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Reverses of Féjer's Inequalities for Convex Functions

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Abstract. Let f be a convex function on I and $a,\ b\in I$ with $a<b$. If $p:[a,b]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t)=p(t)$ for all $t\in[a,b]$, then we show in this paper that

$$\begin{aligned}0&\le\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right|p(t)\,dt\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\\&\le\int_a^b p(t)f(t)\,dt-\left(\int_a^b p(t)\,dt\right)f\left(\frac{a+b}{2}\right)\\&\le\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right|p(t)\,dt\left[f'_-(b)-f'_+(a)\right]\end{aligned}$$

and

$$\begin{aligned}0&\le\frac{1}{2}\int_a^b\left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right]p(t)\,dt\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\\&\le\left(\int_a^b p(t)\,dt\right)\frac{f(a)+f(b)}{2}-\int_a^b p(t)f(t)\,dt\\&\le\frac{1}{2}\int_a^b\left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right]p(t)\,dt\left[f'_-(b)-f'_+(a)\right].\end{aligned}$$

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Keywords: Convex functions, Integral inequalities, Hermite-Hadamard inequality, Féjer's inequalities.

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1. Introduction

The following inequality holds for any convex function f defined on $\mathbb{R}$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}, \quad a,\ b \in \mathbb{R},\ a<b. \quad (1.1)$$

It was firstly discovered by Ch. Hermite in 1881 in the journal *Mathesis* (see [7]). But this result was nowhere mentioned in the mathematical literature and was not widely known as Hermite's result.

E. F. Beckenbach, a leading expert on the history and the theory of convex functions, wrote that this inequality was proven by J. Hadamard in 1893 [1]. In 1974, D. S. Mitrinović found Hermite's note in *Mathesis* [7]. Since (1.1) was known as Hadamard's inequality, the inequality is now commonly referred as the *Hermite-Hadamard inequality.* For a monograph devoted to this result see [5]. The recent survey paper [4] provides other related results.

Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ and assume that $f'_+(a)$ and $f'_-(b)$ are finite. We recall the following improvement and reverse inequality for the first Hermite-Hadamard result that has been established in [2]

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a) \qquad (1.2)\\ &\leq \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \leq \frac{1}{8}(b-a)\left[f'_-(b) - f'_+(a)\right]. \end{aligned}$$

The following inequality that provides a reverse and improvement of the second Hermite-Hadamard result has been obtained in [3]

$$\begin{aligned} 0 &\leq \frac{1}{8}\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right](b-a) \qquad (1.3)\\ &\leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{1}{8}(b-a)\left[f'_-(b) - f'_+(a)\right]. \end{aligned}$$

The constant $\frac{1}{8}$ is best possible in both (1.2) and (1.3).

In 1906, Fejér [6], while studying trigonometric polynomials, obtained inequalities which generalize that of Hermite & Hadamard:

Theorem 1.1. *Consider the integral* $\int_a^b f(t)\,p(t)\,dt$*, where* f *is a convex function in the interval* (a,b) *and* p *is a positive function in the same interval such that*

$$p(a+t) = p(b-t), \quad 0 \leq t \leq \frac{1}{2}(b-a),$$

i.e., $y = p(t)$ is a symmetric curve with respect to the straight line which contains the point $\left(\frac{1}{2}(a+b), 0\right)$ and is normal to the t-axis. Under those conditions the following inequalities are valid:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \le \int_a^b f(t)\,p(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{1.4}$$

If f is concave on (a,b), then the inequalities reverse in (1.4).

Clearly, for $p(t) \equiv 1$ on $[a,b]$ we get 1.1.

If we take $p(t) = \left|t - \frac{a+b}{2}\right|$, $t \in [a,b]$ in Theorem 1.1, then we have

$$\frac{1}{4}f\left(\frac{a+b}{2}\right)(b-a)^2 \le \int_a^b \left|t - \frac{a+b}{2}\right| f(t)\,dt \le \frac{f(a)+f(b)}{8}(b-a)^2, \tag{1.5}$$

for any convex function $f : [a,b] \to \mathbb{R}$.

We observe that, if we take $p(t) = (b-t)(t-a)$, $t \in [a,b]$, then p satisfies the conditions in Theorem 1.1, and by (1.4) we have the following inequality as well

$$\frac{1}{6}f\left(\frac{a+b}{2}\right)(b-a)^3 \le \int_a^b (b-t)(t-a)\,f(t)\,dt \le \frac{f(a)+f(b)}{12}(b-a)^3, \tag{1.6}$$

for any convex function $f : [a,b] \to \mathbb{R}$.

Motivated by the above results, in this paper we obtain an improvement and a reverse for each inequality in (1.4) and therefore generalize the Hermite-Hadamard inequalities (1.2) and (1.3).

2. Improvements and Reverse of Féjer Inequalities

Following Roberts and Varberg [8, p. 5], we recall that if $f : I \to \mathbb{R}$ is a convex function, then for any $x_0 \in \mathring{I}$ (the interior of the interval I) the limits

$$f'_-(x_0) := \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \text{ and } f'_+(x_0) := \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and $f'_-(x_0) \le f'_+(x_0)$. The functions f'_- and f'_+ are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [8, p. 7]).

From the monotonicity of the lateral derivatives f'_- and f'_+ we also have *the gradient inequality*

$$f'_-(x)(x-y) \ge f(x) - f(y) \ge f'_+(y)(x-y)$$

for any $x, y \in \mathring{I}$.

If $I = [a,b]$, then at the end points we also have the inequalities

$$f(x) - f(a) \ge f'_+(a)(x-a)$$

for any $x \in (a,b]$ and

$$f(y) - f(b) \geq f'_{-}(b)(y-b)$$

for any $y \in [a,b)$.

We have the following refinement and reverse of Fejer's first inequality:

Theorem 2.1. *Let f be a convex function on I and a, $b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned}
0 &\leq \frac{1}{2}\int_a^b \left|t - \frac{a+b}{2}\right| p(t)\,dt \left[f'_{+}\left(\frac{a+b}{2}\right) - f'_{-}\left(\frac{a+b}{2}\right)\right] \qquad (2.1)\\
&\leq \int_a^b p(t) f(t)\,dt - \left(\int_a^b p(t)\,dt\right) f\left(\frac{a+b}{2}\right)\\
&\leq \frac{1}{2}\int_a^b \left|t - \frac{a+b}{2}\right| p(t)\,dt \left[f'_{-}(b) - f'_{+}(a)\right].
\end{aligned}$$

Proof. Let a, $b \in I$, with $a < b$. Using the integration by parts formula for Lebesgue integral, we have

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^b \left(\int_t^b p(s)\,ds\right) f'(t)\,dt\\
&= \left(\int_t^b p(s)\,ds\right) f(t)\Bigg]_{\frac{a+b}{2}}^b + \int_{\frac{a+b}{2}}^b p(t) f(t)\,dt\\
&= -\left(\int_{\frac{a+b}{2}}^b p(s)\,ds\right) f\left(\frac{a+b}{2}\right) + \int_{\frac{a+b}{2}}^b p(t) f(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}} \left(\int_a^t p(s)\,ds\right) f'(t)\,dt\\
&= \left(\int_a^t p(s)\,ds\right) f(t)\Bigg]_a^{\frac{a+b}{2}} - \int_a^{\frac{a+b}{2}} p(t) f(t)\,dt\\
&= \left(\int_a^{\frac{a+b}{2}} p(s)\,ds\right) f\left(\frac{a+b}{2}\right) - \int_a^{\frac{a+b}{2}} p(t) f(t)\,dt.
\end{aligned}$$

By subtracting the second identity from the first, we get

$$\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)f'\left(t\right)dt-\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)f'\left(t\right)dt$$
$$=\int_{\frac{a+b}{2}}^{b}p\left(t\right)f\left(t\right)dt+\int_{a}^{\frac{a+b}{2}}p\left(t\right)f\left(t\right)dt$$
$$-\left(\int_{\frac{a+b}{2}}^{b}p\left(s\right)ds\right)f\left(\frac{a+b}{2}\right)-\left(\int_{a}^{\frac{a+b}{2}}p\left(s\right)ds\right)f\left(\frac{a+b}{2}\right).$$

By the symmetry of p we get

$$\int_{\frac{a+b}{2}}^{b}p\left(s\right)ds=\int_{a}^{\frac{a+b}{2}}p\left(s\right)ds=\frac{1}{2}\int_{a}^{b}p\left(s\right)ds$$

and then we can state the following identity of interest in itself

$$\int_{a}^{b}p\left(t\right)f\left(t\right)dt-f\left(\frac{a+b}{2}\right)\int_{a}^{b}p\left(s\right)ds \tag{2.2}$$
$$=\int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}p\left(s\right)ds\right)f'\left(t\right)dt-\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}p\left(s\right)ds\right)f'\left(t\right)dt.$$

By the monotonicity of the derivative we have

$$f'_{+}\left(a\right)\leq f'\left(t\right)\leq f'_{-}\left(\frac{a+b}{2}\right),\ \text{for almost every } t\in\left(a,\frac{a+b}{2}\right)$$

and

$$f'_{+}\left(\frac{a+b}{2}\right)\leq f'\left(t\right)\leq f'_{-}\left(b\right),\ \text{for almost every } t\in\left(\frac{a+b}{2},b\right).$$

This implies

$$f'_{+}\left(a\right)\left(\int_{a}^{t}p\left(s\right)ds\right)\leq f'\left(t\right)\left(\int_{a}^{t}p\left(s\right)ds\right)$$
$$\leq f'_{-}\left(\frac{a+b}{2}\right)\left(\int_{a}^{t}p\left(s\right)ds\right),\ t\in\left[a,\frac{a+b}{2}\right]$$

and

$$f'_{+}\left(\frac{a+b}{2}\right)\left(\int_{t}^{b}p\left(s\right)ds\right)\leq f'\left(t\right)\left(\int_{t}^{b}p\left(s\right)ds\right)$$
$$\leq f'_{-}\left(b\right)\left(\int_{t}^{b}p\left(s\right)ds\right),\ t\in\left[\frac{a+b}{2},b\right],$$

and by integration

$$f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(\int_t^b p(s)\,ds\right)dt \le \int_{\frac{a+b}{2}}^{b}\left(\int_t^b p(s)\,ds\right)f'(t)\,dt \le f'_-(b)\int_{\frac{a+b}{2}}^{b}\left(\int_t^b p(s)\,ds\right)dt$$

and

$$-f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)dt \le -\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)f'(t)\,dt \le -f'_+(a)\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)dt.$$

If we add these inequalities, then we get

$$\begin{aligned} & f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}\left(\int_t^b p(s)\,ds\right)dt - f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)dt \\ & \le \int_{\frac{a+b}{2}}^{b}\left(\int_t^b p(s)\,ds\right)f(t)\,dt - \int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)f(t)\,dt \\ & \le f'_-(b)\int_{\frac{a+b}{2}}^{b}\left(\int_t^b p(s)\,ds\right)dt - f'_+(a)\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)dt. \end{aligned} \tag{2.3}$$

Integrating by parts in the Lebesgue integral, we have

$$\begin{aligned} \int_{\frac{a+b}{2}}^{b}\left(\int_t^b p(s)\,ds\right)dt &= \left(\int_t^b p(s)\,ds\right)t\bigg]_{\frac{a+b}{2}}^{b} + \int_{\frac{a+b}{2}}^{b} tp(t)\,dt \\ &= \int_{\frac{a+b}{2}}^{b} tp(t)\,dt - \frac{a+b}{2}\int_{\frac{a+b}{2}}^{b} p(s)\,ds \\ &= \int_{\frac{a+b}{2}}^{b}\left(t-\frac{a+b}{2}\right)p(t)\,dt = \frac{1}{2}\int_a^b \left|t-\frac{a+b}{2}\right| p(t)\,dt, \end{aligned}$$

where for the last equality we used the symmetry of p.

Similarly,

$$\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)dt=\left(\int_a^t p(s)\,ds\right)t\Big]_a^{\frac{a+b}{2}}-\int_a^{\frac{a+b}{2}}p(t)\,tdt$$
$$=\frac{a+b}{2}\int_a^{\frac{a+b}{2}}p(s)\,ds-\int_a^{\frac{a+b}{2}}p(t)\,tdt$$
$$=\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)p(t)\,dt=\frac{1}{2}\int_a^b\left|t-\frac{a+b}{2}\right|p(t)\,dt.$$

Then by (2.3) we obtain the desired result (2.1). □

Remark 2.2. If we take $p\equiv 1$ in (2.1) and since $\int_a^b\left|t-\frac{a+b}{2}\right|=\frac{1}{4}(b-a)^2$, hence by (2.1) we recapture the inequalities (1.2) from Introduction.

We also have the following refinement and reverse of Fejer's second inequality:

Theorem 2.3. *Let f be a convex function on I and $a,\ b\in I$, with $a<b$. If $p:[a,b]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t)=p(t)$ for all $t\in[a,b]$, then*

$$0\le\frac{1}{2}\int_a^b\left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right]p(t)\,dt\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]\tag{2.4}$$
$$\le\left(\int_a^b p(t)\,dt\right)\frac{f(a)+f(b)}{2}-\int_a^b p(t)\,f(t)\,dt$$
$$\le\frac{1}{2}\int_a^b\left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right]p(t)\,dt\left[f'_-(b)-f'_+(a)\right].$$

Proof. Using the integration by parts for Lebesgue integral, we have

$$\int_a^b\left(\int_a^t p(s)\,ds-\frac{1}{2}\int_a^b p(s)\,ds\right)f'(t)\,dt$$
$$=\left(\int_a^t p(s)\,ds-\frac{1}{2}\int_a^b p(s)\,ds\right)f(t)\Big]_a^b-\int_a^b p(t)\,f(t)\,dt$$
$$=\left(\int_a^b p(s)\,ds-\frac{1}{2}\int_a^b p(s)\,ds\right)f(b)+\left(\frac{1}{2}\int_a^b p(s)\,ds\right)f(a)$$
$$-\int_a^b p(t)\,f(t)\,dt$$
$$=\left(\int_a^b p(t)\,dt\right)\frac{f(a)+f(b)}{2}-\int_a^b p(t)\,f(t)\,dt.$$

We also have

$$\begin{aligned}
&\int_a^b \left(\int_a^t p(s)\,ds - \frac{1}{2}\int_a^b p(s)\,ds\right) f'(t)\,dt \\
&= \int_a^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&= \int_a^{\frac{a+b}{2}} \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&+ \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&= \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f(t)\,dt \\
&- \int_a^{\frac{a+b}{2}} \left(\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^t p(s)\,ds\right) f(t)\,dt.
\end{aligned}$$

Observe that

$$\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds \geq 0 \text{ for } t \in \left[\frac{a+b}{2}, b\right]$$

and

$$\int_a^{\frac{a+b}{2}} p(s)\,ds - \int_a^t p(s)\,ds \geq 0 \text{ for } t \in \left[a, \frac{a+b}{2}\right].$$

By the monotonicity of the derivative we have

$$\begin{aligned}
&f'_+\left(\frac{a+b}{2}\right) \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) dt \\
&\leq \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) f'(t)\,dt \\
&\leq f'_-(b) \int_{\frac{a+b}{2}}^b \left(\int_a^t p(s)\,ds - \int_a^{\frac{a+b}{2}} p(s)\,ds\right) dt
\end{aligned}$$

and

$$
\begin{aligned}
& -f'_{-}\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}\left(\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_a^t p(s)\,ds\right)dt \\
& \leq -\int_a^{\frac{a+b}{2}}\left(\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_a^t p(s)\,ds\right)f'(t)\,dt \\
& \leq -f'_{+}(a)\int_a^{\frac{a+b}{2}}\left(\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_a^t p(s)\,ds\right)dt.
\end{aligned}
$$

If we add these inequalities, then we get

$$
\begin{aligned}
& \left[f'_{+}\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^b\left(\int_a^t p(s)\,ds-\int_a^{\frac{a+b}{2}} p(s)\,ds\right)dt\right. \\
& \left.-f'_{-}\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}\left(\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_a^t p(s)\,ds\right)dt\right] \\
& \leq \int_{\frac{a+b}{2}}^b\left(\int_a^t p(s)\,ds-\int_a^{\frac{a+b}{2}} p(s)\,ds\right)f'(t)\,dt \\
& -\int_{\frac{a+b}{2}}^b\left(\int_a^t p(s)\,ds-\int_a^{\frac{a+b}{2}} p(s)\,ds\right)f'(t)\,dt \\
& \leq f'_{-}(b)\int_{\frac{a+b}{2}}^b\left(\int_a^t p(s)\,ds-\int_a^{\frac{a+b}{2}} p(s)\,ds\right)dt \\
& -f'_{+}(a)\int_a^{\frac{a+b}{2}}\left(\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_a^t p(s)\,ds\right)dt.
\end{aligned} \tag{2.5}
$$

Observe that

$$\begin{aligned}
&\int_{\frac{a+b}{2}}^{b}\left(\int_a^t p(s)\,ds-\int_a^{\frac{a+b}{2}} p(s)\,ds\right)dt\\
&=\int_{\frac{a+b}{2}}^{b}\left(\int_a^t p(s)\,ds\right)dt-\frac{b-a}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds\\
&=\left(\int_a^t p(s)\,ds\right)t\Big]_{\frac{a+b}{2}}^{b}-\int_{\frac{a+b}{2}}^{b} tp(t)\,dt-\frac{b-a}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds\\
&=b\int_a^b p(s)\,ds-\frac{a+b}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_{\frac{a+b}{2}}^{b} tp(t)\,dt-\frac{b-a}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds\\
&=b\int_a^b p(s)\,ds-b\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_{\frac{a+b}{2}}^{b} tp(t)\,dt\\
&=b\int_{\frac{a+b}{2}}^{b} p(s)\,ds-\int_{\frac{a+b}{2}}^{b} tp(t)\,dt=\int_{\frac{a+b}{2}}^{b}(b-t)\,p(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^{\frac{a+b}{2}}\left(\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_a^t p(s)\,ds\right)dt\\
&=\frac{b-a}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds-\int_a^{\frac{a+b}{2}}\left(\int_a^t p(s)\,ds\right)dt\\
&=\frac{b-a}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds-\left(\left(\int_a^t p(s)\,ds\right)t\Big]_a^{\frac{a+b}{2}}-\int_a^{\frac{a+b}{2}} tp(t)\,dt\right)
\end{aligned}$$

$$\begin{aligned}
&=\frac{b-a}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds-\frac{a+b}{2}\int_a^{\frac{a+b}{2}} p(s)\,ds+\int_a^{\frac{a+b}{2}} tp(t)\,dt\\
&=\int_a^{\frac{a+b}{2}} tp(t)\,dt-a\int_a^{\frac{a+b}{2}} p(s)\,ds=\int_a^{\frac{a+b}{2}}(t-a)\,p(t)\,dt.
\end{aligned}$$

If we change the variable $s=b+a-t$, then

$$\int_a^{\frac{a+b}{2}}(t-a)\,p(t)\,dt=\int_{\frac{a+b}{2}}^{b}(b-s)\,p(b+a-s)\,ds=\int_{\frac{a+b}{2}}^{b}(b-s)\,p(s)\,ds.$$

Finally, observe that

$$\frac{1}{2}\int_a^b \left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right] p(t)\,dt$$
$$=\frac{1}{2}\int_a^{\frac{a+b}{2}} \left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right] p(t)\,dt$$
$$+\frac{1}{2}\int_{\frac{a+b}{2}}^b \left[\frac{1}{2}(b-a)-\left|t-\frac{a+b}{2}\right|\right] p(t)\,dt$$
$$=\frac{1}{2}\int_a^{\frac{a+b}{2}} \left[\frac{1}{2}(b-a)-\frac{a+b}{2}+t\right] p(t)\,dt$$
$$+\frac{1}{2}\int_{\frac{a+b}{2}}^b \left[\frac{1}{2}(b-a)-t+\frac{a+b}{2}\right] p(t)\,dt$$
$$=\frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt+\frac{1}{2}\int_{\frac{a+b}{2}}^b (b-t)\,p(t)\,dt$$
$$=\frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt+\frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt=\int_a^{\frac{a+b}{2}} (t-a)\,p(t)\,dt$$

and by (2.5) we get (2.4). □

Remark 2.4. Observe that for $p \equiv 1$ we recapture the inequalities (1.3) from Introduction.

If we consider the symmetric weight $p(t)=\left|t-\frac{a+b}{2}\right|$, $t\in[a,b]$ we obtain from Theorem 2.1 that

$$\begin{aligned}0 &\le \frac{1}{24}(b-a)^3\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \\ &\le \int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt-\frac{1}{4}(b-a)^2 f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{24}(b-a)^3\left[f'_-(b)-f'_+(a)\right]\end{aligned} \tag{2.6}$$

and from Theorem 2.3 that

$$\begin{aligned}0 &\le \frac{1}{48}(b-a)^3\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right] \\ &\le (b-a)^2\frac{f(a)+f(b)}{8}-\int_a^b \left|t-\frac{a+b}{2}\right| f(t)\,dt \\ &\le \frac{1}{48}(b-a)^3\left[f'_-(b)-f'_+(a)\right],\end{aligned} \tag{2.7}$$

where f is convex on $[a,b]$. These provide refinements and reverses of the inequalities (1.5).

If we consider the symmetric weight $p(t) = (t-a)(b-t)$, $t \in [a,b]$ we obtain from Theorem 2.1 that

$$0 \le \frac{1}{64}(b-a)^4 \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] \tag{2.8}$$
$$\le \int_a^b (t-a)(b-t) f(t)\, dt - \frac{1}{6}(b-a)^3 f\left(\frac{a+b}{2} \right)$$
$$\le \frac{1}{64}(b-a)^4 \left[f'_-(b) - f'_+(a) \right]$$

and from Theorem 2.3 that

$$0 \le \frac{5}{192}(b-a)^4 \left[f'_+ \left(\frac{a+b}{2} \right) - f'_- \left(\frac{a+b}{2} \right) \right] \tag{2.9}$$
$$\le (b-a)^3 \frac{f(a)+f(b)}{12} - \int_a^b (t-a)(b-t) f(t)\, dt$$
$$\le \frac{5}{192}(b-a)^4 \left[f'_-(b) - f'_+(a) \right],$$

where f is convex on $[a,b]$. These provide refinements and reverses of the inequalities (1.6).

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