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Some fixed point results for weakly $b-(\varphi, G)$ contraction in $b-$metric spacs endowed with a graph

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Abstract. In this paper, we introduce the concept of weakly $b-(\varphi, G)$ contraction mapping in b-metric spaces endowed with a graph and give some fixed point results for such contractions.

Keywords: Fixed point, b-Metric spaces, Weakly $b-(\varphi, G)$ contraction.

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1. Introduction

The concept of b-metric spaces were firstly obtained in 1989 by Bakhtin [1]. In 2010, Khamsi and Hussain [7] reintroduced the notion of a b-metric under the name metric-type. After that, many authors have carried out further studies on b-metric space. For further works and results in b-metric spaces, see, e.g., [4, 9]. Espinola et al. [3] proved some results on combining graph theory and fixed point theory. Later, Jachymski [6] proved the contraction principal for mappings on a metric space endowed with a graph. In this direction several authors obtained further results in metric spaces endowed with graph(see e.g. [2, 5]). We recall some of the basic definitions and results in the sequel. Let $G = (V(G), E(G))$ be a directed graph such that $V(G)$ is the set of vertices and $E(G)$ is edges of G. Let $\Delta \subset E(G)$, where $\Delta = \{(x, x) :$

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$x \in X\}$. Also, suppose that G has no parallel edges. We denote the conversion of a graph G by G^{-1}. Let $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges, that is, we get $E(\tilde{G}) = E(G) \bigcup E(G^{-1})$. Let x and y are vertices in a graph G. A path in G from x to y of length m is a sequence $\{x_n\}_{n=0}^{m}$ of $m+1$ vertices such that $x_0 = x$ and $x_m = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, ..., m$. The graph G is called connected if there is a path between any two vertices of G and graph G is weakly connected if $\tilde{G}$ is connected. For $x \in X$ we set $[x]_{\tilde{G}}$ which is the equivalence class of the following relation R defined on $V(G)$ by the rule: yRz if there is a path in G from y to z.

Definition 1.1. [9] We say that sequances $\{x_n\}, \{y_n\}$ in X are equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 1.2. [9] Let $f : X \to X$ and the sequence $\{f^n(x)\}$ in X be such that $f^n(x) \to x^*$ with $(f^{n+1}(x), f^n(x)) \in E(G)$ for $n \in \mathbb{N}$ and $x, x^* \in X$.

(1) The graph G is called (C_f)-graph if there exists a subsequence $\{f^{n_k}(x)\}$ of $\{f^n(x)\}$ and $k_0 \in \mathbb{N}$ such that $(f^{n_k}(x), x^*) \in E(G)$ for all $k \geq k_0$.

(2) The graph G is called (H_f)-graph if $f^n(x) \in [x^*]_{\tilde{G}}$ for $n \in \mathbb{N}$, then $r(f^n(x), x^*) \to 0$ as $n \to \infty$, where $r(f^n(x), x^*) = \sum_{i=1}^{M_n} s^i d(z_{i-1}, z_i)$ and $\{z_i\}_{i=0}^{M_n}$ is a path from $f^n(x)$ to x^* in $\tilde{G}$.

Definition 1.3. [8]. (Altering Distance Function) A function $\varphi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(1) φ is continuous and strictly increasing,

(2) $\varphi(t) = 0$ if and only if $t = 0$.

2. Main result

Now, we introduce one new type of contractive mappings in the context of b-metric spaces endowed with a graph and prove the corresponding new result. Throughout this section we assume that (X, d) is a $b-$metric space endowed with directed graph G, which $V(G) = X$ and $\Delta \subset E(G)$.

Definition 2.1. Let (X, d) be b-metric space and f be a self-mapping on X. We say that f is a weakly b-(φ, G) contraction if for every $x, y \in X$, we have

$$(f(x), f(y)) \in E(G) \text{ } whenever \text{ } (x, y) \in E(G), \tag{2.1}$$

$$d(fx, fy) \leq \frac{d(x, y)}{s^2} - \varphi(d(x, y)) \text{ } whenever \text{ } (x, y) \in E(G), \tag{2.2}$$

where φ is an altering distance function.

Example 2.2. Let $G = (X, \Delta)$ and f be a self-mapping on X. Then f is a weakly b-(φ, G) contraction.

Example 2.3. Let $f : X \to X$ be a constant mapping. Then f is a weakly $b - (\varphi, G)$ contraction for any graph G with $V(G) = X$.

Proposition 2.4. *Let (X, d) be a b-metric space with parameter $s \geq 1$ and $f : X \to X$ be a weakly b-(φ, G) contraction. Then*

(1) *f is a weakly b-$(\varphi, \tilde{G})$ contraction and also a b-(φ, G^{-1}) contraction.*
(2) *$f\big([x_0]_{\tilde{G}}\big) \subseteq [x_0]_{\tilde{G}}$ and $f|_{[x_0]_{\tilde{G}}}$ is a weakly b-$(\varphi, \tilde{G_{x_0}})$ contraction provided $x_0 \in X$ is such that $f(x_0) \in [x_0]_{\tilde{G}}$.*

Proof. The proof is similar to Proposition 12.1.[9], therefore we omit it. □

Lemma 2.5. *Let X be a b-metric space with $s \geq 1$ and $f : X \to X$ be a weakly b-(φ, G) contraction. Then for any $x \in X$ and $y \in [x]_{\tilde{G}}$, the sequences $\{f^n(x)\}$ and $\{f^n(y)\}$ are equivalent.*

Proof. Let $x \in X$ and $y \in [x]_{\tilde{G}}$. Then by definition of equivalence class, there exists a path $\{z_i\}_{i=0}^k$ from x to y in $\tilde{G}$ such that $x = z_0, ..., y = z_k$ and $(z_{i-1}, z_i) \in E(\tilde{G})$. By proposition 2.4, f is weakly b-$(\varphi, \tilde{G})$ contraction. Then, for all $n \in \mathbb{N}$, we have $\big(f^n(z_{i-1}), f^n(z_i)\big) \in E(\tilde{G})$. From (2.2), we get

$$\begin{aligned} d(f^n(z_{i-1}), f^n(z_i)) \quad &\leq \quad \frac{d(f^{n-1}(z_{i-1}), f^{n-1}(z_i))}{s^2} \\ &- \quad \varphi\big(d(f^{n-1}(z_{i-1}), f^{n-1}(z_i))\big), \end{aligned} \tag{2.3}$$

where $n \in \mathbb{N}$ and $i = 1, .., k$. Then, we get

$$d\big(f^n(z_{i-1}), f^n(z_i)\big) \leq d\big(f^{n-1}(z_{i-1}), f^{n-1}(z_i)\big),$$

for all $n \in \mathbb{N}$ and $i = 1, ..., k$. Thus $d\big(f^n(z_{i-1}), f^n(z_i)\big)$ is a nonincreasing sequence and hence it is convergent. Let $d\big(f^n(z_{i-1}), f^n(z_i)\big) \to r$, where $r \geq 0$. Letting $n \to \infty$ in (2.3) and using the continuity of φ, we have

$$r \leq \frac{r}{s^2} - \varphi(r) \leq r.$$

Since φ is altering distance function, we obtain $r = 0$, that is,

$$lim_{n\to\infty} d(f^n(z_{i-1}), f^n(z_i)) = 0. \tag{2.4}$$

By b-triangular inequality, we have

$$d(f^n(x), f^n(y)) \leq \sum_{i=1}^{k} s^i d\big(f^n(z_{i-1}), f^n(z_i)\big) \tag{2.5}$$

for all $n \in \mathbb{N}$. Using (2.4), (2.5) and passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0$$

□

Proposition 2.6. *Let (X, d) be a b-metric space with parameter $s > 1$ and $f : X \to X$ be a weakly b-(φ, G) contraction. Suppose $x_0 \in X$ and $f(x_0) \in [x_0]_{\tilde{G}}$. Then $\{f^n(x_0)\}$ is a Cauchy sequence in X.*

Proof. Since $f(x_0) \in [x_0]_{\tilde{G}}$, then there exists path $\{z_i\}_{i=0}^k$ from x_0 to $f(x_0)$ in $\tilde{G}$ such that $x_0 = z_0, ..., f(x_0) = z_k$ and $(z_{i-1}, z_i) \in E(\tilde{G})$. Then by a similar argument to that of the previous one can show that

$$\begin{aligned} d\big(f^n(x_0), f^{n+1}(x_0)\big) &\leq \sum_{i=1}^{k} s^i d\big(f^n(z_{i-1}), f^n(z_i)\big) \\ &\leq \sum_{i=1}^{k} s^i \frac{d\big(z_{i-1}, z_i\big)}{s^{2n}} \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}$. Let $m > n \geq 1$ and $p \geq 1$. Then by b-triangular inequality and (2.6), we have

$$\begin{aligned} d\big(f^n(x_0), f^{n+p}(x_0)\big) &\leq sd\big(f^n(x_0), f^{n+1}(x_0)\big) + s^2 d\big(f^{n+1}(x_0), f^{n+2}(x_0)\big) \\ &\quad + ... + s^p d\big(f^{n+p-1}(x_0), f^{n+p}(x_0)\big) \\ &\leq \frac{1}{s^{n-1}} \Big[\sum_{j=n}^{n+p-1} s^j d\big(f^j(x_0), f^{j-1}(x_0)\big) \Big] \\ &\leq \frac{1}{s^{n-1}} \Big[\sum_{i=1}^{k} s^i \sum_{j=n}^{n+p-1} s^j \frac{d\big(z_{i-1}, z_i\big)}{s^{2j}} \Big] \\ &\leq \frac{1}{s^{n-1}} \Big[\sum_{i=1}^{k} s^i d(z_{i-1}, z_i) \sum_{j=n}^{n+p-1} \frac{1}{s^j} \Big]. \end{aligned}$$

Since $\sum_{j=0}^{\infty} \dfrac{1}{s^j} = \frac{s}{s-1}$, we get $\lim_{n\to\infty} d\big(f^n(x_0), f^{n+p}(x_0)\big) = 0$. This is $\{f^n(x_0)\}$ is a Cauchy sequence. □

Theorem 2.7. *Let (X, d) be a complete b-metric space with parameter $s > 1$ and f be a weakly b-(φ, G) contraction. Assume G is a (C_f)-graph* *and there is $z_0 \in X$ for which $(z_0, f(z_0)) \in E(\tilde{G})$. Then f has a unique fixed point $x^* \in [z_0]_{\tilde{G}}$ and $f^n(y) \to x^*$ for any $y \in [z_0]_{\tilde{G}}$. Also if G is a weakly connected, then f is Picard operator.*

Proof. From Proposition 2.6, $\{f^n(z_0)\}$ is a Cauchy sequence in X. Since X is complete there exists $x^* \in X$ such that $\lim_{n\to\infty} f^n(z_0) = x^*$. Since G is a (C_f) graph and $(f^n(z_0), f^{n+1}(z_0)) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{f^{n_k}(z_0)\}$ of $\{f^n(z_0)\}$ and $p \in \mathbb{N}$ such that $(f^{n_j}(z_0), x^*) \in E(G)$ for all $j \geq p$. Therefore

$$\big(z_0, f(z_0), f^2(z_0), ..., f^{n_1}(z_0), ..., f^{n_p}(z_0), x^*\big)$$

is a path in $\tilde{G}$. This implies that $x^* \in [z_0]_{\tilde{G}}$. Using (2.2) we have

$$\begin{aligned} d(f^{n_j+1}(z_0), f(x^*)) &\leq \frac{d(f^{n_j}(z_0), x^*)}{s^2} - \varphi\big(d(f^{n_j}(z_0), x^*)\big) \\ &\leq d(f^{n_j}(z_0), x^*). \end{aligned}$$

Passing to limit when $j \to \infty$, we obtain $f^{n_j+1}(z_0) \to f(x^*)$. Since $\lim_{n\to\infty} f^n(z_0) = x^*$ and $\{f^{n_j}(z_0)\}$ is a subsequence of $\{f^n(z_0)\}$, Thus $f(x^*) = x^*$. Now let $y \in [z_0]_{\tilde{G}}$. Then by Lemma 2.5, we have

$$\lim_{n\to\infty} d\big(f^n(y), f^n(z_0)\big) = 0. \tag{2.7}$$

By b-triangular inequality, we have

$$d(f^n(y), x^*) \leq s\big(d(f^n(y), f^n(z_0)) + d(f^n(z_0), x^*)\big).$$

From (2.7) and passing to limit when $n \to \infty$, we obtain $\lim_{n\to\infty} f^n(y) = x^*$. To prove the uniqueness of the fixed point, suppose that y^* is another fixed point of f. By b-triangular inequality, we obtain

$$d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leq s\big(d(f^n(x^*), f^n(z_0)) + d(f^n(y^*), f^n(z_0))\big),$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we have $x^* = y^*$. Then f has a unique fixed point. □

Theorem 2.8. *Let (X, d) be a complete b-metric space with parameter $s > 1$ and f be a weakly b-(φ, G) contraction. Suppose G is a connected weakly (H_f)-graph and also there is $z_0 \in X$ such that $(z_0, f(z_0)) \in E(\tilde{G})$. Then f has a unique fixed point $x^* \in X$ and $f^n(y) \to x^*$ for $y \in X$.*

Proof. By Proposition 2.6, the sequence $\{f^n(z_0)\}$ is Cauchy in X and since X is complete, there exists $z^* \in X$ such that $\lim_{n\to\infty} f^n(z_0) = z^*$. Since G is (H_f)-graph, we have

$$\lim_{n\to\infty} r\big(f^n(z_0), z^*\big) = 0.$$

Let $\{x_i^n\}$ be a path from $f^n(z_0)$ to z^* in $\tilde{G}$ for all $n \in \mathbb{N}$ and $i = 0, 1, ..., K_n$. Now, we prove that z^* is a fixed point of f. Using (2.2) and b-triangular inequality, we have

$$\begin{aligned} d(z^*, f(z^*)) &\leq s\big(d(z^*, f^{n+1}(z_0)) + d(f^{n+1}(z_0), f(z^*))\big) \\ &\leq s\big(d(z^*, f^{n+1}(z_0)) + \sum_{i=1}^{K_n} s^i d\big(f(x_{i-1}^n), f(x_i^n)\big)\big) \\ &\leq s\big(d(z^*, f^{n+1}(z_0)) + \sum_{i=1}^{K_n} s^i \big(\frac{d(x_{i-1}^n, x_i^n)}{s^2} - \varphi(d(x_{i-1}^n, x_i^n))\big)\big) \\ &\leq s\big(d(z^*, f^{n+1}(z_0)) + \sum_{i=1}^{K_n} s^i d(x_{i-1}^n, x_i^n)\big) \\ &= s\big(d(z^*, f^{n+1}(z_0)) + r(f^n(z_0), z^*)\big). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we obtain $f(z^*) = z^*$. Now, let $y \in [z_0]_{\tilde{G}} = X$ be arbitrary. Using lemma 2.5 and b-triangular inequality, we have $f^n(y) \to x^*$. □

References

[1] I. A. Bakhtin, The contraction mapping principle in almost metric space, *Functional analysis [Ulyanovsk. Gos. Ped. Inst., Ulyanovsk],* **30***(1989), 26-37.*

[2] B. S. Choudhury and D. Khatua and N. Metiya and P. Saha, End point results of generalized setvalued almost contractions in metric spaces endowed with a graph, *Bol. Soc. Mat. Mex.* **26***(1) (2020), 211-222.*

[3] R. Espinola and W.A. Kirk, Fixed point theorems in R-trees with applications to graph theory, *Topology Appl,* **153***(2006), 1046-1055.*

[4] H. Faraji and K. Nourouzi and D. O'Regan, A fixed point theorem in uniform spaces generated by a family of b-pseudometrics, *Fixed Point Theory,* **20***(1)(2019), 177-183.*

[5] G. Gwozdz-Lukawska and J. Jachymski, IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem, *J. Math. Anal. Appl,* **356***(2)(2009), 453-463.*

[6] J. Jachymski, The Contraction Principal for mappings on a metric space with a graph, *Proc. Amer. Math. Soc,* **36***(4)(2008), 1359-1373.*

[7] M. A. Khamsi and N. Hussain, KKM mappings in metric type spaces, *Nonlinear Anal,* **73***(9)(2010), 3123-3129.*

[8] M.S. Khan and M. Swaleh and S. Sessa, Fixed point theorems by altering distances between the points, *Bull. Austral. Math. Soc,* **30***(1)(1984), 1-9.*

[9] W. Kirk and N. Shahzad, Fixed point theory in distance spaces, Springer, New York, 2014.