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Application of spectral methods to solve nonlinear buckling analysis of an elastic beam

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Abstract. This study is concerned with exploiting the spectral method to solve the fourth-order boundary value problem (BVP). Such equations frequently arise in the study and modeling of large amplitude transverse buckling in an elastic beam. To this end, the properties of shifted Legendre polynomial together with its operational matrix of the derivative and the spectral method is utilized to reduce BVP to a system of algebraic equations. Numerical results turn out the efficiency and accuracy of the propounded technique.

Keywords: Hinged Beam, Spectral method, Nonlinear fourth order boundary value problem, Shifted Legendre polynomials, Operational matrix of derivative.

2000 Mathematics subject classification: 4B16, 34B40, 65M70.

1. Introduction

This paper has been devoted to finding a solution of the following nonlinear BVP for fourth-order differential equations:

$$y^{(4)}(x) - \varepsilon y^{''}(x) - \frac{2}{\pi}\left(\int_0^{\pi}(y^{'})^2 dx\right) y^{''} = p(x), \quad 0 \le x \le \pi, \qquad (1.1)$$

where ε is a constant, $p(x)$ is a continuous, nonpositive or nonnegative function on the interval $[0, \pi]$. For definiteness, we will assume that

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$p(x) \leq 0$ for all $x \in [0, \pi]$. The equation (1.1) arises in the study of transverse vibrations of an elastic beam, where the significant difficulty lies in a nonlinear term under the integral sign. For a detailed overview of this applied subject, one may refer to [10]. The boundary conditions that are often imposed are

$$y(0) = y(\pi) = y^{'}(0) = y^{'}(\pi) = 0, \tag{1.2}$$

which correspond to both ends being clamped, and

$$y(0) = y(\pi) = y^{''}(0) = y^{''}(\pi) = 0, \tag{1.3}$$

that correspond to hinged ends when there is no bending moment at the ends. In the present paper, we consider the boundary conditions in equation (1.3).

In recent years, several authors considered finite approximations of the problem and proposed iterative schemes for solving the system of nonlinear equations obtained. A finite-element approximation based on the space of piecewise linear polynomials on a uniform grid is represented in [11]. An error estimate for the approximation is also given without any numerical results. In [6], Dang and Luan according to the proposed method by Shin in [11] reduce the fourth-order boundary value problem to two second-order boundary value problems and applied the Newton-type iterative method to obtain an approximate solution.

In this vein, we propounded two efficient spectral approaches to obtain a rough solution of BVP. Among numerical methods, spectral methods are very powerful tools to approximate the solutions of many kinds of equations which are raised in various fields of science and engineering [4, 2] and spectral methods such as Galerkin, tau and pseudo-spectral methods [5, 3] are based on the solution of differential equation as a sum of certain basis functions. In what follows, first, the unknown function y is expanded in terms of shifted Legendre polynomials with unknown coefficients. Afterward, by applying collocation and Galerkin methods and properties of shifted Legendre polynomials together with utilizing the operational matrix of derivative, we reach a system of algebraic equations. In the end, computations can be handled simply way and the unknown coefficients will be obtained by solving algebraic equations.

The rest of the paper is organized as follows. In section 2, we describe the basic formulation of shifted Legendre polynomial and its operational matrix of derivative for our subsequent development. Section 3 is devoted to spectral methods based on shifted Legendre polynomials to obtain a approximate solution of the BVP. In section 4, we report our numerical findings and demonstrate the accuracy of the proposed scheme by considering a numerical example. Finally, Section 5 ends this paper with a brief conclusion.

2. Reviewing some attributes of shifted Legendre polynomials

In this section, we recall some properties and concepts for our subsequent development. The Legendre polynomials $P_i(x)$ are the eigenfunctions of the singular Sturm-Liouville problem [5]

$$((1-x^2)L_i^{'}(x))^{'} + i(i+1)L_i(x) = 0. \quad i = 0, 1, 2, \cdots . \tag{2.1}$$

These polynomials on the interval $[-1, 1]$ are defined by the following recursive formula [1]

$$L_{m+1}(t) = \frac{2m+1}{m+1} t \; L_m(t) - \frac{m}{m+1} L_{m-1}(t), \quad m = 1, 2, 3, \cdots , \tag{2.2}$$

$$L_0(t) = 1, \quad L_1(t) = t,$$

also, they are orthogonal with respect to L^2 inner product on the interval $[-1, 1]$ with the weight function $w(x) = 1$

$$\int_{-1}^{1} L_i(x)L_j(x)dx = \frac{2}{2i+1}\delta_{ij}, \tag{2.3}$$

where δ_{ij} is the Kronecker delta. To operate the Legendre polynomials on an arbitrary interval $[a, b]$, we define the so-called shifted Legendre polynomials by using the following change of variable

$$t = \frac{2(x-a)-h}{h}, \quad a \leq x \leq b, \tag{2.4}$$

where $h = b-a$. The shifted Legendre polynomials in x are then defined by

$$\phi_0(x) = 1, \quad \phi_1(x) = \frac{2(x-a)-h}{h}, \tag{2.5}$$

and for $m = 1, 2, 3, \cdots$,

$$\phi_{m+1}(x) = \frac{2m+1}{h(m+1)}(2(x-a)-h)\phi_m(x) - \frac{m}{m+1}\phi_{m-1}(x). \tag{2.6}$$

Let us assume that $\psi_i = \sqrt{\frac{2i+1}{h}}\phi_i$. Straightforward computations show that

$$\int_{a}^{b} \psi_i(x)\psi_j(x)dx = \begin{cases} 1 & for \;\; i = j, \\ 0 & for \;\; i \neq j. \end{cases} \tag{2.7}$$

The analytical form of shifted Legendre polynomials in the interval $[a, b]$ is [8]

$$\psi_i(x) = \sqrt{\frac{2i+1}{h}} \sum_{k=0}^{i} \frac{(-1)^{i+k}(i+k)!(x-a)^k}{(i-k)!(k!)^2 h^k}, \quad i = 0, \cdots , M. \tag{2.8}$$

The unknown function $y(x)$ can be estimated by shifted Legendre polynomials as a form of

$$y(x) = \sum_{i=0}^{M} y_i\psi_i(x) = Y^T\psi(x), \tag{2.9}$$

where

$$Y = [y_0, y_1, \cdots, y_M]^T, \qquad \psi = [\psi_0, \psi_1, \cdots, \psi_M]^T. \tag{2.10}$$

Theorem 2.1. [9] *Let $y(t) \in H^k(-1,1)$ (Sobolev space) and $\sum_{i=0}^{M} y_i\psi_i(x)$ be the best approximation polynomial of $y(t)$ in L_2-norm. Then*

$$\|y(x) - \sum_{i=0}^{M} y_i\psi_i(x)\|_{L_2[-1,1]} \leq C_0 M^{-k}\|y(x)\|_{H^k(-1,1)},$$

where C_0 is a positive constant, which depends on the selected norm and is independent of $y(x)$ and M.

Remark 1. The computational interval can be transformed from $[-1,1]$ to $[a,b]$ via an affine transformation.

2.1. **The operational matrix of the derivative.** Recently, operational matrix approaches have received considerable attention due to their advantageous properties. Several authors have elaborated them for solving various kinds of differential or integral equations (see for example [12] and the references therein). Among them, operational matrices of integration and derivative are the most prominent matrices. The main strategies is that the integral or derivative operator will be replaced by the related matrix, so the main problem is converted to a system of algebraic equations. Especially in numerical solution of nonlinear equations, as there is no need to use any approximation to eliminate the differential part, one prefer to use the operational matrix of the derivative. For the vector ψ in equation (2.10), the operational matrix of the derivative is defined by

$$\frac{d\psi}{dt} = D\psi, \tag{2.11}$$

where $D \in \mathbb{R}^{(M+1)\times(M+1)}$.
For Legendre polynomials, straightforward computations on (2.8) regarding (2.11) show that each element of D, d_{ij} for $i, j = 1, \ldots, M+1$, is given by

$$d_{ij} = (\frac{1}{\pi})\sqrt{2i+1}\sqrt{2j+1}\sum_{k=0}^{i}\sum_{l=1}^{j}\frac{(-1)^{i+k+l+j}(i+k)!(j+l)!l}{(i-k)!(k!)^2(j-l)!(l!)^2(k+l)}.$$

This matrix for $M=5$ is obtained as follows

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2\sqrt{3}}{\pi} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{15}}{\pi} & 0 & 0 & 0 & 0 \\ \frac{2\sqrt{7}}{\pi} & 0 & \frac{2\sqrt{35}}{\pi} & 0 & 0 & 0 \\ 0 & \frac{6\sqrt{3}}{\pi} & 0 & \frac{6\sqrt{7}}{\pi} & 0 & 0 \\ \frac{2\sqrt{11}}{\pi} & 0 & \frac{2\sqrt{55}}{\pi} & 0 & \frac{6\sqrt{11}}{\pi} & 0 \end{bmatrix}. \tag{2.12}$$

3. Solution of BVP

This section, demonstrated how shifted Legendre polynomials can be employed to reduce BVP to a system of algebraic equations. In this regards, we approximate the unknown functions y, y', y'' and $y^{(4)}$ in equation (1.1) with shifted Legendre polynomials. So, we obtain

$$y(x) = Y^T\psi(x) = \sum_{i=0}^{M} y_i\psi_i(x), \;\; y'(x) = Y^T D\psi(x), \;\; y''(x) = Y^T D^2\psi(x),$$

$$y^{(4)} = Y^T D^4\psi(x), \tag{3.1}$$

where the unknown vectors Y and ψ are defined in equation (2.10), M is the order of the Legendre polynomial and D is the operational matrix of derivative defined in equation (2.11). To evaluate a rough solution for $y(x)$, we exploited the spectral method,

3.1. **Method I.** By substituting equations (3.1) in (1.1),

$$Y^T D^4\psi(x) - \varepsilon Y^T D^2\psi(x) - \frac{2}{\pi}\left(\int_0^{\pi} (Y^T D\psi(x))^2\right) Y^T D^2\psi(x) = p(x), \tag{3.2}$$

Also from the boundary conditions,

$$\begin{aligned} &y(0) = Y^T\psi(0) = 0, \quad y(\pi) = Y^T\psi(\pi) = 0, \\ &y''(0) = Y^T D^2\psi(0) = 0, \quad y''(\pi) = Y^T D^2\psi(\pi) = 0. \end{aligned} \tag{3.3}$$

To evaluate an approximate solution for $y(x)$ in (3.2), by collocating this equation at $M-3$, Chebyshev-Gauss-Lobatto nodes on the interval $[0,\pi]$. These $M-3$ nodes that we utilize as collocation nodes are specified as

$$t_i = \frac{\pi}{2}\left(cos(\frac{i\pi}{M}) + 1\right), \quad i = 3, \ldots, M-1,$$

which are in the interval $[0,\pi]$. These equations together with (3.3) generate $(M+1)$ nonlinear equations which can be solved by Newton's iterative method.

3.2. **Method II.** Here we applied an analogous strategy, assume that $p(x)$ approximated by Legendre polynomials as

$$p(x) = P^T \psi(x). \tag{3.4}$$

Substituting equation (3.1) and (3.4) in (1.1) yields

$$Y^T D^4 \psi(x) - \varepsilon Y^T D^2 \psi(x) - \frac{2}{\pi} \left(\int_0^\pi (Y^T D \psi(x))^2 \right) Y^T D^2 \psi(x) = P^T \psi(x). \tag{3.5}$$

By multiplying equation (3.5) in $\psi^T(x)$, integrating from 0 and π and using the orthogonality properties of the Legendre polynomials given in equation (2.7), equation (3.5) is simplified as

$$\mathcal{M} = \mathcal{N}, \tag{3.6}$$

which is a system of nonlinear equations in terms of y_i for $i = 0, \ldots, M$. The parameters $\mathcal{M}$ and $\mathcal{N}$ in equation (3.6) are defined as

$$\begin{aligned} \mathcal{M} &= Y^T D^4 - \varepsilon Y^T D^2 - \tfrac{2}{\pi}(Y^T D D^T Y) Y^T D^2, \\ \mathcal{N} &= P^T. \end{aligned} \tag{3.7}$$

We omit the last 4 rows of the vectors $\mathcal{M}$ and $\mathcal{N}$ in equations (3.7) to have $M - 3$ equations. These $M - 3$ equations are added up together with (3.3) to have the outcome of a nonlinear system of equations with $M+1$ equations and unknowns which can be solved by Newton's iterative method.

4. Numerical results

In this section, we demonstrate the applicability, efficiency, and accuracy of our proposed schemes by considering an example. This illustrative example is implemented in Mathematica 7. As in [6] and [7], we take $\varepsilon = 2$.

Example 1. In this example, we assume that $p(x) = -4sin(x)$. It can be verified that the exact solution of this example is $y(x) = -sin(x)$. Table 1 and Figures 1 and 2 exhibit numerical results of this example by using different values of M. In table 1, our proposed methods are also compared with the method of [11] on the uniform grid with the number of nodes $N = 80$. This method is an iterative method of second-order approximation. In table 1, y_k shows the approximate solution in iteration k. Also, we define

$$E^y = y_{exact} - y_{approx}.$$

Now, we state what we have seen from numerical results as follows:

M	8	10	12	14
$Method\ I\|E^y\|_\infty$	9.63542×10^{-6}	8.7486×10^{-08}	5.0395×10^{-10}	2.0508×10^{-12}
$Method\ II\|E^y\|_\infty$	9.68287×10^{-8}	1.2102×10^{-09}	1.7820×10^{-11}	6.0713×10^{-12}
k[11]	12	23	29	
$\|y - y_k\|_\infty$	1.5200×10^{-04}	1.2852×10^{-04}	1.285177×10^{-04}	

TABLE 1. Numerical results of example 1

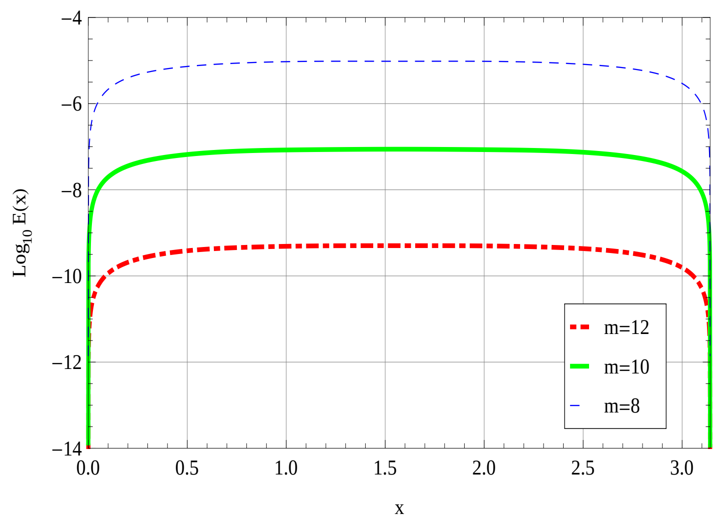

FIGURE 1. Numerical results for Method I ($M = 8, M = 10, M = 12$).

- ⧫ As seen in numerical findings, the proposed approaches with a little amount of M have good and accuracy results.
- ⧫ One of the significant advantages of utilizing the derivative operational matrix is that the matrix has large numbers of zero elements. Hence, the presented method is very attractive and reduces the CPU time and computer memory at the same time keeping the accuracy of the solution.
- ⧫ The propounded approaches lead to rapid convergence as the order of Legendre polynomials increases.

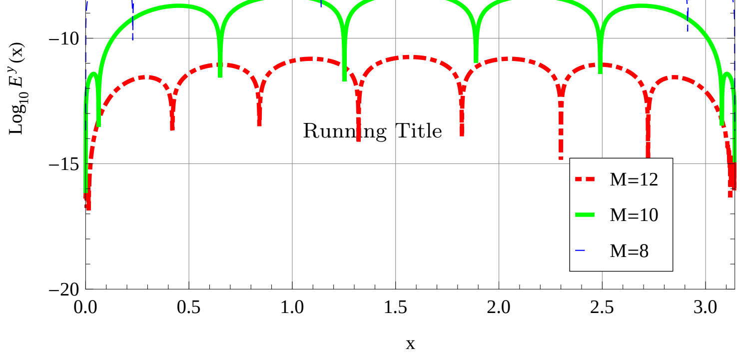

FIGURE 2. `Numerical results for Method II` $(M = 8, M = 10, M = 12)$.

5. Conclusions

The main aim of the current paper is to represent two spectral schemes for solving the nonlinear fourth order BVP. This equation arises in the study of transverse vibrations of a hinged beam. Our propounded approaches are based upon parameterizing the unknown function y, then by utilizing the spectral methods, the considered BVP can be converted into a system of algebraic equations which can be solved by Newton's iterative method. In the presented numerical simulation, the sparse operational matrix of the derivative has been made quick and efficient methods. The numerical results indicate that the current approaches are very effective, accurate, and easy to implement for obtaining the numerical solution of considered BVP.

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