

## Coefficient estimates and Fekete-Szegő coefficient inequality for new subclasses of Bi-univalent functions

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**ABSTRACT.** In this paper, we investigate on two new subclasses  $\mathcal{S}_\sigma^*(a, b)$  and  $\nu_\sigma(a, b)$  of  $\sigma$  consisting of analytic and bi-univalent functions satisfying subordinations in the open unit disk  $\mathbb{U}$ . We consider the Fekete-Szegő inequalities for these new subclasses. Also, we establish estimates for the coefficient for these subclasses.

**Keywords:** Bi-univalent functions, Coefficient estimates, Fekete-Szegő coefficient inequality.

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### 1. INTRODUCTION

Let  $\mathcal{A}$  denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disc  $\mathbb{U} = \{z | z \in \mathbb{C} : |z| < 1\}$ . Further, by  $\mathcal{S}$  we shall denote the class of all functions in  $\mathcal{A}$  which are univalent in  $\mathbb{U}$ . The Koebe one-quarter theorem [3] states that the image of  $\mathbb{U}$  under every function  $f$  from  $\mathcal{S}$  contains a disk of radius  $\frac{1}{4}$ . Thus every

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such univalent function has an inverse  $f^{-1}$  which satisfies

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}).$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}).$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

A function  $f \in A$  is said to be bi-univalent in  $\mathbb{U}$  if both  $f$  and  $f^{-1}$  are univalent in  $\mathbb{U}$ . Let  $\sigma$  denote the class of bi-univalent functions defined in the unit disk  $\mathbb{U}$ .

Although, the familiar Koebe function is not in the class of  $\sigma$ , there are some examples of functions member of  $\sigma$ , such as [8]

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2} \log\left(\frac{1+z}{1-z}\right)$$

so on. Other common examples of functions in  $\mathcal{S}$  for example

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of  $\sigma$ .

Let  $\Omega$  be the family functions  $\omega(z)$  in the unit disc  $\mathbb{U}$  satisfying the conditions

$\omega(0) = 0$ ,  $|\omega(z)| < 1$  for  $z \in \mathbb{U}$ . Note that  $f(z) \prec g(z)$  if there is a function  $\omega(z) \in \Omega$  such that  $f(z) = g(\omega(z))$ , (see [3]).

Recently, Srivastava et al.[8] and Frasian and Aouf [4] and Caglar et al. [2] have introduced and have investigated various subclasses of bi-univalent functions and found estimates on the coefficients  $|a_2|$  and  $|a_3|$  for functions in these classes. In this paper we introduce two subclasses  $\mathcal{S}_\sigma^*(a, b)$  and  $\nu_\sigma(a, b)$  of bi-univalent functions and estimates on the coefficients  $|a_2|$ ,  $|a_3|$  and  $|a_4|$  and also Fekete-Szegö coefficient inequality for functions in these subclasses are given.

**Definition 1.1.** A functions  $f$  given by 1.1 is said to be in class  $\mathcal{S}_\sigma^*(a, b)$ , if the following condition are satisfied:

$$f \in \sigma, \quad \left| \frac{zf'(z)}{f(z)} - a \right| < b, \quad |a-1| < b \leq a, \quad (1.2)$$

and

$$\left| \frac{wg'(w)}{g(w)} - a \right| < b, \quad |a-1| < b \leq a, \quad (1.3)$$

where  $w = f(z)$ ,  $g = f^{-1}$ ,  $w \in \Delta$  and  $z \in \Delta$ .

Also,

**Definition 1.2.** A functions  $f$  given by 1.1 is said to be in class  $\nu_\sigma(a, b)$ , if the following condition are satisfied:

$$f \in \sigma, \quad \left| \left( \frac{z}{f(z)} \right)^2 f'(z) - a \right| < b, \quad |a - 1| < b \leq a, \quad (1.4)$$

and

$$\left| \left( \frac{w}{g(w)} \right)^2 g'(w) - a \right| < b, \quad |a - 1| < b \leq a, \quad (1.5)$$

where  $w = f(z)$ ,  $g = f^{-1}$ ,  $w \in \Delta$  and  $z \in \Delta$ .

**Lemma 1.3.** (see[5] or [3]) If  $p \in \mathcal{P}$  then  $|p_k| \leq 2$  for for each  $k$  and  $|p_2 - \frac{1}{2}p_1^2| \leq 2 - \frac{1}{2}|p_1|^2$ , where  $\mathcal{P}$  is the family of all functions  $p$  analytic in  $\mathbb{U}$  for which  $\text{Re}p(z) > 0$ ,  $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots$  for  $z \in \mathbb{U}$ .

## 2. COEFFICIENTS ESTIMATES FOR THE FUNCTIONS IN $\mathcal{S}_\sigma^*(a, b)$ AND $\nu_\sigma(a, b)$

In the section by using subordination structure and definitions of  $\mathcal{S}_\sigma^*(a, b)$  and  $\nu_\sigma(a, b)$  the coefficients of functions in these classes are obtained.

**Theorem 2.1.** Let  $(a - b) = \beta$  and  $f(z)$  given by 1.1 be in the class  $\mathcal{S}_\sigma^*(a, b)$ ,  $|a - 1| < b \leq a$ . Then

$$|a_2| \leq \sqrt{2(1 - \beta)}, \quad (2.1)$$

$$|a_3| \leq (1 - \beta) + 4(1 - \beta)^2 \quad (2.2)$$

and

$$|a_4| \leq \frac{2(1 - \beta)}{3}(1 + 4(1 - \beta)). \quad (2.3)$$

*Proof.* It follows from 1.2 and 1.3 that

$$\frac{zf'(z)}{f(z)} \prec \beta + (1 - \beta)z$$

and

$$\frac{wg'(w)}{g(w)} \prec \beta + (1 - \beta)w$$

then

$$\frac{zf'(z)}{f(z)} = \beta + (1 - \beta)p(z) \quad (2.4)$$

and

$$\frac{wg'(w)}{g(w)} = \beta + (1 - \beta)q(w) \quad (2.5)$$

where  $p(z)$  and  $q(w)$  in  $\mathcal{P}$  and have the forms

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots \quad (2.6)$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \dots \quad (2.7)$$

where

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (2a_3 - a_2^2)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \dots$$

and

$$\frac{wg'(w)}{g(w)} = 1 - a_2w + (3a_2^2 - 2a_3)w^2 - (3a_4 - 12a_2a_3 + 10a_2^3)w^3 + \dots$$

Now, equating the coefficients in 2.4 and 2.5, we get

$$a_2 = p_1(1 - \beta), \quad (2.8)$$

$$2a_3 - a_2^2 = (1 - \beta)p_2, \quad (2.9)$$

$$2a_3 = (1 - \beta)p_2 + (1 - \beta)^2p_1^2, \quad (2.10)$$

$$3a_4 - 3a_2a_3 + a_2^3 = (1 - \beta)p_3, \quad (2.11)$$

$$a_2 = -q_1(1 - \beta) \quad (2.12)$$

and

$$3a_2^2 - 2a_3 = (1 - \beta)q_2, \quad (2.13)$$

$$4a_2^2 - 2a_3 = (1 - \beta)q_2 + (1 - \beta)^2q_1^2 \quad (2.14)$$

and

$$-(10a_2^3 - 12a_2a_3 + 3a_4) = (1 - \beta)q_3. \quad (2.15)$$

From 2.8 and 2.12, we get

$$p_1 = -q_1 \quad (2.16)$$

and

$$2a_2^2 = (1 - \beta)^2(p_1^2 + q_1^2). \quad (2.17)$$

Now from 2.10, 2.14 and 2.17, we obtain

$$\begin{aligned} 4a_2^2 &= (1 - \beta)(p_2 + q_2) + (1 - \beta)^2(p_1^2 + q_1^2) \\ &= (1 - \beta)(p_2 + q_2) + 2a_2^2 \end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{1}{2}(1 - \beta)(p_2 + q_2).$$

Applying Lemma 1.3 for the coefficients  $p_2$  and  $q_2$ , we immediately have

$$|a_2| \leq \sqrt{2(1-\beta)}.$$

This gives the bound on  $|a_2|$  as asserted in 2.1. Next, in order to find the bound on  $|a_3|$ , by subtracting 2.14 from 2.10, we get

$$\begin{aligned} 4a_3 - 4a_2^2 &= (1-\beta)p_2 + (1-\beta)^2p_1^2 - ((1-\beta)q_2 \\ &\quad + (1-\beta)^2q_1^2). \end{aligned} \quad (2.18)$$

It follows from 2.14-2.18 that

$$4a_3 = (1-\beta)(p_2 - q_2) + 2(1-\beta)^2(p_1^2 + q_1^2)$$

or, equivalently,

$$a_3 = \frac{(1-\beta)(p_2 - q_2)}{4} + \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2}.$$

Applying Lemma 1.3 once again for the coefficients  $p_1, p_2, q_1$  and  $q_2$ , we readily get

$$|a_3| \leq (1-\beta) + 4(1-\beta)^2.$$

Addition of 2.9 with 2.13 yields:

$$2a_2^2 = (1-\beta)(p_2 + q_2). \quad (2.19)$$

Putting  $a_2 = (1-\beta)p_1$  from 2.8 we have after simplification:

$$p_1^2 = \frac{p_2 + q_2}{2(1-\beta)}. \quad (2.20)$$

Next, we subtract 2.13 from 2.9, add the equations 2.11 and 2.15 and get respectively:

$$4a_3 = 4a_2^2 + (1-\beta)(p_2 + q_2) \quad (2.21)$$

and

$$-9a_2^3 + 9a_2a_3 = (1-\beta)(p_3 + q_3) \quad (2.22)$$

We shall now find an estimate on  $|a_4|$ . We wish express  $a_4$  in terms of the first three coefficients of  $p(z)$  and  $q(w)$ . For this we subtract 2.15 from 2.11, and get

$$\begin{aligned} 6a_4 &= -11a_2^3 + 15a_2a_3 + (1-\beta)(p_3 - q_3) \\ &= -9a_2^3 + 9a_2a_3 - 2a_2^3 + 6a_2a_3 + (1-\beta)(p_3 - q_3). \end{aligned}$$

We replace  $-9a_2^3 + 9a_2a_3$  by the right hand side of 2.22, put  $a_3 = (1-\beta)^2p_1^2 + \frac{(1-\beta)}{4}(p_2 - q_2)$  (see 2.21) and  $a_2 = (1-\beta)p_1$ . Thus, we

have:

$$\begin{aligned}
6a_4 &= (1 - \beta)(p_3 + q_3) - 2(1 - \beta)^3 p_1^3 \\
&\quad + 6(1 - \beta)p_1((1 - \beta)^2 p_1^2 + \frac{(1 - \beta)}{4}(p_2 - q_2)) \\
&\quad + (1 - \beta)(p_3 - q_3) \\
&= 2(1 - \beta)p_3 + 4(1 - \beta)^3 p_1^3 + \frac{6(1 - \beta)^2}{4} p_1(p_2 - q_2).
\end{aligned}$$

Next replacing  $p_1^2$  by (2.20) we finally get

$$\begin{aligned}
6a_4 &= 2(1 - \beta)p_3 + 4(1 - \beta)^3 p_1 \frac{p_2 + q_2}{2(1 - \beta)} + \frac{6(1 - \beta)^2}{4} p_1(p_2 - q_2) \\
&= 2(1 - \beta)p_3 + 2(1 - \beta)^2 p_1(p_2 + q_2) + \frac{3(1 - \beta)^2}{2} p_1(p_2 - q_2) \\
&= 2(1 - \beta)p_3 + \frac{7(1 - \beta)^2}{2} p_1 p_2 + \frac{(1 - \beta)^2}{2} p_1 q_2.
\end{aligned}$$

By applying the inequalities  $|p_1| \leq 2$ ,  $|p_2| \leq 2$ ,  $|p_3| \leq 2$  and  $|q_2| \leq 2$  we have

$$\begin{aligned}
6|a_4| &\leq 2(1 - \beta)|p_3| + \frac{7(1 - \beta)^2}{2} |p_1||p_2| + \frac{(1 - \beta)^2}{2} |p_1||q_2|. \\
&\leq 4(1 - \beta) + 16(1 - \beta)^2.
\end{aligned}$$

Or equivalently:

$$|a_4| \leq \frac{2(1 - \beta)}{3} (1 + 4(1 - \beta)).$$

□

**Theorem 2.2.** *Let  $(a - b) = \beta$  and  $f(z)$  given by 1.1 be in the class  $\nu_\sigma(a, b)$ ,  $|a - 1| < b \leq a$ . Then*

$$|a_2| \leq 1, \tag{2.23}$$

$$|a_3| \leq 3 - 2\beta \tag{2.24}$$

and

$$|a_4| \leq \frac{9}{2} - 4\beta. \tag{2.25}$$

*Proof.* It follows from 1.4 and 1.5 that

$$\left(\frac{z}{f(z)}\right)^2 f'(z) \prec \beta + (1 - \beta)z$$

and

$$\left(\frac{w}{g(w)}\right)^2 g'(w) \prec \beta + (1 - \beta)w$$

then

$$\begin{aligned} \left(\frac{z}{f(z)}\right)^2 f'(z) &= \frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' \\ &= \beta + (1 - \beta)p(z) \end{aligned} \quad (2.26)$$

and

$$\begin{aligned} \left(\frac{w}{g(w)}\right)^2 g'(w) &= \frac{w}{g(w)} - w\left(\frac{w}{g(w)}\right)' \\ &= \beta + (1 - \beta)q(w) \end{aligned} \quad (2.27)$$

and

$$\frac{z}{f(z)} - z\left(\frac{z}{f(z)}\right)' = 1 + (a_3 - a_2^2)z^2 + 2(a_4 - 2a_2a_3 + a_2^3)z^3 + \dots$$

and

$$\frac{w}{g(w)} - w\left(\frac{w}{g(w)}\right)' = 1 - (a_3 - a_2^2)w^2 - 2(a_4 - 3a_2a_3 + 2a_2^3)w^3 + \dots$$

where  $p(z)$  and  $q(w)$  in  $\mathcal{P}$  and have the forms

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \dots \quad (2.28)$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \dots \quad (2.29)$$

Now, equating the coefficients in 2.26 and 2.27, we get

$$0 = p_1(1 - \beta), \quad (2.30)$$

$$(a_3 - a_2^2) = p_2(1 - \beta), \quad (2.31)$$

$$2(a_4 - 2a_2a_3 + a_2^3) = p_3(1 - \beta) \quad (2.32)$$

and

$$0 = q_1(1 - \beta), \quad (2.33)$$

$$-(a_3 - a_2^2) = q_2(1 - \beta), \quad (2.34)$$

$$-2(a_4 - 3a_2a_3 + 2a_2^3) = q_3(1 - \beta). \quad (2.35)$$

From 2.31 from 2.34, we have

$$p_2 = -q_2 \quad (2.36)$$

By subtracting 2.34 from 2.31, we get

$$2(a_3 - a_2^2) = (1 - \beta)(p_2 - q_2). \quad (2.37)$$

Addition of 2.32 with 2.35 yields:

$$-2a_2^3 + 2a_2a_3 = (1 - \beta)(p_3 + q_3) \quad (2.38)$$

substituting from 2.37 into 2.38 we get

$$a_2(1 - \beta)(p_2 - q_2) = (1 - \beta)(p_3 + q_3).$$

Applying Lemma 1.3 for the coefficients  $p_2$ ,  $q_2$ ,  $p_3$  and  $q_3$ , we immediately have

$$|a_2| \leq \frac{|p_3| + |q_3|}{|p_2| + |q_2|}$$

we obtain

$$|a_2| \leq 1.$$

which is the bound on  $|a_2|$  as given in 2.23. Next, in order to find the bound on  $|a_3|$ , by following from 2.37 we have

$$a_3 = \frac{1}{2}((1 - \beta)(p_2 - q_2)) + a_2^2$$

Applying Lemma 1.3 for the coefficients  $p_2$  and  $q_2$ , so for the inequality  $|a_2| \leq 1$  we get:

$$\begin{aligned} |a_3| &\leq \frac{1}{2}((1 - \beta)(|p_2| + |q_2|)) + |a_2|^2 \\ &\leq 2(1 - \beta) + 1. \end{aligned}$$

Therefore, we have

$$|a_3| \leq 3 - 2\beta.$$

This is precisely the assertion of 2.24. We shall next find an estimate on  $|a_4|$ .

By subtracting 2.35 from 2.32, we get

$$4a_4 + 6a_2^3 - 8a_2a_3 = (1 - \beta)(p_3 - q_3)$$

now after simplification and from 2.37 we have

$$4a_4 - 2a_2^3 - 4(1 - \beta)(p_3 + q_3) = (1 - \beta)(p_3 - q_3)$$

Therefore, we get

$$a_4 = (1 - \beta)(p_3 + q_3 + \frac{1}{4}p_3 - \frac{1}{4}q_3) + \frac{1}{2}a_2^3$$

Applying Lemma 1.3 for the coefficients  $p_3$  and  $q_3$ , so for the inequality  $|a_2| \leq 1$  we get:

$$\begin{aligned} |a_4| &\leq \frac{1}{4}((1 - \beta)(5|p_3| + 3|q_3|)) + \frac{1}{2}|a_2|^3 \\ &\leq 4(1 - \beta) + \frac{1}{2}. \end{aligned}$$

Therefore, we have

$$|a_4| \leq \frac{9}{2} - 4\beta.$$

□



### 3. FEKETE-SZEGŐ COEFFICIENT INEQUALITY FOR THE FUNCTION CLASS $\mathcal{S}_\sigma^*(a, b)$ AND THE FUNCTION CLASS $\nu_\sigma(a, b)$

In the last section by using the analytic functions  $r$  and  $s$  and definitions of  $\mathcal{S}_\sigma^*(a, b)$  and  $\nu_\sigma(a, b)$  the Fekete-Szegő inequality on functions in these classes are investigated.

**Theorem 3.1.** *Let  $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$ , ( $z \in \mathbb{U}$ ), where  $B_n$  are real with  $B_1 > 0$  and  $B_2 \geq 0$ . If  $(a - b) = \beta$  and  $f(z)$  given by 1.1 belongs to  $\mathcal{S}_\sigma^*(a, b)$ , then*

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}[2|B_2|(1 - \beta) + (7 - 4\mu)B_1^2(1 - \beta)^2]. \quad (3.1)$$

*Proof.* If  $f(z) \in \mathcal{S}_\sigma^*(a, b)$ , there exist two analytic functions  $r, s : \mathbb{U} \rightarrow \mathbb{U}$ , with

$r(0) = 0 = s(0)$  such that,

$$\frac{zf'(z)}{f(z)} = \beta + (1 - \beta)\varphi(r(z)). \quad (3.2)$$

$$\frac{wg'(w)}{g(w)} = \beta + (1 - \beta)\varphi(s(w)). \quad (3.3)$$

Define the functions  $p$  and  $q$  by

$$p(z) = \frac{1 + r(z)}{1 - r(z)} = 1 + p_1z + p_2z^2 + \dots$$

and

$$q(w) = \frac{1 + s(w)}{1 - s(w)} = 1 + q_1w + q_2w^2 + \dots$$

or equivalently,

$$\begin{aligned} r(z) = \frac{p(z) - 1}{p(z) + 1} &= \frac{1}{2}(p_1z + (p_2 - \frac{p_1^2}{2})z^2 \\ &\quad + (p_3 - \frac{p_1^3}{4} - p_1p_2)z^3 + \dots) \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} s(w) = \frac{q(w) - 1}{q(w) + 1} &= \frac{1}{2}(q_1w + (q_2 - \frac{q_1^2}{2})w^2 \\ &\quad + (q_3 - \frac{q_1^3}{4} - q_1q_2)w^3 + \dots) \end{aligned} \quad (3.5)$$

Using 3.4 and 3.5 in 3.2 and 3.3, we have

$$\frac{zf'(z)}{f(z)} = \beta + (1 - \beta)\varphi\left(\frac{p(z) - 1}{p(z) + 1}\right). \quad (3.6)$$

and

$$\frac{wg'(w)}{g(w)} = \beta + (1 - \beta)\varphi\left(\frac{q(w) - 1}{q(w) + 1}\right). \quad (3.7)$$

Again using 3.4 and 3.5 along with  $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$ , it is evident that

$$\begin{aligned} \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) &= 1 + \frac{1}{2}B_1p_1z + \left(\frac{1}{2}B_1\left(p_2 - \frac{p_1^2}{2}\right) \right. \\ &\quad \left. + \frac{B_2p_1^2}{4}\right)z^2 + \dots \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} \varphi\left(\frac{q(w) - 1}{q(w) + 1}\right) &= 1 + \frac{1}{2}B_1q_1w + \left(\frac{1}{2}B_1\left(q_2 - \frac{q_1^2}{2}\right) \right. \\ &\quad \left. + \frac{B_2q_1^2}{4}\right)w^2 + \dots \end{aligned} \quad (3.9)$$

It follows from 3.6, 3.7, 3.8 and 3.9 that

$$a_2 = \frac{B_1p_1}{2}(1 - \beta) \quad (3.10)$$

$$\begin{aligned} a_3 &= \frac{1}{2}\left[\left(\frac{1}{2}B_1\left(p_2 - \frac{1}{2}p_1^2\right) + \frac{1}{4}B_2p_1^2\right)(1 - \beta) \right. \\ &\quad \left. + \frac{1}{4}B_1^2p_1^2(1 - \beta)^2\right] \end{aligned} \quad (3.11)$$

$$a_2 = -\frac{B_1q_1}{2}(1 - \beta) \quad (3.12)$$

$$\begin{aligned} a_3 &= \frac{1}{2}\left[\frac{3}{2}B_1^2q_1^2(1 - \beta)^2 - \left(\frac{1}{2}B_1\left(q_2 - \frac{1}{2}q_1^2\right) \right. \right. \\ &\quad \left. \left. + \frac{1}{4}B_2q_1^2\right)(1 - \beta)\right] \end{aligned} \quad (3.13)$$

Therefore from 3.10 and 3.12 we have,

$$a_2^2 = \frac{1}{8}B_1^2(1 - \beta)^2(p_1^2 + q_1^2)$$

Adding 3.11 and 3.13, we get

$$\begin{aligned} a_3 &= \frac{1}{16}[B_1^2(1 - \beta)^2[p_1^2 + 6q_1^2] + (1 - \beta)[2B_1\left(\left(p_2 \right. \right. \\ &\quad \left. \left. - \frac{1}{2}p_1^2\right) - \left(q_2 - \frac{1}{2}q_1^2\right) + B_2(p_1^2 + q_1^2)\right]] \end{aligned} \quad (3.14)$$

Therefore

$$\begin{aligned} a_3 - \mu a_2^2 &= \frac{1}{16}[B_1^2(1-\beta)^2[p_1^2 + 6q_1^2] + (1-\beta)[2B_1((p_2 - \frac{1}{2}p_1^2) \\ &\quad - (q_2 - \frac{1}{2}q_1^2)) + B_2(p_1^2 + q_1^2)] \\ &\quad - \frac{1}{8}\mu B_1^2(1-\beta)^2(p_1^2 + q_1^2). \end{aligned}$$

Taking the absolute values we obtain:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{16}[B_1^2(1-\beta)^2[|p_1|^2 + 6|q_1|^2] + (1-\beta)[2B_1(|p_2 - \frac{1}{2}p_1^2| + |q_2 - \frac{1}{2}q_1^2|) \\ &\quad + B_2(|p_1|^2 + |q_1|^2)]] - \frac{1}{8}\mu B_1^2(1-\beta)^2(|p_1|^2 + |q_1|^2). \end{aligned}$$

Applying Lemma 1.3 for the coefficients  $p_1, q_1$  and  $|p_2 - \frac{1}{2}p_1^2| \leq 2 - \frac{1}{2}|p_1|^2$ ,  $|q_2 - \frac{1}{2}q_1^2| \leq 2 - \frac{1}{2}|q_1|^2$  we have:

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{1}{16}[B_1^2(1-\beta)^2[|p_1|^2 + 6|q_1|^2] + (1-\beta)[2B_1(2 \\ &\quad - \frac{1}{2}|p_1|^2 + 2 - \frac{1}{2}|q_1|^2) + B_2(|p_1|^2 + |q_1|^2)]] \\ &\quad - \frac{1}{8}\mu B_1^2(1-\beta)^2(|p_1|^2 + |q_1|^2). \end{aligned}$$

Upon simplification we obtain:

$$|a_3 - \mu a_2^2| \leq \frac{1}{4}[2|B_2|(1-\beta) + (7-4\mu)B_1^2(1-\beta)^2].$$

□

**Theorem 3.2.** Let  $(a-b) = \beta$ . further let  $\varphi(z) = 1 + B_1z + B_2z^2 + \dots$ , ( $z \in \mathbb{U}$ ), where  $B_n$  are real with  $B_1 > 0$  and  $B_2 \geq 0$ . If  $f(z)$  given by (1.1) belongs to  $\nu_\sigma(a, b)$ ,

$$\left(\frac{z}{f(z)}\right)^2 f'(z) = \beta + (1-\beta)\varphi(r(z)).$$

$$\left(\frac{w}{g(w)}\right)^2 g'(w) = \beta + (1-\beta)\varphi(s(w)).$$

then

$$|a_3 - a_2^2| \leq |B_2|(1-\beta).$$

#### 4. CONCLUSION

The Fekete-Szegö problem have always been the main interest of researchers in Univalent and bi-Univalent classes. Many studies related to this problem are around analytic normalized functions. Here the the Fekete-Szegö inequality is obtained for functions in  $\mathcal{S}_\sigma^*(a, b)$  and  $\nu_\sigma(a, b)$ .

Using subordination structure. Also by using the integral and differential operators we may obtain the bounds of coefficients and Fekete-Szegö problem in future.

#### REFERENCES

- [1] S. Azizi, A. Ebadian, Sh. Najafzadeh, Coefficient Estimates for a Subclass of Bi-univalent Functions, *Communications on Advanced Computational Science with Applications* 2015, no.1 (2015), 41-44.
- [2] M. Caglar, H. Orhan, N. Yagmur, Coefficient bounds for new subclasses of bi-univalent functions, *Filomat*, **27** (7) (2013), 1165-1171.
- [3] P.L. Duren, Univalent functions, *Grundlehern der Mathematischen Wissenschaften*, **259**, Springer, New York, (1983).
- [4] B. A. Frasin, M. K. Aouf, New subclasses of bi-univalent functions, *Applied Mathematics Letters* **24**(2011) 1569-1573.
- [5] J. M. Jahangiri, On the coefficients of powers of a class of Bazilevich, *Indian J. Pure Appl. Math.* **17**, no. 9, 1140-1144, 1986.
- [6] J. M. Jahangiri, N. Magesh, J. Yamini, Fekete-Szegö inequalities for classes of bi-starlike and bi-convex functions, *Electronic Journal of Mathematical Analysis and Applications*, Vol. **3**(1) Jan. (2015), pp. 133-140.
- [7] H. Rahmatan, Sh. Najafzadeh, A. Ebadian, The norm of Pre-Schwarzian derivativers on bi-univalent functions of order  $\alpha$ , *Bull. Iranian Math. Soc.* Vol **43** (2017), No. 5, pp. 1037-1043.
- [8] H. M. Srivastava, A. K. Mishra, P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett*, **23** (10) (2010) 1188-1192.