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Hamiltonian cycle in the power graph of direct product two p-groups of prime exponents

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Abstract. The power graph $\mathcal{P}(G)$ of a finite group G is a graph whose vertex set is the group G and distinct elements $x, y \in G$ are adjacent if one is a power of the other, that is, x and y are adjacent if $x \in \langle y \rangle$ or $y \in \langle x \rangle$. Suppose that $G = P \times Q$, where P (resp. Q) is a finite p-group (resp. q-group) of exponent p (resp. q) for distinct prime numbers $p < q$. In this paper, we determine necessary and sufficient conditions for existence of Hamiltonian cycles in $\mathcal{P}(G)$.

Keywords: Power graph, Direct product, p-Group, Hamiltonian cycle

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1. Introduction

The *power graph* $\mathcal{P}(G)$ of a group G is a graph with elements of G as its vertices such that two distinct elements x and y are adjacent if $y = x^m$ or $x = y^m$ for some positive integer m. Clearly, two distinct elements x and y are adjacent if and only if $x \in \langle y \rangle$ or $y \in \langle x \rangle$ when G is a finite group.

Power graphs of groups are introduced by Kelarev and Quinn [8, 9]. In [3], Cameron shows that two finite groups with isomorphic power

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graphs must have the same number of elements of equal orders. Furthermore, Cameron and Gosh [4] show that two finite abelian groups with isomorphic power graphs are isomorphic.

For a finite group G, all nonidentity elements are adjacent to the identity element, hence $\mathcal{P}(G)$ is always connected. In [6], we introduced *proper power graph* $\mathcal{P}^*(G)$ to be the induced subgraph of $\mathcal{P}(G)$ whose elements are nontrivial elements of G and investigated whether the proper power graph of a finite group is connected? Also, we computated the number of connected components of the graph $\mathcal{P}^*(G)$ for some classes of finite groups, say nilpotent groups and symmetric groups. The number of connected components of a graph Γ is denoted by $c(\Gamma)$. In this paper, we fix prime numbers $p < q$, and a p-group P and a q-group Q of orders p^m and q^n with exponents p, q, respectively. Let c_p and c_q denote the number of connected components of $\mathcal{P}^*(P)$ and $\mathcal{P}^*(Q)$, respectively. If $x, y \in \mathcal{P}(G)$ are adjacent, then we write $x \sim y$. For a subset X of the group G, $\mathcal{P}(X)$ indicates the induced subgraph of $\mathcal{P}(G)$ with vertex set X. Chakrabarty, Ghosh, and Sen [5] studied power graphs that are complete or Eulerian or Hamiltonian. In this paper, we will give necessary and sufficient conditions for existence of Hamiltonian cycles in the power graph $\mathcal{P}(P \times Q)$ of directe product the groups P and Q.

2. Main result

The following simple condition is necessary for deciding whether a given graph is Hamiltonian (see[2]).

Theorem 2.1. *Let S be a set of vertices of a Hamiltonian graph Γ. Then $c(\Gamma - S) \leq |S|$, where $c(\Gamma - S)$ is the number of connected components of $\Gamma - S$.*

Lemma 2.2. *Suppose that $G = H \times K$, and $c(\mathcal{P}^*(H)) = m$ and $c(\mathcal{P}^*(K)) = n$. If the graph $\mathcal{P}(G)$ is Hamiltonian, then*

(1) $mn \leq |H| + |K| - 1$;
(2) $n \leq |H|$ *and* $m \leq |K|$.

Proof. Let $H_1, \ldots, H_m$ and $K_1, \ldots, K_n$ of the connected components of the graphs $\mathcal{P}^*(H)$ and $\mathcal{P}^*(K)$. For every $1 \leq i \leq m$ and $1 \leq j \leq n$, let

$$G_{i,j} = H_i \times K_j, \quad G_{i,K} = H_i \times K, \quad G_{H,j} = H \times K_j.$$

Now, for every $g' \in G$, one can show that

(1) if $g \in G_{i,j}$ and $g \sim g'$, then $g' \in G_{i,j} \cup \big(H \times \{e\} \cup (\{e\} \times K)\big)$;
(2) if $g \in G_{i,K}$ and $g \sim g'$, then $g' \in G_{i,K} \cup \big(\{e\} \times K\big)$;
(3) if $g \in G_{H,j}$ and $g \sim g'$, then $g' \in G_{H,j} \cup \big(H \times \{e\}\big)$.

By (1), (2), and (3), we can show that the connected components of the graphs $\mathcal{P}(G) \setminus \big(H \times \{e\} \cup (\{e\} \times K)\big)$, $\mathcal{P}(G) \setminus (\{e\} \times K)$, and $\mathcal{P}(G) \setminus (H \times \{e\})$ are $G_{i,j}$, $G_{i,K}$, and $G_{H,j}$, respectively. By theorem 2.1, the results follows. $\square$

In the following theorem [6], the number of connected components of a finite p-group is computed.

Theorem 2.3. *Let G be a finite p-group. Then there exists a one-to-one correspondence between the connected components of $\mathcal{P}^*(G)$ and the minimal cyclic subgroups of G.*

Example 2.4. If P is a finite p-group of exponent p, then $\mathcal{P}^*(P)$ is a union of complete graphs of order $p-1$. Moreover, the number of connected components of $\mathcal{P}^*(P)$ is equal to $(p^m-1)/(p-1)$, where p^m is the order of P.

Theorem 2.5. *Let $G = P \times Q$ and $m, n \geq 2$. If $c_q \leq p^m$ and $c_p \leq q$, then $\mathcal{P}(G)$ is Hamiltonian.*

Proof. Let $H_1, \ldots, H_m$ and $K_1, \ldots, K_n$ be connected components of the graphs $\mathcal{P}^*(H)$ and $\mathcal{P}^*(K)$, respectively. For every $1 \leq r \leq c_p$ and $1 \leq s \leq c_q$, we know that

$$H_r = \langle x_r \rangle \setminus \{e\}, \quad |x_r| = p, \quad K_s = \langle y_s \rangle \setminus \{e\}, \quad |y_s| = q.$$

Put

$$X_{rs} = \{x_r^i y_s^j \mid 1 \leq i \leq p-1, 1 \leq j \leq q-1\}$$

and

$$Y_s = \bigcup_{r=1}^{p+1} X_{rs} \cup \{y_s^j \mid 1 \leq j \leq q-1\}.$$

Note that the subgraph $\mathcal{P}(X_{rs})$ is complete and has a Hamiltonian path

$$L_{rs} : x_r y_s \sim x_r^2 y_s \sim x_r^i y_s^j \sim \cdots \sim x_r^{p-1} y_s^{q-1}.$$

We claim that for every $1 \leq s \leq c_q$, the graph $\mathcal{P}(Y_s)$ has a Hamiltonian path, denoted by L_s, which begins from a vertex of X_{rs} and ends at a vertex of $X_{r's}$, where $r \neq r'$.

For simplicity, let $r = 1$ and $r' = c_p$. Since $c_p \leq q$, we can write the following Hamiltonian path:

$$L^*{}_s : L_{1s} \sim y_s \sim L_{2s} \sim y_s^2 \sim \cdots \sim L_{c_p-1s} \sim y_s^{c_p-1} \sim y_s^{c_p} \sim \cdots \sim y_s^{q-1} \sim L_{c_p s}$$

To prove the claim, it is enough to substitute L_{1s} and $L_{c_p s}$ with L_{rs} and $L_{r's}$ in the path $L^*{}_s$, respectively.

Now, since $c_q \leq p^m$ we can extend the paths L_s to a cycle in the graph $\mathcal{P}(G)$ as

$$\mathcal{C} : e \sim L_1 \sim x_1 \sim L_2 \sim x_2 \sim \cdots \sim x_{c_p} \sim L_{c_p+1} \\ \sim x_1^2 \sim L_{c_p+2} \sim \cdots \sim x_r^i \sim L_s \sim x_{r'}^j \sim \cdots \sim x_{c_p}^{p-1} \sim L_{c_q} \sim e,$$

where L_1 is a Hamiltonian path of Y_1 that ends at a vertex of X_{11} and L_2 is a Hamiltonian path of Y_2 that begins from a vertex of X_{12} and ends at a vertex of X_{22}. Actually, for $s \geq 2$, if L_s is the above cycle between ${x_r}^i$ and $x_{r'}^j$ for $r \neq r'$, then we L_s is a Hamiltonian path of Y_s with beginning from a vertex of X_{rs} and ending at a vertex of $X_{r's}$.

Suppose that $x \in G$ is an element of order p. We know that $x \in \langle x_r \rangle$ for some $1 \leq r \leq c_p$. If x is not in the cycle $\mathcal{C}$, then since $c_p \leq q \leq c_q$, we can join x to x_r in $\mathcal{C}$, hence the cycle $\mathcal{C}$ will be made into a Hamiltonian cycle of $\mathcal{P}(G)$ by continuing this process. □

Corollary 2.6. *Let $G = (\mathbb{Z}_p \times \mathbb{Z}_p) \times (\mathbb{Z}_q \times \mathbb{Z}_q)$. Then the graph $\mathcal{P}(G)$ is Hamltonian if and only if $q \leq p^2 - 1$.*

Proof. Put $m = n = 2$ in Theorm 2.5. □

Example 2.7. The graph $\mathcal{P}(\mathbb{Z}_6 \times \mathbb{Z}_6)$ is Hamiltonian.

The following paths in the $\mathcal{P}(\mathbb{Z}_6 \times \mathbb{Z}_6)$ contain all elements of order 6 and 3:

(1) L_1: $(\overline{0}, \overline{1}) \sim (\overline{0}, \overline{5}) \sim (\overline{0}, \overline{2}) \sim (\overline{3}, \overline{2}) \sim (\overline{3}, \overline{4}) \sim (\overline{0}, \overline{4}) \sim (\overline{3}, \overline{1}) \sim (\overline{3}, \overline{5})$;
(2) L_2: $(\overline{1}, \overline{3}) \sim (\overline{5}, \overline{3}) \sim (\overline{2}, \overline{0}) \sim (\overline{4}, \overline{3}) \sim (\overline{2}, \overline{3}) \sim (\overline{4}, \overline{0}) \sim (\overline{1}, \overline{0}) \sim (\overline{5}, \overline{0})$;
(3) L_3: $(\overline{5}, \overline{4}) \sim (\overline{1}, \overline{2}) \sim (\overline{2}, \overline{4}) \sim (\overline{1}, \overline{5}) \sim (\overline{5}, \overline{1}) \sim (\overline{4}, \overline{2}) \sim (\overline{2}, \overline{1}) \sim (\overline{4}, \overline{5})$;
(4) L_4: $(\overline{2}, \overline{5}) \sim (\overline{4}, \overline{1}) \sim (\overline{2}, \overline{2}) \sim (\overline{1}, \overline{4}) \sim (\overline{5}, \overline{2}) \sim (\overline{4}, \overline{4}) \sim (\overline{5}, \overline{5}) \sim (\overline{1}, \overline{1})$.

Hence, we obtain the Hamiltonian cycle

$$(\overline{0}, \overline{0}) \sim L_1 \sim (\overline{3}, \overline{3}) \sim L_2 \sim (\overline{3}, \overline{0}) \sim L_3 \sim (\overline{0}, \overline{3}) \sim L_4 \sim (\overline{0}, \overline{0}).$$

of $\mathcal{P}(\mathbb{Z}_6 \times \mathbb{Z}_6)$.

Lemma 2.8. *Let $G = P \times Q$. Suppose that $m \geq 3$ and $n \geq 2$. If*

(i) $c_p \leq q^n$,
(ii) $c_q \leq p^m$, *and*
(iii) $c_p c_q \leq p^m + q^n - 1$,

then

(a) $p^m < q^n$.

Also, if $n = 2$, then

(b_1) *$c_p \leq 2q-2$,*
(b_2) *$p^{m-1} < q$,*
(b_3) *$c_p - p < q-2$, and*
(b_4) *the graph $\mathcal{P}(G)$ is Hamiltonian.*

Proof. (a) First suppose that $n=2$. Assume on the contrary that $q^2 \leq p^m$, but it is clear that $q^2 \neq p^m$, then $q^2 < p^m$. From (iii), we conclude that

$$(1+p+\cdots+p^{m-1})(q-p+2) \leq q^2$$

and this results $(q-p+2) \leq p-1$, since otherwise

$$(1+p+\cdots+p^{m-1})(q-p+2) > p^m - 1 \quad \text{or} \quad p^m - 1 < q^2 < p^m$$

which is a contradiction. Hence $q \leq 2p-3$.

Put $q = p+t$ with $0 < t \leq p-3$. Using (iii), we get ether

$$(1+p+\cdots+p^{m-1})(p+t+1) \leq p^m + p^2 + 2pt + t^2 - 1$$

or

$$0 < (1+p+\cdots+p^{m-1}) + (p+p^2+\cdots+p^{m-1}) - p^2 \leq (p-3)[(p-3)+2p-(1+p+\cdots+p^{m-1})].$$

Hence

$$[(p-3)+2p-(1+p+\cdots+p^{m-1})] > 0$$

when $m \geq 3$. Then $2p - p^2 - 4 > 0$ or $p < 2$, which is a contradiction.

Now, suppose that $n \geq 3$. Again, we assume on the contrary that $q^n < p^m$. By (iii) and the fact that $q^n \leq p^m - 1$, we obtain

$$q^n - 1 \leq 2(q-1)(p-1)$$

or equivalently

$$1 + q + \cdots + q^{n-1} \leq 2(p-1).$$

Since $n \geq 3$,

$$p^2 < q^2 < 1 + q + \cdots + q^{n-1} < 2(p-1) < 2p$$

which implies that $p < 2$, a contradiction. Therefore $p^m < q^n$.

In what follows, we assume that $n = 2$.

(b_1) According to (iii) and $p^m < q^2$, we have

$$c_p(q+1) \leq p^m + q^2 - 1 < 2q^2 - 1 \quad \text{or} \quad c_p \leq 2q-2.$$

(b_2) From (a) and $p^m < q^2$, we get

(*) If $m \geq 4$, then $q > p^2$ and by (iii),

$$c_p \leq (q-1) + \frac{p^m}{q+1} < (q-1) + \frac{p^m}{q}.$$

Thus, $c_p - p^{m-2} < q-1 < q$ or

$$p^{m-1} < (1+p+p^2+\cdots+p^{m-2}+p^{m-1}) - p^{m-2} < q.$$

Hence $p^{m-1} < q$.

(**) Suppose that $m = 3$. By (iii), we conclude that

$$q^2 - (p^2 + p + 1)q - (-p^3 + p^2 + p + 2) \geq 0.$$

Hence either $q \leq q_1$ or $q \geq q_2$, where $q_1, q_2 = \frac{1}{2}(p^2 + p + 1 \mp \sqrt{\Delta})$ with $\Delta = p^4 - 2p^3 + 7p^2 + 6p + 9 > 0$.

If $q \leq q_1$, then $c_p = p^2 + p + 1 > 2q$, which is a contradiction by (a). Therefore, $q \geq q_2$. On the other hand, the function $g(p) = q_2 - p^2$ has minimum 2 in $[0, \infty)$. Hence, for $q \geq q_2$, we have $q > p^2$

(b_3) By (iii), we have

$$(1 + q)c_p \leq p^m + q^2 - 1 \leq pq + q^2 - 1 = (q + 1)(p + q - 1) - p$$

or equivalently

$$c_p \leq (p + q - 1) - \frac{p}{q + 1}.$$

Therefore $c_p - p < q - 2$.

(b_4) If $c_p < c_q$, then by Theorem 2.5, $\mathcal{P}(G)$ is Hamiltonian. Hence assume that $c_q \leq c_p$. Let $H_1, H_2, \ldots, H_{c_p}$ and $K_1, K_2, \ldots, K_{c_q}$ be connected components of $\mathcal{P}^*(P)$ and $\mathcal{P}^*(Q)$, respectively. Actually, for every $1 \leq r \leq c_p$ and $1 \leq s \leq c_q$,

$$H_r = \langle x_r \rangle \setminus \{e\}, \quad |x_r| = p \quad \text{and} \quad K_s = \langle y_s \rangle \setminus \{e\}, \quad |y_s| = q.$$

Put

$$X_{rs} = \{x_r^i y_s^j : 1 \leq i \leq p - 1, 1 \leq j \leq q - 1\}$$

and

$$B = \{1, 2, \ldots, c_q\}.$$

If $1 \leq r \leq c_q$, we define the subset $B_r = \{\overline{r-1}, \overline{r}, \ldots, \overline{r+p-2}\}$ of B where $\overline{u}$ denotes the remainder of u indivision by c_q, i.e. $u \equiv \overline{u} \pmod{c_q}$, and $B_r = \varnothing$ for every $c_q + 1 \leq r \leq c_p$.

Note that the subgraph induced by X_{rs} is complete and has a Hamiltonian path

$$L_{rs} : x_r y_s \sim x_r^2 y_s \sim x_r^i y_s^j \sim \cdots \sim x_r^{p-1} y_s^{q-1}.$$

Now, for each $1 \leq r \leq c_q$, we define the path Λ_r as follows

$$\Lambda_r : L_{r\overline{r-1}} \sim x_r \sim L_{r\overline{r+1}} \sim x_r^2 \sim L_{r\overline{r+2}} \sim x_r^3 \sim \cdots \sim x_r^{p-1} \sim L_{r\overline{r}}.$$

The paths Λ_r can be joined to obtain a longer path

$$L : \Lambda_1 \sim y_1 \sim \Lambda_2 \sim y_2 \sim \Lambda_3 \sim \cdots \sim \Lambda_{c_q} \sim y_{c_q}.$$

Next, for every $1 \leq s \leq c_q$, we define the subsets Ω_s and Δ_s of the set $G \setminus L_e$, where L_e is the set of elements used in the path L as

$$\Omega_s = \{y_s^2, y_s^3, \ldots, y_s^{q-1}\} \quad \text{and} \quad \Delta_s = \bigcup X_{rs},$$

where union is on $1 \leq r \leq c_p$ such that $s \notin B_r$. It is clear that $G \setminus L_e$ is partitioned by the subsets Ω_s, Δ_s, and $\{e\}$.

On the other hand, for every $1 \leq s \leq c_q$, w ehave

$$M_s = \{1 \leq r \leq c_p : s \in B_r\} = \{\overline{s-p+2}, \overline{s-p+1}, \ldots, \overline{s}, \overline{s+1}\}.$$

Hence

$$\{X_{rs} : s \notin B_r\}| = c_p - p.$$

Since $c_p - p < q - 2$, we have a path Γ_s containing elements of $G \setminus L_e$ as

$$\Gamma_s : L_{r_1 s} \sim y_s^2 \sim L_{r_2 s} \sim y_s^3 \sim \cdots \sim L_{r_{c_p-p} s} \sim {y_s}^{c_p-p+1} \sim {y_s}^{c_p-p+2} \sim \cdots \sim y_s^{q-1}$$

where $r_i \in M_s$ for $1 \leq i \leq c_p - p$.

Again, by attaching the paths Λ_r, Γ_s, L, and identity element e we obtain the Hamiltonian cycle

$$\mathcal{C} : e \sim \Lambda_1 \sim y_1 \sim \Gamma_1 \sim \Lambda_2 \sim y_2 \sim \Gamma_2 \sim \cdots \sim \Lambda_{c_q} \sim y_{c_q} \sim \Gamma_{c_q} \sim e,$$

as required.

□

Theorem 2.9. *Let $G = P \times Q$, where P and Q are groups of order p^m, q^n and with exponents p, q, respectively. The graph $\mathcal{P}(G)$ is Hamiltonian if and only if*

(i) $c_q \leq p^m$;
(ii) $c_p \leq q^n$; *and*
(iii) $c_p c_q \leq p^m + q^n - 1$.

Proof. Assume that the graph $\mathcal{P}(G)$ is Hamiltonian. Then, by Lemma 2.2, the results hold. For the converse, we discuss on the numbers n, m and show that $\mathcal{P}(G)$ is Hamiltonian.

(1) If $n = 1$ or $m = 1$, then $G = \mathbb{Z}_p \times Q$ or $\mathbb{Z}_q \times P$ and $\mathcal{P}(G)$ is Hamiltonian by Corollary 2.15 [7].

(2) If $n = m = 2$, then by Theorem 2.6, the graph $\mathcal{P}(G)$ is Hamiltonian.

(3) If $n = 2$ and $m \geq 3$, then the graph $\mathcal{P}(G)$ is Hamiltonian by Lemma 2.8.

(4) If $n \geq 3$ and $m \in \mathbb{N}$, then the assumptions yields that $n = 3$. First observe that by Lemma 2.8 and the fact that $p < q$, we get $\dfrac{p^m}{q^n - 1} \leq 1$, hence $p^m - 1 \leq 2(p-1)(q-1)$ by part (iii).

On the other hand, when $c_q \leq p^m$, we get $q(1 + q + \cdots + q^{n-2}) \leq 2(p-1)(q-1)$. Since $p < q$, we have $1 + q + \cdots + q^{n-2} \leq 2q - 2$ and we conclude that $n - 2 \leq 1$ or $n \leq 3$. Thus, $n = 3$ and the proof is complete.

Now, we show that $\mathcal{P}(G)$ is Hamiltonian. By part (iii), we have

$$\frac{q^3}{1+p+p^2+\cdots+p^{m-1}} - q^2 > 0$$

or equivalently $(1+p+p^2+\cdots+p^{m-1}) < q$. Therefore, by Theorem 2.5, the graph $\mathcal{P}(G)$ is Hamiltonian. □

Example 2.10. Suppose that $G = P \times Q$, where P and Q are groups of orders 5^3 and 29^2, respectively. Then the graph $\mathcal{P}(G)$ is Hamiltonian. We have

(1) $c_p = 31 \leq 29^2 = 841$
(2) $c_q = 30, \leq 125 = 5^3$
(3) $31 \times 30 = 930 \leq 5^3 + 29^2 - 1 = 965$

Then by Lemma 2.8, $5^3 = 125 \leq 29^2 = 841$ and the graph $\mathcal{P}(G)$ is Hamiltonian.

Additional Details: Using the symbols introduced in the proof of Lemma 2.8, put $B = \{1, 2, 3, \ldots, 30\}$ and choose the subsets B_r of B for every $1 \leq r \leq 31$ as

$$B_1 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\} = \{30, 1, 2, 3, 4\}, \quad B_2 = \{1, 2, 3, 4, 5\}, \quad B_3 = \{2, 3, 4, 5, 6\}, \ldots,$$
$$B_{28} = \{27, 28, 29, 30, 1\}, \quad B_{29} = \{28, 29, 30, 1, 2\}, \quad B_{30} = \{29, 30, 1, 2, 3\}, \quad B_{31} = \emptyset.$$

For every $1 \leq r \leq 31$ and $1 \leq s \leq 30$,

$$H_r = \langle x_r \rangle \setminus \{e\}, \quad |x_r| = 5 \quad \text{and} \quad K_s = \langle y_s \rangle \setminus \{e\}, \quad |y_s| = 29.$$

Also,

$$X_{rs} = \{x_r^i y_s^j \mid 1 \leq i \leq 4, 1 \leq j \leq 28\}.$$

Corresponding to the subsets B_r, we write the sets X_{rs} in a table

1	$X_{1(30)}$	X_{12}	X_{13}	X_{14}	X_{11}
2	X_{21}	X_{23}	X_{24}	X_{25}	X_{22}
3	X_{32}	X_{34}	X_{35}	X_{36}	X_{33}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
28	$X_{(28)(27)}$	$X_{(28)(29)}$	$X_{(28)(30)}$	$X_{(28)1}$	$X_{(28)(28)}$
29	$X_{(29)(28)}$	$X_{(29)(30)}$	$X_{(29)1}$	$X_{(29)2}$	$X_{(29)(29)}$
30	$X_{(30)(29)}$	$X_{(30)1}$	$X_{(30)2}$	$X_{(30)3}$	$X_{(30)(30)}$

The table of paths

Λ_1	$L_{1(30)} \sim x_1 \sim L_{12} \sim x_1^2 \sim L_{13} \sim x_1^3 \sim L_{14} \sim x_1^4 \sim L_{11}$
Λ_2	$L_{21} \sim x_2 \sim L_{23} \sim x_2^2 \sim L_{24} \sim x_2^3 \sim L_{25} \sim x_2^4 \sim L_{22}$
Λ_3	$L_{32} \sim x_3 \sim L_{34} \sim x_3^2 \sim L_{35} \sim x_3^3 \sim L_{36} \sim x_3^4 \sim L_{33}$
$\vdots$	$\vdots$
Λ_{28}	$L_{(28)(27)} \sim x_{28} \sim L_{(28)(29)} \sim x_{28}^2 \sim L_{(28)(30)} \sim x_{28}^3 \sim L_{(28)1} \sim x_{28}^4 \sim L_{(28)(28)}$
Λ_{29}	$L_{(29)(28)} \sim x_{29} \sim L_{(29)(30)} \sim x_{29}^2 \sim L_{(29)1} \sim x_{29}^3 \sim L_{(29)2} \sim x_{29}^4 \sim L_{(29)(29)}$
Λ_{30}	$L_{(30)(29)} \sim x_{30} \sim L_{(30)1} \sim x_{30}^2 \sim L_{(30)2} \sim x_{30}^3 \sim L_{(30)3} \sim x_{30}^4 \sim L_{(30)(30)}$

The paths Λ_r can be joined to make a longer path

$$L : \Lambda_1 \sim y_1 \sim \Lambda_2 \sim y_2 \sim \Lambda_3 \sim \cdots \sim \Lambda_{30} \sim y_{30}.$$

On the other hand, the X_{r1}'s and X_{r2}'s used above are $X_{11}, X_{21}, X_{(28)1}, X_{(29)1}, X_{(30)1}$ and $X_{12}, X_{22}, X_{32}, X_{(29)2}, X_{(30)2}$, respectively.

Γ_1	$L_{31} \sim y_1^2 \sim L_{41} \sim y_1^3 \sim L_{51} \sim y_1^4 \sim \cdots \sim L_{(27)1} \sim y_1^{26} \sim L_{(30)1} \sim y_1^{27} \sim y_1^{28}$
Γ_2	$L_{42} \sim y_2^2 \sim L_{52} \sim y_2^3 \sim L_{62} \sim y_2^4 \sim \cdots \sim L_{(27)2} \sim y_2^{25} \sim L_{(28)2} \sim y_2^{26} \sim L_{(30)2} \sim y_2^{27} \sim y_2^{28}$
Γ_3	$L_{53} \sim y_3^2 \sim L_{63} \sim y_3^3 \sim L_{73} \sim y_3^4 \sim \cdots \sim L_{(28)3} \sim y_3^{25} \sim L_{(29)3} \sim y_3^{26} \sim L_{(31)3} \sim y_3^{27} \sim y_3^{28}$
$\vdots$	$\vdots$
Γ_{28}	$L_{1(28)} \sim {y_{28}}^2 \sim \cdots \sim L_{(24)(28)} \sim y_{28}^{25} \sim L_{(30)(28)} \sim {y_{28}}^{26} \sim L_{(31)(28)} \sim {y_{28}}^{27} \sim {y_{28}}^{28}$
Γ_{29}	$L_{1(29)} \sim {y_{29}}^2 \sim L_{2(29)} \sim {y_{29}}^3 \sim \cdots \sim L_{(25)(29)} \sim {y_{29}}^{26} \sim L_{(31)(29)} \sim {y_{29}}^{27} \sim {y_{29}}^{28}$
Γ_{30}	$L_{2(30)} \sim {y_{30}}^2 \sim L_{3(30)} \sim {y_{30}}^4 \sim \cdots \sim L_{(26)(30)} \sim y_{30}^{26} \sim L_{(31)(30)} \sim {y_{30}}^{27} \sim {y_{30}}^{28}$

Therefore, we have a Hamiltonian cycle as in the following

$$\mathcal{C} : e \sim \Lambda_1 \sim y_1 \sim \Gamma_1 \sim \Lambda_2 \sim y_2 \sim \Gamma_2 \sim \cdots \sim \Lambda_{30} \sim y_{30} \sim \Gamma_{30} \sim e.$$

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