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On a class of Kirchhoff type systems with singular nonlinearity

Saleh Shakeri[1] [2]

[1] Department of Mathematics, Ayatollah Amoli Branch, Islamic Azad University , Amol, Iran

Abstract. Using the method of sub-super solutions, we study the existence of positive solutions for a class of singular nonlinear semipositone systems involving nonlocal operator.

Keywords: Sub-Supersolution, Infinite semipositone systems, Singular weights, Kirchhoff-type.

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1. Introduction

We study the existence of positive solutions to the singular infinite semipositone system

$$\begin{cases} -M_1\Big(\int_\Omega |\nabla u|^p dx\Big) div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) \\ = |x|^{-(\alpha+1)p+c_1}(a_1 u^{p-1} - f_1(v) - \dfrac{b_1}{u^{\gamma_1}}), & x\in\Omega, \\ -M_2\Big(\int_\Omega |\nabla v|^q dx\Big) div(|x|^{-\beta q}|\nabla v|^{q-2}\nabla v) \\ = |x|^{-(\beta+1)q+c_2}(a_2 v^{q-1} - f_2(u) - \dfrac{b_2}{v^{\gamma_2}}), & x\in\Omega, \\ u = v = 0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

[1]Corresponding author: gurdalmehmet@sdu.edu.tr
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[2]Saleh Shakeri :

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$ with $0 \in \Omega$, $1 < p, q < N$, $0 \leq \alpha < \dfrac{N-p}{p}$, $0 \leq \beta < \dfrac{N-q}{q}$, $\gamma_1, \gamma_2 \in (0,1)$, and $a_1, a_2, b_1, b_2, c_1, c_2$ are positive constants and
$f_i : [0,\infty) \to \mathbb{R}$, $i = 1,2$, are continuous functions and $M_i : [0,\infty] \to \mathbb{R}^+$, $i = 1,2$, aside from being continuous and nondecreasing functions and $0 < M_{i,0} \leq M_i(t) \leq M_{i,\infty}$ for all $t \in [0,\infty)$, verify:

(H) There exist $t_2 > t_1 > 0$ such that $\dfrac{M_i(t_2)}{t_2^{\frac{2}{N-2}}} > \dfrac{M_i(t_1)}{t_1^{\frac{2}{N-2}}}$, see ([10]).

A typical example of a function satisfying this condition is $M_i(t) = M_{i,0} + at$,$i = 1,2$ with $a \geq 0$ and for all $t \geq 0$. We make the following assumptions:
(A1) There exist $L > 0$ and $b > 1$ such that $f_i(u) < Lu^b$, for all $u \geq 0$ and $i = 1,2$.
(A2) There exists a constant $S^* > 0$ such that $\max\{a_1 u^{p-1} - f_1(v), a_2 v^{q-1} - f_2(u)\} < S^*$, for all $u, v \geq 0$.
A simple example of f_i satisfying these assumptions is $f_i(u) = u^b$,$i = 1,2$ for any $b > 1$.

System (1.1) is related to the stationary problem of a model introduced by Kirchhoff [12]. More precisely, Kirchhoff proposed a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 dx \Big) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, P_0, h, E are all constants. This equation extends the classical D'Alembert wave equation. A distinguishing feature of equation (1.2) is that the equation has a nonlocal coefficient $\dfrac{P_0}{h} + \dfrac{E}{2L} \displaystyle\int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 dx$ which depends on the average $\dfrac{1}{2L} \displaystyle\int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 dx$; hence the equation is no longer a pointwise identity. we refer to [19] for additional result on kirchhoff equations.In recent years ,there has been considerable progress on the study of nonlocal problems,(see [15, 17, 18]). Nonlocal problems can be used for modeling, for example, physical and biological systems for which u describes a process which depends on the average of itself, such as the population density. On the other hand, elliptic problems involving more general operator, such as the degenerate quasilinear elliptic operator given by $-div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u)$, were motivated by the following Caffarelli, Kohn and Nirenberg's inequality (see [4, 16, 21]).

The study of this type of problem is motivated by its various applications, for example, in fluid mechanics, in newtonian fluids, in flow through porous media and in glaciology (see [3], [7]). So, the study of positive solutions of singular elliptic problems has more practical meanings. Let $F(h,k) = a_1 h^{p-1} - f_1(k) - \frac{b_1}{h^{\gamma_1}}$, and $G(h,k) = a_2 k^{q-1} - f_2(h) - \frac{b_2}{k^{\gamma_2}}$. Then $\lim\limits_{(h,k)\to(0,0)} F(h,k) = \lim\limits_{(h,k)\to(0,0)} G(h,k) = -\infty$, and hence we refer to (1.1) as an infinite semipositone system. In [13] the authors discussed the single problem (1.1) when $M_1(t) \equiv 1$, $\alpha = 0$, $p = c_1 = 2$, and see [20] for the single equation case when $M_1(t) \equiv 1$. Here we focus on further extending the study in [20, 13] for infinities semipositone Kirchhoff type systems involving singularity. Our approach is based on the method of sub-supersolutions, see [5, 8].

2. Main result

In this paper, we denote by $W_0^{1,p}(\Omega, |x|^{-\alpha p})$, the completion of $C_0^\infty(\Omega)$, with respect to the norm $\|u\| = \left(\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx\right)^{\frac{1}{p}}$. To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -div(|x|^{-sr}|\nabla\phi|^{r-2}\nabla\phi) = \lambda|x|^{-(s+1)r+t}|\phi|^{r-2}\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

For $r = p$, $s = \alpha$ and $t = c_1$, let $\phi_{1,p}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of (2.1) such that $\phi_{1,p}(x) > 0$ in Ω and $\|\phi_{1,p}\|_\infty = 1$ and for $r = q$, $s = \beta$ and $t = c_2$, let $\phi_{1,q}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of (2.1) such that $\phi_{1,q}(x) > 0$ in Ω, and $\|\phi_{1,q}\|_\infty = 1$ (see [14, 22]). It can be shown that $\dfrac{\partial\phi_{1,r}}{\partial n} < 0$ on $\partial\Omega$ for $r = p, q$. Here n is the outward normal. We will also consider the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0(\Omega, |x|^{-\alpha p}) \times W_0(\Omega, |x|^{-\beta q})$ for the system

$$\begin{cases} -div(|x|^{-\alpha p}|\nabla\zeta_p|^{p-2}\nabla\zeta_p) = |x|^{-(\alpha+1)p+c_1}, & x \in \Omega, \\ -div(|x|^{-\beta q}|\nabla\zeta_q|^{q-2}\nabla\zeta_q) = |x|^{-(\beta+1)q+c_2}, & x \in \Omega, \\ \zeta_p = \zeta_q = 0, & x \in \partial\Omega, \end{cases}$$

to discuss our existence result. It is well known that $\zeta_r(x) > 0$ in Ω and $\dfrac{\partial\zeta_r(x)}{\partial n} < 0$ on $\partial\Omega$, for $r = p, q$ (see [14]).

A pair of nonnegative functions (ψ_1, ψ_2), (z_1, z_2) is called a subsolution and super-solution of (1.1) if they satisfy $(\psi_1, \psi_2) = (0, 0) =$

(z_1, z_2) on $\partial\Omega$ and

$$
\begin{aligned}
&M_1\Big(\int_\Omega |\nabla \psi_1|^p dx\Big)\int_\Omega |x|^{-\alpha p}|\nabla \psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx\\
&\le \int_\Omega |x|^{-(\alpha+1)p+c_1}(a_1\psi_1^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^{\gamma_1}})w dx,\\
&M_2\Big(\int_\Omega |\nabla \psi_2|^q dx\Big)\int_\Omega |x|^{-\beta q}|\nabla \psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx\\
&\le \int_\Omega |x|^{-(\beta+1)q+c_2}(a_2\psi_2^{q-1} - f_2(\psi_1) - \frac{b_2}{\psi_2^{\gamma_2}})w dx,\\
&M_1\Big(\int_\Omega |\nabla z_1|^p dx\Big)\int_\Omega |x|^{-\alpha p}|\nabla z_1|^{p-2}\nabla z_1\cdot\nabla w dx\\
&\ge \int_\Omega |x|^{-(\alpha+1)p+c_1}(a_1 z_1^{p-1} - f_1(z_2) - \frac{b_1}{z_1^{\gamma_1}})w dx,\\
&M_2\Big(\int_\Omega |\nabla z_2|^q dx\Big)\int_\Omega |x|^{-\beta q}|\nabla z_2|^{q-2}\nabla z_2\cdot\nabla w dx\\
&\ge \int_\Omega |x|^{-(\beta+1)q+c_2}(a_2 z_2^{q-1} - f_2(z_1) - \frac{b_2}{z_2^{\gamma_2}})w dx,
\end{aligned}
$$

for all $w \in W = \{w \in C_0^\infty(\Omega) \mid w \ge 0, x \in \Omega\}$.

A key role in our arguments will be played by the following auxiliary result. Its proof is similar to that presented in [6], the reader can consult further the papers [1, 2, 11].

Lemma 2.1. *Assume that $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and increasing, and there exists $m_0 > 0$ such that $M(t) \ge m_0$ for all $t \in \mathbb{R}_0^+$. If the functions $u, v \in W_0^{1,p}(\Omega, |x|^{-\alpha p})$ satisfy*

$$
\begin{aligned}
&M\Big(\int_\Omega |x|^{-\alpha p}|\nabla u|^p dx\Big)\int_\Omega |x|^{-\alpha p}|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi dx\\
&\le M\Big(\int_\Omega |x|^{-\alpha p}|\nabla v|^p dx\Big)\int_\Omega |x|^{-\alpha p}|\nabla v|^{p-2}\nabla v\cdot\nabla\varphi dx
\end{aligned}
$$

for all $\varphi \in W_0^{1,p}(\Omega, |x|^{-\alpha p})$, $\varphi \ge 0$, then $u \le v$ in Ω.

From Lemma 2.1 we can establish the basic principle of the sub-and supersolution method for nonlocal systems. Indeed, we consider the following nonlocal system

$$\begin{cases} -M_1\Big(\int_\Omega |\nabla u|^p dx\Big) div(|x|^{-\alpha p}|\nabla u|^{p-2}\nabla u) = |x|^{-(\alpha+1)p+c_1} h(x,u,v), & x\in\Omega,\\ -M_2\Big(\int_\Omega |\nabla v|^q dx\Big) div(|x|^{-\beta q}|\nabla v|^{q-2}\nabla v) = |x|^{-(\beta+1)q+c_2} k(x,u,v), & x\in\Omega,\\ u=v=0, & x\in\partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $h,k:\bar\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy the following conditions:
$(HK1)$ $h(x,s,t)$ and $k(x,s,t)$ are caratheodory functions and they are bounded if s,t belong to bounded sets.
$(HK2)$ There exists a function $g:\mathbb{R}\to\mathbb{R}$ being continuous, nondecreasing, with $g(0)=0$, $0\le g(s)\le c(1+|s|^{\min\{p,q\}})$ for some $c>0$, and applications $s\mapsto h(x,s,t)+g(s)$ and $t\mapsto k(x,s,t)+g(t)$ are nondecreasing, for a.e $x\in\Omega$.
If $u,v\in L^\infty(\Omega)$, with $u(x)\le v(x)$ for a.e $x\in\Omega$, we denote by $[u,v]$ the set $\{w\in L^\infty(\Omega): u(x)\le w(x)\le v(x)$ for a.e $x\in\Omega\}$. Using Lemma 2.1 and the method as in the Proof of Theorem 2.4 of [14] *(see also section 4 of* [5]*), we can establish a version of the abstract lower and upper-solution method for our class of the operators as follows.*

Proposition 2.2. *Let $M_i:\mathbb{R}_0^+\to\mathbb{R}^+$, $i=1,2$, are two continuous and increasing functions $0<M_i\le M_i(t)\le M_{i,\infty}$ for all $t\in\mathbb{R}^+$. Assume that the functions h,k satisfy the conditions (HK_1) and (HK_2). Assume that $(\underline{u},\underline{v})$, $(\overline{u},\overline{v})$ are respectively, a weak subsolution and a weak supersolution of system (2.2) with $\underline{u}(x)\le\overline{u}(x)$ and $\underline{v}(x)\le\overline{v}(x)$ for a.e $x\in\Omega$. Then there exist a minimal (u_*,v_*) (and, respectively, a maximal (u^*,v^*)) weak solution for system (2.2) in the set $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. In particular, every weak solution $(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ of system (2.2) satisfies $u_*(x)\le u(x)\le u^*(x)$ and $v_*(x)\le v(x)\le v^*(x)$ for a.e $x\in\Omega$.*

Theorem 2.3. *Assume if $a_1>M_{1,\infty}\Big(\dfrac{p}{p-1+\gamma_1}\Big)^{p-1}\lambda_{1,p}$, $a_2>M_{2,\infty}\Big(\dfrac{q}{q-1+\gamma_2}\Big)^{q-1}\lambda_{1,q}$, then there exists $c>0$ such that if $\max\{b_1,b_2\}\le c$, then the system (1.1) admits a positive solution.*

Proof. We start with the construction of a positive subsolution for (1.1). To get a positive subsolution, we can apply an anti-maximum principle (see [9]), from which we know that there exist a $\delta_1>0$ and a solution z_λ of

$$\begin{cases} -div(|x|^{-sr}|\nabla z|^{r-2}\nabla z) = |x|^{-(s+1)r+t}(\lambda z^{r-1}-1), & x\in\Omega,\\ z=0 & x\in\partial\Omega, \end{cases} \tag{2.3}$$

for $\lambda \in (\lambda_{1,r}, \lambda_{1,r} + \delta_1)$, for $r = p, q$, $s = \alpha, \beta$ and $t = c_1, c_2$.
Fix $\hat{\lambda}_1 \in \Big(\lambda_{1,p}, \min\Big\{(\frac{p-1+\gamma_1}{p})^{p-1}a_1, \lambda_{1,p}+\delta_1\Big\}\Big)$ and

$$\hat{\lambda}_2 \in \Big(\lambda_{1,q}, \min\Big\{(\frac{q-1+\gamma_2}{q})^{q-1}a_2, \lambda_{1,q}+\delta_1\Big\}\Big).$$

Let $\theta_i = \|z_{\hat{\lambda}_i}\|$ for $i = 1, 2$. It is well known that $z_{\hat{\lambda}_1}, z_{\hat{\lambda}_2} > 0$ in Ω and $\frac{\partial z_{\hat{\lambda}_1}}{\partial n}, \frac{\partial z_{\hat{\lambda}_2}}{\partial n} < 0$ on $\partial\Omega$, where n is the outer unit normal to Ω. Hence there exist positive constants $\epsilon, \delta, \sigma_p, \sigma_q$ such that

$$|x|^{-sr}|\nabla z_{\hat{\lambda}_i}|^r \geq \epsilon, \quad x \in \overline{\Omega_\delta}, \tag{2.4}$$

$$z_{\hat{\lambda}_i} \geq \sigma_r, \quad x \in \Omega_0 = \Omega \backslash \overline{\Omega_\delta}, \tag{2.5}$$

with $r = p, q$; $s = \alpha, \beta$; $i = 1, 2$ and $\overline{\Omega_\delta} = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. Choose $\eta_1, \eta_2 > 0$ such that $\eta_1 \leq \min|x|^{-(s+1)r+t}$, and $\eta_2 \geq \max|x|^{-(s+1)r+t}$, in $\overline{\Omega}_\delta$, for $r = p, q$, $s = \alpha, \beta$ and $t = c_1, c_2$. We construct a subsolution (ψ_1, ψ_2) of (1.1) using $z_{\hat{\lambda}_1}, z_{\hat{\lambda}_2}$. Define $(\psi_1, \psi_2) = \Big(M(\frac{p-1+\gamma_1}{p})z_{\hat{\lambda}_1}^{\frac{p}{p-1+\gamma_1}}, M(\frac{q-1+\gamma_2}{q})z_{\hat{\lambda}_2}^{\frac{q}{q-1+\gamma_2}}\Big)$, where

$$M = \min\left\{ \left(\frac{M_{1,\infty}\left(\frac{q}{q-1+\gamma_2}\right)^b \theta_1^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}}{L\theta_2^{\frac{qb}{q-1+\gamma_2}}} \right)^{\frac{1}{b-p+1}}, \right.$$

$$\left(\frac{M_{2,\infty}\left(\frac{p}{p-1+\gamma_1}\right)^b \theta_2^{\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}}}{L\theta_1^{\frac{pb}{p-1+\gamma_1}}} \right)^{\frac{1}{b-q+1}},$$

$$\left(\frac{\left(\frac{p-1}{Lp}\right)\theta_1^{\frac{p(p-1)}{p-1+\gamma_1}}\left[\left(\frac{p-1+\gamma_1}{p}\right)^{p-1}a_1 - M_{1,\infty}\hat{\lambda_1}\right]}{\left(\frac{q-1+\gamma_2}{q}\right)^b\theta_2^{\frac{qb}{q-1+\gamma_2}}} \right)^{\frac{1}{b-p+1}},$$

$$\left.\left(\frac{\left(\frac{q-1}{Lq}\right)\theta_2^{\frac{q(q-1)}{q-1+\gamma_2}}\left[\left(\frac{q-1+\gamma_2}{q}\right)^{q-1}a_2 - M_{2,\infty}\hat{\lambda_2}\right]}{\left(\frac{p-1+\gamma_1}{p}\right)^b\theta_1^{\frac{pb}{p-1+\gamma_1}}} \right)^{\frac{1}{b-q+1}} \right\}.$$

Let $w \in W$. Then a calculation shows that

$$\nabla\psi_1 = Mz_{\hat{\lambda}_1}^{\frac{1-\gamma_1}{p-1+\gamma_1}}\nabla z_{\hat{\lambda}_1},$$

$$
\begin{aligned}
&M_1\Big(\int_\Omega |\nabla\psi_1|^p dx\Big)\int_\Omega |x|^{-\alpha p}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx\\
&\le M_{1,\infty}M^{p-1}\int_\Omega |x|^{-\alpha p} z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\lambda_1}}|\nabla z_{\hat{\lambda_1}}|^{p-2}\nabla z_{\hat{\lambda_1}}\nabla w dx\\
&= M_{1,\infty}M^{p-1}\int_\Omega |x|^{-\alpha p}|\nabla z_{\hat{\lambda_1}}|^{p-2}\nabla z_{\hat{\lambda_1}}\left[\nabla\Big(z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}w\Big)-\Big(\nabla z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}\Big)w\right]dx\\
&= M_{1,\infty}M^{p-1}\int_\Omega \Big[|x|^{-(\alpha+1)p+c_1} z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}(\hat{\lambda_1}z_{\hat{\lambda_1}}^{p-1}-1)\\
&-|x|^{-\alpha p}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda_1}}|^p}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx\\
&= M_{1,\infty}\int_\Omega \Big[|x|^{-(\alpha+1)p+c_1}M^{p-1}\hat{\lambda_1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}-|x|^{-(\alpha+1)p+c_1}M^{p-1}z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}\\
&-|x|^{-\alpha p}M^{p-1}\tfrac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda_1}}|^p}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx,
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
&\int_\Omega |x|^{-(\alpha+1)p+c_1}\Big[a_1\psi_1^{p-1}-f_1(\psi_2)-\frac{b_1}{\psi_1^{\gamma_1}}\Big]wdx=\\
&\int_\Omega \Big[|x|^{-(\alpha+1)p+c_1}a_1M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}\\
&-|x|^{-(\alpha+1)p+c_1}f_1\Big(M(\frac{q-1+\gamma_2}{q})z_{\hat{\lambda_2}}^{\frac{q}{q-1+\gamma_2}}\Big)\\
&-|x|^{-(\alpha+1)p+c_1}\frac{b_1}{M^{\gamma_1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{\gamma_1}z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx.
\end{aligned}
\tag{2.7}
$$

Similarly

$$
\begin{aligned}
&M_2\Big(\int_\Omega |\nabla\psi_2|^q dx\Big)\int_\Omega |x|^{-\beta q}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla w dx\\
&\le M_{2,\infty}\textstyle\int_\Omega \Big[|x|^{-(\beta+1)q+c_2}M^{q-1}\hat{\lambda_2}z_{\hat{\lambda_2}}^{\frac{q(q-1)}{q-1+\gamma_2}}-\\
&|x|^{-(\beta+1)q+c_2}M^{q-1}z_{\hat{\lambda_2}}^{\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}}-|x|^{-\beta q}M^{q-1}\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}\frac{|\nabla z_{\hat{\lambda_2}}|^q}{z_{\hat{\lambda_2}}^{\frac{\gamma_2 q}{q-1+\gamma_2}}}\Big]wdx,
\end{aligned}
\tag{2.8}
$$

and

$$\begin{aligned}
&\int_\Omega |x|^{-(\beta+1)q+c_2}\Big[a_2\psi_2^{q-1} - f_2(\psi_1) - \frac{b_2}{\psi_2^{\gamma_2}}\Big]wdx \\
&= \int_\Omega \Big[|x|^{-(\beta+1)q+c_2}a_2M^{q-1}\Big(\tfrac{q-1+\gamma_2}{q}\Big)^{q-1} z_{\hat{\lambda_2}}^{\frac{q(q-1)}{q-1+\gamma_2}} \\
&-|x|^{-(\beta+1)q+c_2} f_2\Big(M\Big(\frac{p-1+\gamma}{p}\Big)z_{\hat{\lambda_1}}^{\frac{p}{p-1+\gamma_1}}\Big) \\
&-|x|^{-(\beta+1)q+c_2}\frac{b_2}{M^{\gamma_2}\Big(\frac{q-1+\gamma_2}{q}\Big)^{\gamma_2} z_{\hat{\lambda_2}}^{\frac{\gamma_2 q}{q-1+\gamma_2}}}\Big]wdx
\end{aligned} \tag{2.9}$$

Let $c = \min\Big\{M_{1,\infty}M^{p-1+\gamma_1}\dfrac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\Big(\dfrac{p-1+\gamma_1}{p}\Big)^{\gamma_1}\dfrac{\epsilon}{\eta_2},$
$M_{2,\infty}M^{q-1+\gamma_2}\frac{(1-\gamma_2)(q-1)}{q-1+\gamma_2}\Big(\frac{q-1+\gamma_2}{q}\Big)^{\gamma_2}\frac{\epsilon}{\eta_2},$
$\dfrac{M^{p-1+\gamma_1}}{p}\Big(\dfrac{p-1+\gamma_1}{p}\Big)^{\gamma_1}\sigma_p^p\Big[\Big(\dfrac{p-1+\gamma_1}{p}\Big)^{p-1}a_1 - M_{1,\infty}\hat{\lambda_1}\Big],$
$\frac{M^{q-1+\gamma_2}}{q}\Big(\frac{q-1+\gamma_2}{q}\Big)^{\gamma_2}\sigma_q^q\Big[\Big(\frac{q-1+\gamma_2}{q}\Big)^{q-1}a_2 - M_{2,\infty}\hat{\lambda_2}\Big]\Big\}.$

First we consider the case when $x\in\overline{\Omega}_\delta$. We have $|x|^{-\alpha p}|\nabla\phi_{1,p}|\geq\epsilon$ on $\overline{\Omega}_\delta$. Since $M_{1,\infty}\Big(\dfrac{p}{p-1+\gamma_1}\Big)^{p-1}\hat{\lambda_1}\leq a_1$, we have

$$\begin{aligned}
&|x|^{-(\alpha+1)p+c_1}M_{1,\infty}M^{p-1}\hat{\lambda_1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} \\
&\leq |x|^{-(\alpha+1)p+c_1}a_1M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}},
\end{aligned} \tag{2.10}$$

and from the choice of M, we know that

$$LM^{b-p+1}\theta_2^{\frac{qb}{q-1+\gamma_2}} \leq M_{1,\infty}\Big(\frac{q}{q-1+\gamma_2}\Big)^b\theta_1^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}. \tag{2.11}$$

By (2.11) and (A_1) we have $-|x|^{-(\alpha+1)p+c_1}M_{1,\infty}M^{p-1}z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}$
$\leq -|x|^{-(\alpha+1)p+c_1}LM^b\Big(\frac{q-1+\gamma_2}{q}\Big)^b z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}}$

$$\leq -|x|^{-(\alpha+1)p+c_1}f_1\Big(M\Big(\frac{q-1+\gamma_2}{q}\Big)z_{\hat{\lambda_2}}^{\frac{q}{q-1+\gamma_2}}\Big). \tag{2.12}$$

Next, from (2.4) and definition of c, we have

$$|x|^{-\alpha p}M_{1,\infty}M^{p-1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}|\nabla z_{\hat{\lambda_1}}|^p \geq |x|^{-(\alpha+1)p+c_1}\frac{b_1}{M^{\gamma_1}(\frac{p-1+\gamma_1}{p})^{\gamma_1}},$$

and

$$-|x|^{-\alpha p}M_{1,\infty}M^{p-1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda_1}}|^p}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}$$
$$\leq -|x|^{-(\alpha+1)p+c_1}\frac{b_1}{M^{\gamma_1}\left(\frac{p-1+\gamma_1}{p}\right)^{\gamma_1}z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}. \tag{2.13}$$

Hence by using (2.10) , (2.12) and (2.13) for $b_1 \leq c$, we have

$$M_1\Big(\int_{\bar{\Omega}}|\nabla\psi_1|^p dx\Big)\int_{\overline{\Omega}_\delta}|x|^{-\alpha p}|\nabla\psi_1|^{p-2}\nabla\psi_1\cdot\nabla w dx$$
$$\leq \int_{\overline{\Omega}_\delta}\Big[|x|^{-(\alpha+1)p+c_1}a_1M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}$$
$$-|x|^{-(\alpha+1)p+c_1}f_1\Big(M\Big(\frac{q-1+\gamma_2}{q}\Big)z_{\hat{\lambda_2}}^{\frac{q}{q-1+\gamma_2}}\Big)-$$
$$|x|^{-(\alpha+1)p+c_1}\frac{b_1}{M^{\gamma_1}\left(\frac{p-1+\gamma_1}{p}\right)^{\gamma_1}z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx$$
$$=\int_{\overline{\Omega}_\delta}|x|^{-(\alpha+1)p+c_1}\Big[a_1\psi_1^{p-1}-f_1(\psi_2)-\frac{b_1}{\psi_1^{\gamma_1}}\Big]wdx. \tag{2.14}$$

Similarly

$$M_2\Big(\int_{\overline{\Omega}_\delta}|\nabla\psi_2|^q dx\Big)\int_{\overline{\Omega}_\delta}|x|^{-\beta q}|\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w dx$$
$$\leq \int_{\overline{\Omega}_\delta}\Big[|x|^{-(\beta+1)q+c_2}a_2M^{q-1}\left(\frac{q-1+\gamma_2}{q}\right)^{q-1}z_{\hat{\lambda_2}}^{\frac{q(q-1)}{q-1+\gamma_2}}$$
$$-|x|^{-(\beta+1)q+c_2}f_2\Big(M\Big(\frac{p-1+\gamma_1}{p}\Big)z_{\hat{\lambda_1}}^{\frac{p}{p-1+\gamma_1}}\Big)-$$
$$|x|^{-(\beta+1)q+c_2}\frac{b_2}{M^{\gamma_2}\left(\frac{q-1+\gamma_2}{q}\right)^{\gamma_2}z_{\hat{\lambda_2}}^{\frac{\gamma_2 q}{q-1+\gamma_2}}}\Big]wdx$$
$$=\int_{\overline{\Omega}_\delta}|x|^{-(\beta+1)q+c_2}\Big[a_2\psi_2^{q-1}-f_2(\psi_1)-\frac{b_2}{\psi_2^{\gamma_2}}\Big]wdx. \tag{2.15}$$

On the other hand, on $\Omega_0=\Omega\backslash\overline{\Omega}_\delta$, we have $z_{\hat{\lambda_1}}\geq\sigma_p$ and $z_{\hat{\lambda_2}}\geq\sigma_q$, for some $0<\sigma_p,\sigma_q<1$, and from the definition of c, for $b_1\leq c$ we have

$$\frac{b_1}{M^{\gamma_1}\left(\frac{p-1+\gamma_1}{p}\right)^{\gamma_1}}\leq\frac{1}{p}M^{p-1}\sigma_p^p\Big[\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}a_1-M_{1,\infty}\hat{\lambda_1}\Big]$$
$$\leq\frac{1}{p}M^{p-1}z_{\hat{\lambda_1}}^p\Big[\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}a_1-M_{1,\infty}\hat{\lambda_1}\Big]. \tag{2.16}$$

Also from the choice of M, we have

$$LM^{b-p+1}\Big(\frac{q-1+\gamma_2}{q}\Big)^b z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}} \leq z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}\frac{p-1}{p}\Big[\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}a_1 - M_{1,\infty}\hat{\lambda_1}\Big]. \tag{2.17}$$

Hence from (2.16) and (2.17) we have

$$M_1\Big(\int_{\Omega_0}|\nabla\psi_1|^p dx\Big)\int_{\Omega_0}|x|^{-\alpha p}|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla w dx$$

$$\leq M_{1,\infty}\int_{\Omega_0}\Big[|x|^{-(\alpha+1)p+c_1}M^{p-1}\hat{\lambda_1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} - |x|^{-(\alpha+1)p+c_1}M^{p-1}z_{\hat{\lambda_1}}^{\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}}$$

$$-|x|^{-\alpha p}M^{p-1}\frac{(1-\gamma_1)(p-1)}{p-1+\gamma_1}\frac{|\nabla z_{\hat{\lambda_1}}|^p}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx$$

$$\leq M_{1,\infty}\int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}M^{p-1}\hat{\lambda_1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}wdx$$

$$= M_{1,\infty}\int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}\frac{1}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big[\frac{1}{p}\hat{\lambda_1}M^{p-1}z_{\hat{\lambda_1}}^p + \frac{p-1}{p}\hat{\lambda_1}M^{p-1}z_{\hat{\lambda_1}}^p\Big]wdx$$

$$\leq \int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}\frac{1}{z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big[\Big(\frac{1}{p}M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}a_1 z_{\hat{\lambda_1}}^p - \frac{b_1}{M^{\gamma_1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{\gamma_1}}\Big)+$$

$$M^{p-1}z_{\hat{\lambda_1}}^p\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}\Big(\frac{(p-1)a_1}{p}$$

$$-LM^{b-p+1}\Big(\frac{q-1+\gamma_2}{q}\Big)^b\Big(\frac{p-1+\gamma_1}{p}\Big)^{1-p}\frac{z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}}}{z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}}}\Big)\Big]wdx$$

$$= \int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}\Big[a_1M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} - LM^b\Big(\frac{q-1+\gamma_2}{q}\Big)^b z_{\hat{\lambda_2}}^{\frac{qb}{q-1+\gamma_2}} -$$

$$\frac{b_1 z_{\hat{\lambda_1}}^{\frac{-\gamma_1 p}{p-1+\gamma_1}}}{M^{\gamma_1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{\gamma_1}}\Big]wdx$$

$$\leq \int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}\Big[a_1M^{p-1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{p-1}z_{\hat{\lambda_1}}^{\frac{p(p-1)}{p-1+\gamma_1}} - f_1\Big(M\Big(\frac{q-1+\gamma_2}{q}\Big)z_{\hat{\lambda_2}}^{\frac{q}{q-1+\gamma_2}}\Big)$$

$$-\frac{b_1}{M^{\gamma_1}\Big(\frac{p-1+\gamma_1}{p}\Big)^{\gamma_1}z_{\hat{\lambda_1}}^{\frac{\gamma_1 p}{p-1+\gamma_1}}}\Big]wdx = \int_{\Omega_0}|x|^{-(\alpha+1)p+c_1}\Big[a_1\psi_1^{p-1} - f_1(\psi_2) - \frac{b_1}{\psi_1^{\gamma_1}}\Big]wdx. \tag{2.18}$$

Similarly

$$M_2\Big(\int_{\Omega_0}|\nabla\psi_2|^q dx\Big)\int_{\Omega_0}|x|^{-\beta q}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla w dx$$
$$\leq \int_{\Omega_0}|x|^{-(\beta+1)q+c_2}\Big[a_2\psi_2^{q-1}-f_2(\psi_1)-\frac{b_2}{\psi_2^{\gamma_2}}\Big]w dx. \tag{2.19}$$

By using (2.14), (2.15), (2.18) and (2.19) we see that (ψ_1,ψ_2) is a sub-solution of (1.1).

Next, we construct a super-solution (z_1,z_2) of (1.1) such that $(z_1,z_2)\geq(\psi_1,\psi_2)$. Let $(z_1,z_2)=\Big[\Big(\frac{S^*}{M_1}\Big)^{\frac{1}{p-1}}\zeta_p(x),\Big(\frac{S^*}{M_2}\Big)^{\frac{1}{q-1}}\zeta_q(x)\Big]$. By (A_2) and choose a large constant S^*, we shall verify that (z_1,z_2) is a super-solution of (1.1). To this end, let $w\in W$. Then we have

$$M_1\Big(\int_{\Omega}|\nabla z_1|^p dx\Big)\int_{\Omega}|x|^{-\alpha p}|\nabla z_1|^{p-2}\nabla z_1\nabla w dx\geq S^*\int_{\Omega}|x|^{-(\alpha+1)p+c_1}w dx$$
$$\geq\int_{\Omega}|x|^{-(\alpha+1)p+c_1}\Big[a_1z_1^{p-1}-f_1(z_2)-\frac{b_1}{z_1^{\gamma_1}}\Big]w dx. \tag{2.20}$$

Similarly,

$$M_2\Big(\int_{\Omega}|\nabla z_2|^q dx\Big)\int_{\Omega}|x|^{-\beta q}|\nabla z_2|^{q-2}\nabla z_2\nabla w dx$$
$$\geq\int_{\Omega}|x|^{-(\beta+1)q+c_2}\Big[a_2z_2^{q-1}-f_2(z_1)-\frac{b_2}{z_2^{\gamma_2}}\Big]w dx. \tag{2.21}$$

Thus (z_1,z_2) is a super-solution of (1.1). Finally, we can choose $S^*\gg 1$ such that $(\psi_1,\psi_2)\geq(z_1,z_2)$ in Ω. Hence, if $\max\{b_1,b_2\}\leq c$, by Lemma 2.1 there exists a positive solution (u,v) of (1.1) such that $(\psi_1,\psi_2)\leq(u,v)\leq(z_1,z_2)$. This completes the proof of Theorem 2.3. □

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Email address: s.shakeri@iauamol.ac.ir