end{cases}$$

so, $z_2 = w_1$ and $\begin{cases} \mu_R(w_1, h_4, d_1) > \theta, \\ \nu_R(w_1, h_4, d_1) < 1-\theta \end{cases}$. Then, $w_1 H \sim dH$ and consequently $[wH] = [dH]$.

(2) $([aH]o[eH])^{IF} = (\mu_{([aH]o[eH])}, \nu_{([aH]o[eH])})$ and $([eH]o[aH])^{IF} = (\mu_{([eH]o[aH])}, \nu_{([eH]o[aH])}) \Rightarrow$

$$\forall a \in G, \mu_{([aH]o[eH])}([aH]) \geq \mu_R(a, e, a) > \theta,\ \mu_{([eH]o[aH])}([aH]) \geq \mu_R(e, a, a) > \theta$$
$$\forall a \in G, \nu_{([aH]o[eH])}([aH]) \leq \nu_R(a, e, a) < 1-\theta,\ \nu_{([eH]o[aH])}([aH]) \leq \nu_R(e, a, a) < 1-\theta$$

(3) $([aH]o[a^{-1}H])^{IF} = (\mu_{([aH]o[a^{-1}H])}, \nu_{([aH]o[a^{-1}H])})$ and $([a^{-1}H]o[aH])^{IF} = (\mu_{([a^{-1}H]o[aH])}, nu_{([a^{-1}H]o[aH])}) \Rightarrow$

$$\mu_{([aH]o[a^{-1}H])}([eH]) \geq \mu_R(a, a^{-1}, e) > \theta,\ \mu_{([a^{-1}H]o[aH])}([eH]) \geq \mu_R(a^{-1}, a, e) > \theta$$
$$\nu_{([aH]o[a^{-1}H])}([eH]) \leq \nu_R(a, a^{-1}, e) < 1-\theta,\ \nu_{([a^{-1}H]o[aH])}([eH]) \leq \nu_R(a^{-1}, a, e) < 1-\theta$$

Hence, $(\frac{G}{H}, \bar{R})$ is IF group. □

Definition 2.11. Let (G_1, R_1) and (G_2, R_2) be two IF group and let $f : G_1 \longrightarrow G_2$ be a mapping. If

$$\mu_{R_1}(a, b, c) > \theta \Longrightarrow \mu_{R_2}(f(a), f(b), f(c)) > \theta$$

$$\nu_{R_1}(a, b, c) < 1-\theta \Longrightarrow \nu_{R_2}(f(a), f(b), f(c)) < 1-\theta$$

then f is called an *IF (group) homomorphism.* IF f is 1-1, it is called an IF epimorphism. If f is both 1-1 and onto, it is called an IF isomorphism. Let $G = (\mu_G, \nu_G)$ be an IF binary operation on R. Then we have a mapping

$$\alpha_G : IF(R) \times IF(R) \longrightarrow IF(R)$$
$$(A, B) \longmapsto \alpha_G(A, B)$$

where $IF(R) = \left\{ A = (\mu_A, \nu_A) | \begin{array}{l} \mu_A : R \to [0,1] \\ \nu_A : R \to [0,1] \end{array} \text{ is a mapping} \right\}$ and

$\mu_G(A, B)(c) = \bigvee_{a,b \in R}(\mu_A(a) \wedge \mu_B(b) \wedge \mu_G(a, b, c))$

$\nu_G(A, B)(c) = \bigwedge_{a,b \in R}(\nu_A(a) \vee \nu_B(b) \vee \nu_G(a, b, c))$

Let $A=\chi^{IF}_{\{A\}}=(\chi_{\{A\}},\chi^{C}_{\{A\}})$ and $B=\chi^{IF}_{\{B\}}=(\chi_{\{B\}},\chi^{c}_{\{B\}})$ and Let $G(A,B)$ and $H(A,B)$ be denoted as aob and $a*b$, respectively. Then for aob and $a*b=(\mu_{a*b},\nu_{a*b})$
$\mu_{(aob)}(c)=\mu_G(a,b,c)\ ,\ \forall c\in R,$
$\nu_{(aob)}(c)=\nu_G(a,b,c)\ ,\ \forall c\in R.$
$\mu_{(a*b)}(c)=\mu_H(a,b,c)\ ,\ \forall c\in R,$
$\nu_{(a*b)}(c)=\nu_H(a,b,c)\ ,\ \forall c\in R.$

$$\begin{cases}\mu_{((aob)oc)}(z)=\bigvee_{d\in G}(\mu_G(a,b,d)\wedge\mu_G(d,c,z))\\ \nu_{((aob)oc)}(z)=\bigwedge_{d\in G}(\nu_G(a,b,d)\vee\nu_G(d,c,z))\end{cases}$$

$$\begin{cases}\mu_{(ao(boc))}(z)=\bigvee_{d\in G}(\mu_G(b,c,d)\wedge\mu_G(a,d,z))\\ \nu_{(ao(boc))}(z)=\bigwedge_{d\in G}(\nu_G(b,c,d)\vee\nu_G(a,d,z))\end{cases}$$

$$a*(boa)=(\mu_{a*(boa)},\nu_{a*(boa)})\Rightarrow\begin{cases}\mu_{(a*(boc))}(z)=\bigvee_{d\in G}(\mu_G(b,c,d)\wedge\mu_H(a,d,z))\\ \nu_{(a*(boc))}(z)=\bigwedge_{d\in G}(\nu_G(b,c,d)\vee\nu_H(a,d,z))\end{cases}$$

for $((a*b)o(a*c))=(\mu_{((a*b)o(a*c))},\nu_{((a*b)o(a*c))})\Rightarrow$

$$\begin{cases}\mu_{((a*b)o(a*c))}(z)=\bigvee_{d\in G}(\mu_H(a,b,d)\wedge\mu_H(a,c,e)\wedge\mu_G(d,e,z))\\ \nu_{((a*b)o(a*c))}(z)=\bigwedge_{d\in G}(\nu_H(a,b,d)\vee\nu_H(a,c,e)\vee\nu_G(d,e,z))\end{cases}$$

Definition 2.12. Let R be a nonempty set and let G and H be two IF binary operations on R. Then (R,G,H) is called *IF ring* if the following conditions hold for $((a*b)*c)^{IF}=(\mu_{((a*b)*c)},\ \nu_{((a*b)*c)})$, $(a*(b*c))^{IF}=(\mu_{(a*(b*c)),\nu_{(a*(b*c))}})$, $((aob)*c)^{IF}=(\mu_{((aob)*c)},\nu_{((aob)*c)})$, $((a*c)o(b*c))^{IF}=(\mu_{((a*c)o(b*c))},\nu_{((a*c)o(b*c))})$ and $((a*b)oc)^{IF}=(\mu_{((a*b)oc)},\nu_{((a*b)oc)})$
(1) (R,G) is an abelian IF group;
(2) $\begin{cases}\forall a,b,c,z_1,z_2\in R,\ \mu_{((a*b)*c)}(z_1)>\theta \text{ and } \mu_{(a*(b*c))}(z_2)>\theta\\ \forall a,b,c,z_1,z_2\in R,\ \nu_{((a*b)*c)}(z_1)<1-\theta \text{ and } \nu_{(a*(b*c))}(z_2)<1-\theta\end{cases}$ imply $z_1=z_2$
(3) $\begin{cases}\forall a,b,c,z_1,z_2\in R,\ \mu_{((aob)*c)}(z_1)>\theta \text{ and } \mu_{((a*c)o(b*c))}(z_2)>\theta\\ \forall a,b,c,z_1,z_2\in R,\ \nu_{((aob)*c)}(z_1)<1-\theta \text{ and } \nu_{((a*c)o(b*c))}(z_2)<1-\theta\end{cases}$ imply $z_1=z_2$
$\begin{cases}\forall a,b,c,z_1,z_2\in R,\ \mu_{((a*b)oc)}(z_1)>\theta \text{ and } \mu_{((a*b)o(a*c))}(z_2)>\theta\\ \forall a,b,c,z_1,z_2\in R,\ \nu_{((a*b)oc)}(z_1)<1-\theta \text{ and } \nu_{((a*b)o(a*c))}(z_2)<1-\theta\end{cases}$ imply $z_1=z_2$

Definition 2.13. Let (R,G,H) be a IF ring.
(1) If $\begin{cases}\mu_{(a*b)}(u)>\theta\Longleftrightarrow\mu_{(b*a)}(u)>\theta\\ \nu_{(a*b)}(u)<1-\theta\Longleftrightarrow\nu_{(b*a)}(u)<1-\theta\end{cases}$ then (R,G,H) is said to be a commutative IF ring.
(2) If $\exists e_*\in R$ such that for $(a*e_*)^{IF}=(\mu_{(a*e_*)},\nu_{(a*e_*)})$, $(e_**a)^{IF}=(\mu_{(e_**a)},\ \nu_{(e_**a)})$ $\begin{cases}\mu_{(a*e_*)}(a)>\theta\\ \nu_{(a*e_*)}(a)<1-\theta\end{cases}$ and $\begin{cases}\mu_{(e_**a)}(a)>\theta\\ \nu_{(e_**a)}(a)<1-\theta\end{cases}$ for every $a\in R$, then (R,G,H) is said to be IF ring with identity.

(3) Let (R, G, H) be an IF ring with identity. If for a member $a \in R$ there exists $b \in R$ such that $\begin{cases} \mu_{(a*b)}(e_*) > \theta \\ \nu_{(a*b)}(e_*) < 1-\theta \end{cases}$ and $\begin{cases} \mu_{(b*a)}(e_*) > \theta \\ \nu_{(b*a)}(e_*) < 1-\theta \end{cases}$, then b is said to be an inverse element of a and is denoted by a^{-1}.

Proposition 2.14. *Let (R, G, H) be an IF ring and let S be an nonempty subset of R. Then (S, G, H) is an IF subring of R if and only if*

(1) $\begin{cases} \mu_{(aob)}(c) > \theta \\ \nu_{(aob)}(c) < 1-\theta \end{cases}$ implies $c \in S$ and $\begin{cases} \mu_{(a*b)}(c) > \theta \\ \nu_{(a*b)}(c) < 1-\theta \end{cases}$ implies $c \in S$ for all $a, b \in S,\ c \in R$;

(2) $a \in S$ implies $a^{-1} \in S$.

Definition 2.15. A nonempty subset $I = (\mu_I, \nu_I)$ of a IF ring (R, G, H) is called a *IF ideal* of R if the following conditions are satisfied.

(1) $\begin{cases} \forall x, y \in I,\ \mu_{(xoy)}(z) > \theta \\ \forall x, y \in I,\ \nu_{(xoy)}(z) < 1-\theta \end{cases} \Longrightarrow z \in I$ for all $z \in R$

(2) $\{\ \forall x \in I,\ x^{-1} \in I;$

(3) $\begin{cases} \forall s \in I, \text{ for all } r \in R,\ \mu_{(r*s)}(x) > \theta \Longrightarrow x \in I \text{ and } \mu_{(s*r)}(y) > \theta \\ \forall s \in I, \text{ for all } r \in R,\ \nu_{(r*s)}(x) < 1-\theta \Longrightarrow x \in I \text{ and } \nu_{(s*r)}(y) < 1-\theta \end{cases} \Longrightarrow$

$$\text{y} \in I,\ x,\ y \in R.$$

3. IF Modules over IF Rings

Let $(R,\ G,\ H)$ be a IF ring and $(M,\ J)$ be an abelian IF group and let α_P be IF function $R \times M$ into M. Then we have a mapping

$$P:\ IF(R) \times IF(M) \longrightarrow IF(M)$$

$$(A, N) \longmapsto P(A, N)$$

$$P = (\mu_P,\ \nu_P) \Rightarrow \begin{cases} \mu_P(A, N)(x) = \bigvee\limits_{(r,n) \in A \times N} (A(r) \wedge N(n) \wedge p(r, n, x)), \\ \nu_P(A, N)(x) = \bigwedge\limits_{(r,n) \in A \times N} (A(r) \vee N(n) \vee p(r, n, x)), \end{cases}$$

where $IF(R) = \left\{ A = (\mu_A, \nu_A) \mid \begin{array}{l} \mu_A : R \longrightarrow [0,1] \\ \nu_A : R \longrightarrow [0,1] \end{array} \right\}$ and

$IF(M) = \left\{ N = (\mu_N, \nu_N) \mid \begin{array}{l} \mu_N : M \longrightarrow [0,1] \\ \nu_N : M \longrightarrow [0,1] \end{array} \right\}$.

Let $A = \{r\}$ and $N = \{M\}$, and let $P(A, N)$ and $J(a, b)$ be denoted as $r \odot m$ and $a \oplus b$, respectively. Then

$$(r \odot m)(x) = P(r, m, x),\ \forall x \in M,$$

$$(r \odot (m_1 \oplus m_2))^{IF} = (\mu_{(r \odot (m_1 \oplus m_2))}, \nu_{(r \odot (m_1 \oplus m_2))}) \Longrightarrow$$

$$\begin{cases} \mu_{(r\odot(m_1\oplus m_2))}(x) = \bigvee_{m\in M}(\mu_J(m_1,m_2,m)\wedge\mu_p(r,m,x)) \\ \nu_{(r\odot(m_1\oplus m_2))}(x) = \bigwedge_{m\in M}(\nu_J(m_1,m_2,m)\vee\nu_p(r,m,x)) \end{cases}$$

$((r_1 or_2)\odot m)^{IF} = (\mu_{((r_1 or_2)\odot m)}, \nu_{((r_1 or_2)\odot m)})$,
$((r_1 * r_2)\odot m)^{IF} = (\mu_{((r_1*r_2)\odot m)},\ \nu_{((r_1*r_2)\odot m)})$ and
$((r_1\odot(r_2\odot m))^{IF} = (\mu_{((r_1\odot(r_2\odot m))},\ \nu_{((r_1\odot(r_2\odot m))}) \Longrightarrow$

$$\begin{cases} \mu_{((r_1 or_2)\odot m)}(x) = \bigvee_{r\in R}(\mu_G(r_1,r_2,r)\wedge\mu_P(r,m,x)) \\ \nu_{((r_1 or_2)\odot m)}(x) = \bigwedge_{r\in R}(\nu_G(r_1,r_2,r)\vee\nu_P(r,m,x)) \end{cases},$$

$$\begin{cases} \mu_{((r_1*r_2)\odot m)}(x) = \bigvee_{r\in R}(\mu_H(r_1,r_2,r)\wedge\mu_P(r,m,x)) \\ \nu_{((r_1*r_2)\odot m)}(x) = \bigwedge_{r\in R}(\nu_H(r_1,r_2,r)\vee\nu_P(r,m,x)) \end{cases},$$

$$\begin{cases} \mu_{((r_1\odot(r_2\odot m))}(x) = \bigvee_{m_1\in M}(\mu_P(r_2,m,m_1)\wedge\mu_P(r_1,m_1,x)) \\ \nu_{((r_1\odot(r_2\odot m))}(x) = \bigwedge_{m_1\in M}(\nu_P(r_2,m,m_1)\vee\nu_P(r_1,m_1,x)) \end{cases},$$

$(r\odot(m_1\oplus m_2))^{IF} = (\mu_{(r\odot(m_1\oplus m_2))}, \nu_{(r\odot(m_1\oplus m_2))}) \Longrightarrow$

$$\begin{cases} \mu_{((r\odot m_1)\oplus(r\odot m_2))}(x) = \bigvee_{x_1,x_2\in M}(\mu_P(r,m_1,x_1)\wedge\mu_P(r,m_2,x_2)\wedge\mu_J(x_1,x_2,x)) \\ \nu_{((r\odot m_1)\oplus(r\odot m_2))}(x) = \bigwedge_{x_1,x_2\in M}(\nu_P(r,m_1,x_1)\vee\nu_p(r,m_2,x_2)\vee\nu_J(x_1,x_2,x)) \end{cases}$$

Definition 3.1. Let (R,G,H) be an IF ring and Let (M,J) be an abelian IF group. M is called an (left) IF module over R or (left) $R-$IFmodule together with an IF function $P : R\times M \longrightarrow M$, if the following conditions hold, for all $r, r_1, r_2 \in R$ and for all $m, m_1, m_2 \in M$, denote $(r\odot(m_1\oplus m_2))^{IF} = (\mu_{(r\odot(m_1\oplus m_2))}, \nu_{(r\odot(m_1\oplus m_2))}) \Longrightarrow$

(1) $\begin{cases} \mu_{(r\odot(m_1\oplus m_2))}(x) > \theta \text{ and } \mu_{((r\odot m_1)\oplus(r\odot m_2))}(y) > \theta \text{ imply } x = y \\ \nu_{(r\odot(m_1\oplus m_2))}(x) < 1-\theta \text{ and } \nu_{((r\odot m_1)\oplus(r\odot m_2))}(y) < 1-\theta \text{ imply } x = y \end{cases}$

denote $((r_1 or_2)\odot m)^{IF} = (\mu_{((r_1 or_2)\odot m)}, \nu_{((r_1 or_2)\odot m)}) \Longrightarrow$

(2) $\begin{cases} \mu_{((r_1 or_2)\odot m)}(x) > \theta \text{ and } \mu_{((r_1\odot m)\oplus(r_2\odot m))}(y) > \theta \text{ imply } x = y \\ \nu_{((r_1 or_2)\odot m)}(x) < 1-\theta \text{ and } \nu_{((r_1\odot m)\oplus(r_2\odot m))}(y) < 1-\theta \text{ imply } x = y \end{cases}$

denote $((r_1 * r_2)\odot m)^{IF} = (\mu_{((r_1*r_2)\odot m)}, \nu_{((r_1*r_2)\odot m)})$

(3) $\begin{cases} \mu_{((r_1*r_2)\odot m)}(x) > \theta \textit{ and } \mu_{(r_1\odot(r_2\odot m))}(y) > \theta \text{ imply } x = y \\ \nu_{((r_1*r_2)\odot m)}(x) < 1-\theta \textit{ and } \nu_{(r_1\odot(r_2\odot m))}(y) < 1-\theta \text{ imply } x = y \end{cases}$.

Proposition 3.2. *Let (R,G,H) be an IF ring and let (M,J) be an $R-$IFmodule; then for all $r, r_1, r_2 \in R, m, m_1, m_2 \in M$,*

(1) $\begin{cases} \mu_{(r\odot(m_1\oplus m_2))}(x) > \theta \iff \mu_{((r\odot m_1)\oplus(r\odot m_2))}(x) < 1-\theta \\ \nu_{(r\odot(m_1\oplus m_2))}(x) < 1-\theta \iff \nu_{((r\odot m_1)\oplus(r\odot m_2))}(x) < 1-\theta \end{cases}$.

(2) $\begin{cases} \mu_{((r_1 or_2)\odot m)}(x) > \theta \iff \mu_{((r_1\odot m)\oplus(r_2\odot m))}(x) > \theta \\ \nu_{((r_1 or_2)\odot m)}(x) < 1-\theta \iff \nu_{((r_1\odot m)\oplus(r_2\odot m))}(x) < 1-\theta \end{cases}$.

(3) $\begin{cases} \mu_{((r_1 * r_2)\odot m)}(x) > \theta \iff \mu_{(r_1\odot(r_2\odot m))}(y) > \theta \\ \nu_{((r_1 * r_2)\odot m)}(x) < 1-\theta \iff \nu_{(r_1\odot(r_2\odot m))}(y) < 1-\theta \end{cases}$.

Proof. It is clear by definitions. □

Remark 3.3. Let (G, R) be an IF group, then $(aob)(d) > 0$ and $(aoc)(d) > 0$ imply $b = c$.

Proof. Let b be an inverse element of a, then with $((boa)oa)^{IF} = (\mu_{((boa)oa)}, \nu_{((boa)oa)})$ and $(bo(aoa))^{IF} = (\mu_{(bo(aoa))},\ \nu_{(bo(aoa))})$, we have

$$\mu_{((boa)oa)}(a) \geq \mu_R(b, a, e) \wedge \mu_R(e, a, a) > \theta,$$
$$\nu_{((boa)oa)}(a) \leq \nu_R(b, a, e) \vee \nu_R(e, a, a) < 1-\theta,$$
$$\mu_{(bo(aoa))}(e) \geq \mu_R(a, a, a) \wedge \mu_R(b, a, e) > \theta,$$
$$\nu_{(bo(aoa))}(e) \leq \nu_R(a, a, a) \vee \nu_R(b, a, e) < 1-\theta.$$

It follows that $a = e$. □

Proposition 3.4. *Let (R, G, H) be an IF ring with zero element e_0 and (M, J) be a left $R-$fmodule with identity element e_j. Then for all $r \in R$, $m \in M$*

(1) $(r \odot e_j)^{IF} = (\mu_{(r\odot e_j)},\ \nu_{(r\odot e_j)}) \Rightarrow \begin{cases} \mu_{(r\odot e_j)}(e_j) > \theta \\ \nu_{(r\odot e_j)}(e_j) < 1-\theta \end{cases}$

(2) $(e_0 \odot m)^{IF} = (\mu_{(e_0\odot m)},\ \nu_{(e_0\odot m)}) \begin{cases} \mu_{(e_0\odot m)}(e_j) > \theta \\ \nu_{(e_0\odot m)}(e_j) < 1-\theta \end{cases}$

(3) $(r\odot m)^{IF} = (\mu_{(r\odot m)},\ \nu_{(r\odot m)}) \Rightarrow \begin{cases} \mu_{(r\odot m)}(x) > \theta \Longrightarrow \mu_{(r\odot m^{-1})}(x^{-1}) > \theta \\ \nu_{(r\odot m)}(x) < 1-\theta \Longrightarrow \nu_{(r\odot m^{-1})}(x^{-1}) < 1-\theta \end{cases}$

(4) $(r\odot m)^{IF} = (\mu_{(r\odot m)},\ \nu_{(r\odot m)}) \Rightarrow \begin{cases} \mu_{(r\odot m)}(x) > \theta \Longrightarrow \mu_{(r^{-1}\odot m)}(x^{-1}) > \theta \\ \nu_{(r\odot m)}(x) < 1-\theta \Longrightarrow \nu_{(r^{-1}\odot m)}(x^{-1}) < 1-\theta \end{cases}$

Proof. Let $x \in M$ such that $\begin{cases} \mu_{(r\odot e_j)}(e_j) > \theta \\ \nu_{(r\odot e_j)}(e_j) < 1-\theta \end{cases}$. then by $(r \odot (e_J \oplus e_J))^{IF} = (\mu_{(r\odot(e_J\oplus e_J))},\ \nu_{(r\odot(e_J\oplus e_J))})$

$$\begin{cases} \mu_{(r\odot(e_J\oplus e_J))}(x) > \mu_J(e_J, e_J, e_J) \wedge \mu_p(r, e_j, x) > \theta \\ \nu_{(r\odot(e_J\oplus e_J))}(x) < \nu_J(e_J, e_J, e_J) \vee \nu_p(r, e_j, x) < 1-\theta \end{cases}$$

it follows that $((r\odot e_j)\oplus(r\odot e_j))^{IF} = (\mu_{((r\odot e_j)\oplus(r\odot e_j))},\ \nu_{((r\odot e_j)\oplus(r\odot e_j))}) \Rightarrow$
$\begin{cases} \mu_{((r\odot e_j)\oplus(r\odot e_j))}(x) > \theta \\ \nu_{((r\odot e_j)\oplus(r\odot e_j))}(x) < 1-\theta \end{cases}$ from Proposition **??**. then

$$\begin{cases} \mu_{((r\odot e_j)\oplus(r\odot e_j))}(x) > \mu_P(r, e_J, x) \wedge \mu_P(r, e_J, x) \wedge \mu_J(x, x, x) > \theta \\ \nu_{((r\odot e_j)\oplus(r\odot e_j))}(x) < \nu_P(r, e_J, x) \vee \nu_P(r, e_J, x) \vee \nu_J(x, x, x) < 1-\theta \end{cases}$$

Thus $\begin{cases} \mu_J(x, x, x) > \theta \\ \nu_J(x, x, x) < 1-\theta \end{cases}$ and $x = e_j$ from Remark **??**.

(2) Let $x \in M$ such that $\begin{cases} \mu_{(e_0 \odot m)}(e_j) > \theta \\ \nu_{(e_0 \odot m)}(e_j) < 1-\theta \end{cases}$. then by $((e_0 o e_0) \odot m)^{IF} = (\mu_{((e_0 o e_0) \odot m)},\ \nu_{((e_0 o e_0) \odot m)})$

$$\begin{cases} \mu_{((e_0 o e_0) \odot m)}(x) > \mu_G(e_0, e_0, e_0) \wedge \mu_p(e_0, m, x) > \theta \\ \nu_{((e_0 o e_0) \odot m)}(x) < \nu_G(e_0, e_0, e_0) \vee \nu_p(e_0, m, x) < 1-\theta \end{cases}$$

It follows that

$((e_0 \odot m) \oplus (e_0 \odot m))^{IF} = (\mu_{((e_0 \odot m) \oplus (e_0 \odot m))},\ \nu_{((e_0 \odot m) \oplus (e_0 \odot m))}) \Rightarrow \begin{cases} \mu_{((e_0 \odot m) \oplus (e_0 \odot m))}(x) > \theta \\ \nu_{((e_0 \odot m) \oplus (e_0 \odot m))}(x) < 1-\theta \end{cases}$

from Proposition **??**. Then

$$\begin{cases} \mu_{((e_0 \odot m) \oplus (e_0 \odot m))}(x) > \mu_p(e_0, m, x) \wedge \mu_P(e_0, m, x) \wedge \mu_J(x, x, x) > \theta \\ \nu_{((e_0 \odot m) \oplus (e_0 \odot m))}(x) < \nu_p(e_0, m, x) \vee \nu_P(e_0, m, x) \vee \nu_J(x, x, x) < 1-\theta \end{cases}$$

Thus similar to (1), $\begin{cases} \mu_J(x, x, x) > \theta \\ \nu_J(x, x, x) < 1-\theta \end{cases}$ and so $x = e_J$.

(3) Let $\begin{cases} \mu_P(r, m, x) > \theta \\ \nu_P(r, m, x) < 1-\theta \end{cases}$ and let $y \in M$ such that $\begin{cases} \mu_P(r, m^{-1}, y) > \theta \\ \nu_P(r, m^{-1}, y) < 1-\theta \end{cases}$

$(r \odot (m \oplus m^{-1}))^{IF} = (\mu_{(r \odot (m \oplus m^{-1}))},\ \nu_{(r \odot (m \oplus m^{-1}))}) \Rightarrow$

$$\begin{cases} \mu_{(r \odot (m \oplus m^{-1}))}(e_J) > \mu_J(m, m^{-1}, e_J) \wedge \mu_P(r, e_J, e_J) > \theta \\ \nu_{(r \odot (m \oplus m^{-1}))}(e_J) < \nu_J(m, m^{-1}, e_J) \vee \nu_P(r, e_J, e_J) < 1-\theta \end{cases}$$

by Proposition **??** we have $((r \odot m) \oplus (r \odot m^{-1}))^{IF} = (\mu_{((r \odot m) \oplus (r \odot m^{-1}))},\ \nu_{((r \odot m) \oplus (r \odot m^{-1}))})$ that $\begin{cases} \mu_{((r \odot m) \oplus (r \odot m^{-1}))}(e_J) > \theta \\ \nu_{((r \odot m) \oplus (r \odot m^{-1}))}(e_J) < 1-\theta \end{cases}$.

Therefore $\begin{cases} \mu_J(x, y, e_j) > \theta \\ \nu_J(x, y, e_j) < 1-\theta \end{cases}$ and consequently $y = x^{-1}$.

(4) It is obtain similar to (3).

□

Proposition 3.5. *If (R, G, H) is a IF ring and K is any IF subring of R, then R is a $K-$IFmodule.*

Proof. Let (R, G, H) is a IF ring and let K is any IF subring of R Consider the mapping $\left\{ P = (\mu_P, \nu_P) \mid \begin{array}{l} \mu_P : \mu_K \times \mu_R \longrightarrow \mu_R \\ \nu_P : \nu_K \times \nu_R \longrightarrow \nu_R \end{array} \right.$, defined by $P(k, r) = H(k, r)$. It is obviously a IF function which satisfies the conditions in Definition **??**. Moreover observe that (R, G) is necessarily an abelian IF group. consequently R is a left $K-$ifmodule. □

4. IF Submodule and IF Module Homomorphisms

Definition 4.1. Let (R, G, H) be an IF ring, (M, J) an $R-$IFmodule, and N a nonempty subset of M. If (N, J) is a $R-$IFmodule, N is called an *IF submodule* of M.

By definitions we have the following Proposition trivially.

Proposition 4.2. *Let (R, G, H) be an IF ring, (M, J) a $R-$IFmodule, and N a nonempty subset of M. Then N is an IF submodule of M if and only if*

(1) *(N, J) is an IF subgroup of (M, J);*

(2) *for all $r \in R$, $b \in N$, $(r \odot b)^{IF} = \begin{cases} \mu_{(r\odot b)}(c) > \theta \\ \nu_{(r\odot b)}(c) < 1-\theta \end{cases}$ implies $c \in N$.*

Proposition 4.3. *If $\{N_i \mid i \in I\}$ is a family of IF submodules of an IF module M, then $\bigcap\limits_{i\in I} N_i$ is an IF submodule of M.*

Proof. It is clear. □

Definition 4.4. Let A and B be two IF modules over a IF ring (R, G, H) with a function $P : R \times M \longrightarrow M$. A function $f : A \longrightarrow B$ is an $R-$IFmodule homomorphism which provided that, for all $a, b \in A$ and $r \in R$,

(1) $\begin{cases} \mu_G(a, b, x) > \theta \\ \nu_G(a, b, x) < 1-\theta \end{cases}$ implies $\begin{cases} \mu_G(f(a), f(b), f(x)) > \theta \\ \nu_G(f(a), f(b), f(x)) < 1-\theta \end{cases}$;

(2) $\begin{cases} \mu_P(r, a, x) > \theta \\ \nu_P(r, a, x) < 1-\theta \end{cases}$ implies $\begin{cases} \mu_P(r, f(a), f(x)) > \theta \\ \nu_P(r, f(a), f(x)) < 1-\theta \end{cases}$.

Clearly, a $R-$IFmodule homomorphism $f : A \longrightarrow B$ is necessarily an abelian IF group homomorphism. Consequently the same terminology is used for IF modules: F is a $R-$IFmodule monomorphism (resp., epimorphism, isomorphism) if it is injective (resp., surjective, bijective) as IF group homomorphisms.

Let $f : A \longrightarrow B$ be an $R-$IFmodule homomorphism.Then the kernel and the image of f as IF group homomorphisms are denoted by

$$Kerf = \{a \in A \mid f(a) = e_B\},$$
$$Imf = \{b \in B \mid b = f(a), a \in A\},$$

respectively.

Theorem 4.5. *Let (R, G, H) be an IF ring and let $f : A \longrightarrow B$ be a $R-$IFmodule homomorphism.Then*

(1) *f is an $R-$IFmodule monomorphism if and only if $Kerf = \{e_A\}$*

(2) *$f : A \longrightarrow B$ is an $R-$IFmodule isomorphism if and only if there exists an IF module homomorphism $G : B \longrightarrow A$ such that $gf = e_A$ and $fg = e_B$.*

Theorem 4.6. *Let $f : (G_1, R_1) \longrightarrow (G_2, R_2)$ be an IF group homomorphism, then if H_2 is a fuzzy subgroup of G_2, then $f^{-1}(H_2)$ is a fuzzy subgroup of G_1.*

Proposition 4.7. *Let $f : A \longrightarrow B$ be a $R-$IFmodule homomorphism. Then*

(1) *$Kerf$ is an IF submodule of A;*

(2) *Imf is an IF submodule of B;*
(3) *if C is any IF submodule of B, then $f^{-1}(C) = \{a \in A | f(a) \in C\}$ is an IF submodule of A.*

Proof.
(1) $Kerf$ is a fuzzy subgroup of the abelian fuzzy group A from Theorem 26 in [?] . Let $r \in R$ and $a \in Kerf$ such that $\{ \begin{array}{c} \mu_P(r,a,x) > \theta, \\ \nu_P(r,a,x) < 1-\theta \end{array}$

Since f is a $R-$IFmodule homomorphism, $\{ \begin{array}{c} \mu_P(r,f(a),f(x)) > \theta, \\ \nu_P(r,f(a),f(x)) < 1-\theta \end{array}$.

On the other hand, as $a \in Kerf$ we have $f(a) = e_B$. Therefore $\{ \begin{array}{c} \mu_P(r,e_B,f(x)) > \theta, \\ \nu_P(r,e_B,f(x)) < 1-\theta \end{array}$ and so $f(x) = e_B$ from Proposition **??**. So $x \in Kerf$ is obtained.

(2) Imf is a IF subgroup of the abelian IF group A from Theorem 26 in [?]. For any $r \in R$, $b \in Imf$ there exists $a \in A$ such that $b = f(a)$. Let $x \in A$, $H = (\mu_H, \nu_H)$ such that $\left\{ \begin{array}{c} \mu_H(r,\ a,\ x) > \theta \\ \nu_H(r,\ a,\ x) < 1-\theta \end{array} \right.$. Since f is an $R-$ifmodule homomorphism, $\left\{ \begin{array}{c} \mu_H(r,\ f(A),\ f(x)) > \theta \\ \nu_H(r,\ f(A),\ f(x)) < 1-\theta \end{array} \right.$ which means $\left\{ \begin{array}{c} \mu_H(r,\ b,\ f(x)) > \theta \\ \nu_H(r,\ b,\ f(x)) < 1-\theta \end{array} \right.$ $andso$f(x)$\in B$.

(3) $f^{-1}(C)$ is an IF subgroup of the abelian IF group A from Theorem **??**. Let $r \in R$ and $x \in f^{-1}(C)$ such that $\left\{ \begin{array}{c} \mu_H(r,\ x,\ u) > \theta \\ \nu_H(r,\ x,\ u) < 1-\theta \end{array} \right.$. Since $\left\{ \begin{array}{c} \mu_H(r,\ f(x),\ f(u)) > \theta \\ \nu_H(r,\ f(x),\ f(u)) < 1-\theta \end{array} \right.$ and $f(x) \in C$ we have that $f(u) \in C$ and $u \in f^{-1}(C)$. This completes the proof. □

.

References

[1] H. Aktas and N. A. Cagman, A type of fuzzy ring, *rchive for Mathematical Logic.* **46(3-4)***(2007), 165-177.*

[2] K. T. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems.* **20***(1986), 87-96.*

[3] R. Biswas, Intuitionistic fuzzy subgroups, *Math. Forum.* **10***(1989), 37-46.*

[4] D. Coker, An introduction to intuitionistic fuzzy topological spaces, *Fuzzy Sets and Systems.* **88***(1997), 81-89.*

[5] V. N. Dixit, R. Kumar and N. Ajmal, On fuzzy rings, *Fuzzy Sets and Systems.* **49(2)***(1992), 205-213.*

[6] W.-J. Liu, Fuzzy invariant subgroups and fuzzy ideals, *Fuzzy Sets and Systems.* **8(2)***(1982), 133-139.*

[7] D. S. Malik and J. N. Mordeson, Fuzzy homomorphisms of rings, *Fuzzy Sets and Systems.* **46(1)***(1992), 139-146.*

[8] T. K. Mukherjee and M. K. Sen, On fuzzy ideals of a ring, *Fuzzy Sets and Systems.* **21(1)***(1987), 99-104.*
[9] C. V. Negoita and D. A. Ralescue, Applications of Fuzzy Sets and Systems Analysis, *Birkhauser, Basel. (1987).*
[10] A. Rosenfeld, Fuzzy groups, *Journal of Mathematical Analysis and applications.* **35***(1971), 512-517.*
[11] X. Yuan and E. S. Lee, Fuzzy group based on fuzzy binary operation, *Computers and Mathematics with Applications.* **47(4-5)***(2004), 631-641.*
[12] L. A. Zadeh, Fuzzy Sets, *Information and Computation.* **8***(1965), 338-353.*