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Length and Mean Fuzzy UP-Subalgebras of UP-Algebras

Narupon Tacha [2], Phongsakon Phayapsiang [2] and Aiyared Iampan [2,1]
[2] Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand

Abstract. The aim of this paper is to introduce the notions of the length and the mean of a hyper structure in UP-algebras. The notions of length fuzzy UP-subalgebras and mean fuzzy UP-subalgebras of UP-algebras are introduced, and related properties are investigated. Characterizations of length fuzzy UP-subalgebras and mean fuzzy UP-subalgebras are discussed. Relations between length fuzzy UP-subalgebras (resp., mean fuzzy UP-subalgebras) and hyperfuzzy UP-subalgebras are established. Moreover, we discuss the relationships among length fuzzy UP-subalgebras (resp., mean fuzzy UP-subalgebras) and upper level subsets, lower level subsets, and equal level subsets of the length (resp., mean) of a fuzzy structure in UP-algebras.

Keywords: UP-algebra, length fuzzy UP-subalgebra, mean fuzzy UP-subalgebra, hyperfuzzy UP-subalgebra.

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1. Introduction

The branch of the logical algebra, UP-algebras were introduced by Iampan [4] in 2017, and it is known that the class of KU-algebras [10] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [16] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [9], Kaijae et al.

[1]Corresponding author: aiyared.ia@up.ac.th
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[8] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, the notion of Q-fuzzy sets in UP-algebras was introduced by Tanamoon et al. [19], Sripaeng et al. [18] introduced the notion anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, the notion of $\mathcal{N}$-fuzzy sets in UP-algebras was introduced by Songsaeng and Iampan [17], Senapati et al. [14, 15] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, Romano [11] introduced the notion of proper UP-filters in UP-algebras, etc.

A fuzzy subset f of a set S is a function from S to a closed interval $[0,1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [20] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

Hyperstructures have a lot of applications in several domains of mathematics and computer science. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The study of fuzzy hyper structures is an interesting research area of fuzzy sets. As a generalization of fuzzy sets and interval-valued fuzzy sets, Ghosh and Samanta [3] introduced the notion of hyperfuzzy sets, and applied it to group theory. Jun et al. [7] applied the hyperfuzzy sets to BCK/BCI-algebras, and introduced the notion of k-fuzzy substructures for $k \in \{1,2,3,4\}$. They introduced the concepts of hyperfuzzy substructures of several types by using k-fuzzy substructures, and investigated their basic properties. They also defined hyperfuzzy subalgebras of type (i,j) for $i,j \in \{1,2,3,4\}$, and discussed relations between the hyperfuzzy substructure/subalgebra and its length. They investigated the properties of hyperfuzzy subalgebras related to upper-level subsets and lower-level subsets.

In this paper, we introduce the notions of the length and the mean of a hyper structure in UP-algebras. The notions of length fuzzy UP-subalgebras and mean fuzzy UP-subalgebras of UP-algebras are introduced, and related properties are investigated. Characterizations of length fuzzy UP-subalgebras and mean fuzzy UP-subalgebras are discussed. Relations between length fuzzy UP-subalgebras (resp., mean fuzzy UP-subalgebras) and hyperfuzzy UP-subalgebras are established. Moreover, we discuss the relationships among length fuzzy UP-subalgebras (resp., mean fuzzy UP-subalgebras) and upper level subsets, lower level subsets, and equal level subsets of the length (resp., mean) of a fuzzy structure in UP-algebras.

2. Preliminaries

Before we begin our study, we will give the definition of a UP-algebra.

Definition 2.1. [4] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1): $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2): $(\forall x \in A)(0 \cdot x = x)$,
(UP-3): $(\forall x \in A)(x \cdot 0 = 0)$, and
(UP-4): $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras (see [10]).

Example 2.2. [13] Let X be a universal set and let $\Omega \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A^C \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A^C \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω. In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type 1*, and $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type 2*.

Example 2.3. [2] Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by

$$(\forall x, y \in \mathbb{N}) \left(x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases} \right)$$

and

$$(\forall x, y \in \mathbb{N}) \left(x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise} \end{cases} \right).$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are UP-algebras.

Example 2.4. [17] Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	0	0	2	3	2	3	6
2	0	1	0	3	1	5	3
3	0	1	2	0	4	1	2
4	0	0	0	3	0	3	3
5	0	0	2	0	2	0	2
6	0	1	0	0	1	1	0

Then $(A, \cdot, 0)$ is a UP-algebra.

For more examples of UP-algebras, see [1, 5, 12, 13].

The following proposition is important for the study of UP-algebras.

Proposition 2.5. [4, 5] *In a UP-algebra $A = (A, \cdot, 0)$, the following properties hold:*

(1) $(\forall x \in A)(x \cdot x = 0)$,
(2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,
(3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,
(4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,
(5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,
(6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,
(7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,
(8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,
(9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,
(10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,
(11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,
(12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, *and*
(13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

From [4], the binary relation $\leq$ on a UP-algebra $A = (A, \cdot, 0)$ is defined as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

Definition 2.6. [4] A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is called a *UP-subalgebra* of A if

$$(\forall x, y \in S)(x \cdot y \in S).$$

Definition 2.7. [20] Let A be a nonempty set. A mapping $f : A \to [0, 1]$ is called a *fuzzy set* in A (or a fuzzy subset of A) where $[0, 1]$ is the unit segment of the real line. An ordered pair (A, f) is called a *fuzzy structure* in A. A fuzzy structure (A, f) in A is said to be *constant* if a fuzzy set f is constant.

Definition 2.8. [3] Let A be a nonempty set. A mapping $\tilde{f} : A \to \tilde{P}([0,1])$ is called a *hyperfuzzy set* over A where $\tilde{P}([0,1])$ is the family of all nonempty subsets of $[0,1]$. An ordered pair $(A, \tilde{f})$ is called a *hyper structure* over A.

Definition 2.9. [6] Given a hyper structure $(A, \tilde{f})$ over a nonempty set A, we define two fuzzy structures $(A, \tilde{f}_{\inf})$ and $(A, \tilde{f}_{\sup})$ in A as follows:

$$\tilde{f}_{\inf} : A \to [0,1], x \mapsto \inf \tilde{f}(x),$$
$$\tilde{f}_{\sup} : A \to [0,1], x \mapsto \sup \tilde{f}(x).$$

In what follows, let A denote a UP-algebra $(A, \cdot, 0)$ unless otherwise specified.

The following is a definition of all 4 types of fuzzy UP-subalgebras which will lead to other definitions.

Definition 2.10. A fuzzy structure (A, f) in A is called

(1) a *fuzzy UP-subalgebra of A with type* 1 (briefly, 1-fuzzy UP-subalgebra of A) if

$$(\forall x, y \in A)(f(x \cdot y) \geq \min\{f(x), f(y)\}).$$

(2) a *fuzzy UP-subalgebra of A with type* 2 (briefly, 2-fuzzy UP-subalgebra of A) if

$$(\forall x, y \in A)(f(x \cdot y) \leq \min\{f(x), f(y)\}).$$

(3) a *fuzzy UP-subalgebra of A with type* 3 (briefly, 3-fuzzy UP-subalgebra of A) if

$$(\forall x, y \in A)(f(x \cdot y) \geq \max\{f(x), f(y)\}).$$

(4) a *fuzzy UP-subalgebra of A with type* 4 (briefly, 4-fuzzy UP-subalgebra of A) if

$$(\forall x, y \in A)(f(x \cdot y) \leq \max\{f(x), f(y)\}).$$

Proposition 2.11. *If (A, f) is a k-fuzzy UP-subalgebra of A for $k = 1, 3$, then*

$$(\forall x \in A)(f(0) \geq f(x)). \tag{2.1}$$

Proof. If (A, f) is a 1-fuzzy UP-subalgebra of A, then for all $x \in A$,

$$f(0) = f(x \cdot x) \geq \min\{f(x), f(x)\} = f(x). \quad \text{(Proposition 2.5 (1))}$$

If (A, f) is a 3-fuzzy UP-subalgebra of A, then for all $x \in A$,

$$f(0) = f(x \cdot x) \geq \max\{f(x), f(x)\} = f(x). \quad \text{(Proposition 2.5 (1))}$$

Therefore, $f(0) \geq f(x)$ for all $x \in A$. □

Proposition 2.12. *If (A, f) is a k-fuzzy UP-subalgebra of A for $k = 2, 4$, then*

$$(\forall x \in A)(f(0) \leq f(x)). \tag{2.2}$$

Proof. If (A, f) is a 2-fuzzy UP-subalgebra of A, then for all $x \in A$,

$$f(0) = f(x \cdot x) \leq \min\{f(x), f(x)\} = f(x). \qquad \text{(Proposition 2.5 (1))}$$

If (A, f) is a 4-fuzzy UP-subalgebra of A, then for all $x \in A$,

$$f(0) = f(x \cdot x) \leq \max\{f(x), f(x)\} = f(x). \qquad \text{(Proposition 2.5 (1))}$$

Therefore, $f(0) \leq f(x)$ for all $x \in A$. □

Theorem 2.13. *Every 3-fuzzy UP-subalgebra of A is a 1-fuzzy UP-subalgebra.*

Proof. Assume that (A, f) is a 3-fuzzy UP-subalgebra of A. Let $x, y \in A$. Then

$$f(x \cdot y) \geq \max\{f(x), f(y)\} \geq \min\{f(x), f(y)\}.$$

Hence, (A, f) is a 1-fuzzy UP-subalgebra of A. □

The following example show that the converse of Theorem 2.13 is not true.

Example 2.14. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	0	3	0
2	0	2	0	3	0
3	0	2	2	0	0
4	0	2	2	3	0

Let (A, f) be a fuzzy structure in A in which f is given as follows:

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.8 & 0.8 & 0.5 & 0.5 \end{pmatrix}.$$

Then (A, f) is 1-fuzzy UP-subalgebra of A. We see that

$$f(0 \cdot 3) = 0.5 \not\geq 0.8 = \max\{0.8, 0.5\} = \max\{f(0), f(3)\}.$$

Thus f is not a 3-fuzzy UP-subalgebra of A

Theorem 2.15. *Every 2-fuzzy UP-subalgebra of A is a 4-fuzzy UP-subalgebra.*

Proof. Assume that (A, f) is a 2-fuzzy UP-subalgebra of A. Let $x, y \in A$. Then

$$f(x \cdot y) \leq \min\{f(x), f(y)\} \leq \max\{f(x), f(y)\}.$$

Hence, (A, f) is a 4-fuzzy UP-subalgebra of A. □

The following example show that the converse of Theorem 2.15 is not true.

Example 2.16. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 2.14. Let (A, f) be a fuzzy structure in A in which f is given as follows:

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.7 \end{pmatrix}.$$

Then (A, f) is a 4-fuzzy UP-subalgebra of A. We see that

$$f(0 \cdot 4) = f(4) = 0.7 \nleq 0.2 = \min\{f(0), f(4)\}.$$

Thus (A, f) is not a 2-fuzzy UP-subalgebra of A.

Theorem 2.17. *A fuzzy structure (A, f) in A is a 2-fuzzy UP-subalgebra of A if and only if it is constant.*

Proof. Assume that (A, f) is a 2-fuzzy UP-subalgebra of A. Then by Proposition 2.12, we have $f(0) \leq f(x)$ for all $x \in A$. By (UP-2), we have $f(x) = f(0 \cdot x) \leq \min\{f(0), f(x)\} = f(0)$ for all $x \in A$. Thus $f(x) = f(0)$ for all $x \in A$, so f is constant. Hence, (A, f) is constant.

Conversely, assume that (A, f) is constant. Then $f(x) = f(0)$ for all $x \in A$. Let $x, y \in A$. Then $f(x \cdot y) = f(0) = \min\{f(0), f(0)\} = (\leq) \min\{f(x), f(y)\}$. Therefore, (A, f) is a 2-fuzzy UP-subalgebra of A. □

Theorem 2.18. *A fuzzy structure (A, f) in A is a 3-fuzzy UP-subalgebra of A if and only if it is constant.*

Proof. Assume that (A, f) is a 3-fuzzy UP-subalgebra of A. Then by Proposition 2.11, we have $f(0) \geq f(x)$ for all $x \in A$. By (UP-2), we have $f(x) = f(0 \cdot x) \geq \max\{f(0), f(x)\} = f(0)$. Thus $f(x) = f(0)$ for all $x \in A$, so f is constant. Hence, (A, f) is constant.

Conversely, assume that (A, f) is constant. Then $f(0) = f(x)$ for all $x \in A$. Let $x, y \in A$. Then $f(x \cdot y) = f(0) = \max\{f(0), f(0)\} = (\geq) \max\{f(x), f(y)\}$. Therefore, (A, f) is a 3-fuzzy UP-subalgebra of A. □

By Theorems 2.17 and 2.18, we obtain that 2-fuzzy UP-subalgebras, 3-fuzzy UP-subalgebras, and constant fuzzy structures coincide.

Definition 2.19. For any $i, j \in \{1, 2, 3, 4\}$, a hyper structure $(A, \tilde{f})$ over A is called an (i, j)-*hyperfuzzy UP-subalgebra* of A if a fuzzy structures $(A, \tilde{f}_{\inf})$ is an i-fuzzy UP-subalgebra of A and a fuzzy structures $(A, \tilde{f}_{\sup})$ is a j-fuzzy UP-subalgebra of A.

3. Length of a hyper structure in UP-algebras

In this section, we introduce the notion of the length of a hyper structure in UP-algebras. The notions of length fuzzy UP-subalgebras of UP-algebras are introduced, and related properties are investigated. Relations between length fuzzy UP-subalgebras and hyperfuzzy UP-subalgebras are established. Moreover, we discuss the relationships among length fuzzy UP-subalgebras and upper level subsets, lower level subsets, and equal level subsets of the length of a hyper structure in UP-algebras.

Definition 3.1. Given a hyper structure $(A, \tilde{f})$ over A, we define a fuzzy structures $(A, \tilde{f}_{\mathrm{l}})$ in A as follows:

$$\tilde{f}_{\mathrm{l}} : A \to [0, 1], x \mapsto \tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x)$$

which is called the *length* of $\tilde{f}$.

Definition 3.2. A hyper structure $(A, \tilde{f})$ over A is called a *length* 1-*fuzzy* (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) *UP-subalgebra* of A if a fuzzy structures $(A, \tilde{f}_{\mathrm{l}})$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy, and 4-fuzzy) UP-subalgebra of A.

Example 3.3. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	0	2	0	4
4	0	0	0	0	0

Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.2, 0.4) \cup [0.5, 1) & (0.5, 0.9] & [0.2, 0.3] \cup (0.4, 0.8] & [0.7, 0.9] & [0.2, 0.3] \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{l}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.4 & 0.6 & 0.2 & 0.1 \end{pmatrix}.$$

Thus $(A, \tilde{f}_{\mathrm{l}})$ is a 1-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A.

Proposition 3.4. *If $(A, \tilde{f})$ is a length k-fuzzy UP-subalgebra of A for $k = 1, 3$, then*

$$(\forall x \in A)(\tilde{f}_l(0) \geq \tilde{f}_l(x)). \tag{3.1}$$

Proof. It is straightforward by Proposition 2.11. □

Proposition 3.5. *If $(A, \tilde{f})$ is a length k-fuzzy UP-subalgebra of A for $k = 2, 4$, then*

$$(\forall x \in A)(\tilde{f}_l(0) \leq \tilde{f}_l(x)). \tag{3.2}$$

Proof. It is straightforward by Proposition 2.12. □

Theorem 3.6. *Every length 3-fuzzy UP-subalgebra of A is a length 1-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 2.13. □

Theorem 3.7. *Every length 2-fuzzy UP-subalgebra of A is a length 4-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 2.15. □

Theorem 3.8. *Length 2-fuzzy UP-subalgebra and length 3-fuzzy UP-subalgebra of A coincide.*

Proof. It is straightforward by Theorems 2.17 and 2.18. □

Theorem 3.9. *Given a UP-subalgebra S of A and $B_1, B_2 \in \tilde{P}([0, 1])$, let $(A, \tilde{f})$ be a hyper structure over A given by*

$$\tilde{f} : A \to \tilde{P}([0, 1]), x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{otherwise.} \end{cases}$$

If $B_1 \subset B_2$, then $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. Also, if $B_2 \subset B_1$, then $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A.

Proof. If $x \in S$, then $\tilde{f}(x) = B_2$ and so

$$\tilde{f}_l(x) = \tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x) = \sup \tilde{f}(x) - \inf \tilde{f}(x) = \sup B_2 - \inf B_2.$$

If $x \notin S$, then $\tilde{f}(x) = B_1$ and so

$$\tilde{f}_l(x) = \tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x) = \sup \tilde{f}(x) - \inf \tilde{f}(x) = \sup B_1 - \inf B_1.$$

Assume that $B_1 \subset B_2$. Then $\sup B_2 - \inf B_2 \geq \sup B_1 - \inf B_1$.

Case 1: Let $x, y \in S$. Then $\tilde{f}_l(x) = \sup B_2 - \inf B_2$ and $\tilde{f}_l(y) = \sup B_2 - \inf B_2$. Thus $\min\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_2 - \inf B_2$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so

$$\tilde{f}_l(x \cdot y) = \sup B_2 - \inf B_2.$$

Thus

$$\tilde{f}_l(x \cdot y) = \sup B_2 - \inf B_2 = (\geq) \min\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Case 2: Let $x, y \notin S$. Then $\tilde{f}_l(x) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y) = \sup B_1 - \inf B_1$, so $\min\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_1 - \inf B_1$. Thus

$$\tilde{f}_l(x \cdot y) \geq \sup B_1 - \inf B_1 = \min\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\tilde{f}_l(x) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y) = \sup B_2 - \inf B_2$, so $\min\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_1 - \inf B_1$. Thus

$$\tilde{f}_l(x \cdot y) \geq \sup B_1 - \inf B_1 = \min\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\tilde{f}_l(x) = \sup B_2 - \inf B_2$ and $\tilde{f}_l(y) = \sup B_1 - \inf B_1$, so $\min\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_1 - \inf B_1$. Thus

$$\tilde{f}_l(x \cdot y) \geq \sup B_1 - \inf B_1 = \min\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Hence, $\tilde{f}_l$ is a 1-fuzzy UP-subalgebra of A and so $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A.

Assume that $B_2 \subset B_1$. Then $\sup B_2 - \inf B_2 \leq \sup B_1 - \inf B_1$.

Case 1: Let $x, y \in S$. Then $\tilde{f}_l(x) = \sup B_2 - \inf B_2$ and $\tilde{f}_l(y) = \sup B_2 - \inf B_2$. Thus $\max\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_2 - \inf B_2$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so

$$\tilde{f}_l(x \cdot y) = \sup B_2 - \inf B_2.$$

Thus

$$\tilde{f}_l(x \cdot y) = \sup B_2 - \inf B_2 = (\leq) \max\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Case 2: Let $x, y \notin S$. Then $\tilde{f}_l(x) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y) = \sup B_1 - \inf B_1$, so $\max\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_1 - \inf B_1$. Thus

$$\tilde{f}_l(x \cdot y) \leq \sup B_1 - \inf B_1 = \max\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\tilde{f}_l(x) = \sup B_1 - \inf B_1$ and $\tilde{f}_l(y) = \sup B_2 - \inf B_2$, so $\max\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_1 - \inf B_1$. Thus

$$\tilde{f}_l(x \cdot y) \leq \sup B_1 - \inf B_1 = \max\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\tilde{f}_l(x) = \sup B_2 - \inf B_2$ and $\tilde{f}_l(y) = \sup B_1 - \inf B_1$, so $\max\{\tilde{f}_l(x), \tilde{f}_l(y)\} = \sup B_1 - \inf B_1$. Thus

$$\tilde{f}_l(x \cdot y) \leq \sup B_1 - \inf B_1 = \max\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Hence, $\tilde{f}_l$ is a 4-fuzzy UP-subalgebra of A and so $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. □

Example 3.10. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 2.14. Then $S = \{0, 1, 2\}$ is a UP-subalgebra of A. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.1, 0.9) & [0.1, 0.9) & [0.1, 0.9) & (0.3, 0.8] & (0.3, 0.8] \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.8 & 0.8 & 0.8 & 0.5 & 0.5 \end{pmatrix}.$$

By Theorem 3.9, we have $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. We see that

$$\tilde{f}_l(0 \cdot 3) = 0.5 \ngeq 0.8 = \max\{0.8, 0.5\} = \max\{\tilde{f}_l(0), \tilde{f}_l(3)\}.$$

Thus $(A, \tilde{f}_l)$ is not a 3-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is not a length 3-fuzzy UP-subalgebra of A. Give a UP-subalgebra $S = \{0, 1, 2, 3\}$ of A, let $(A, \tilde{f})$ be a hyper structure over A given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ (0.3, 0.5) & (0.3, 0.5) & (0.3, 0.5) & (0.3, 0.5) & [0.2, 0.9) \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.7 \end{pmatrix}.$$

By Theorem 3.9, we have $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. We see that

$$\tilde{f}_l(0 \cdot 4) = \tilde{f}_l(4) = 0.7 \nleq 0.2 = \min\{\tilde{f}_l(0), \tilde{f}_l(4)\}.$$

Thus $(A, \tilde{f}_l)$ is not a 2-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is not a length 2-fuzzy UP-subalgebra of A.

Definition 3.11. [16] Let (A, f) be a fuzzy structure in A. For any $t \in [0, 1]$, the sets

$$U(f; t) = \{x \in A \mid f(x) \geq t\},$$
$$L(f; t) = \{x \in A \mid f(x) \leq t\},$$
$$E(f; t) = \{x \in A \mid f(x) = t\}$$

are called *upper t-level subset*, *lower t-level subset*, and *equal t-level subset* of f, respectively.

Theorem 3.12. *A hyper structure $(A, \tilde{f})$ over A is a length 1-fuzzy UP-subalgebra of A if and only if the set $U(\tilde{f}_l; t)$ is a UP-subalgebra of A for all $t \in [0, 1]$ with $U(\tilde{f}_l; t) \neq \emptyset$.*

Proof. Assume that $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. Let $t \in [0,1]$ be such that $U(\tilde{f}_l; t) \neq \emptyset$ and let $x, y \in U(\tilde{f}_l; t)$. Then $\tilde{f}_l(x) \geq t$ and $\tilde{f}_l(y) \geq t$. Since $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A, we have

$$\tilde{f}_l(x \cdot y) \geq \min\{\tilde{f}_l(x), \tilde{f}_l(y)\} \geq t.$$

Thus $x \cdot y \in U(\tilde{f}_l; t)$. Hence, $U(\tilde{f}_l; t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [0,1]$, the set $U(\tilde{f}_l; t)$ is a UP-subalgebra of A if $U(\tilde{f}_l; t) \neq \emptyset$. Let $x, y \in A$. Then $\tilde{f}_l(x), \tilde{f}_l(y) \in [0,1]$. Choose $t = \min\{\tilde{f}_l(x), \tilde{f}_l(y)\}$. Thus $\tilde{f}_l(x) \geq t$ and $\tilde{f}_l(y) \geq t$ and so $x, y \in U(\tilde{f}_l; t) \neq \emptyset$. By assumption, we have $U(\tilde{f}_l; t)$ is a UP-subalgebra of A and so $x \cdot y \in U(\tilde{f}_l; t)$. Thus

$$\tilde{f}_l(x \cdot y) \geq t = \min\{\tilde{f}_l(x), \tilde{f}_l(y)\}.$$

Hence, $(A, \tilde{f}_l)$ is a 1-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. □

Corollary 3.13. *If $(A, \tilde{f})$ is a length 3-fuzzy UP-subalgebra of A, then the set $U(\tilde{f}_l; t)$ is a UP-subalgebra of A for all $t \in [0,1]$ with $U(\tilde{f}_l; t) \neq \emptyset$.*

Proof. It is straightforward by Theorems 4.6 and 3.12. □

The following example show that the converse of Corollary 3.13 is not true.

Example 3.14. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	0
2	0	0	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.1, 0.3) \cup [0.5, 0.8) & (0.5, 0.8] & [0.1, 0.3] \cup (0.5, 0.7] & [0.5, 0.7] & (0.3, 0.5] \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.7 & 0.3 & 0.6 & 0.2 & 0.2 \end{pmatrix}.$$

We have

$$U(\tilde{f}_1;t)=\begin{cases}\emptyset & \text{if } t\in(0.7,1],\\ \{0\} & \text{if } t\in(0.6,0.7],\\ \{0,2\} & \text{if } t\in(0.3,0.6],\\ \{0,1,2\} & \text{if } t\in(0.2,0.3],\\ A & \text{if } t\in[0,0.2]\end{cases}$$

and so $U(\tilde{f}_1;t)$ is a UP-subalgebra of A for all $t\in[0,1]$ with $U(\tilde{f}_1;t)\neq\emptyset$. We see that

$$\tilde{f}_1(0\cdot 4)=\tilde{f}_1(4)=0.2\not\geq 0.7=\max\{\tilde{f}_1(0),\tilde{f}_1(4)\}.$$

Thus $(A,\tilde{f}_1)$ is not a 3-fuzzy UP-subalgebra of A, that is, $(A,\tilde{f})$ is not a length 3-fuzzy UP-subalgebra of A.

Theorem 3.15. *A hyper structure $(A,\tilde{f})$ over A is a length 4-fuzzy UP-subalgebra of A if and only if the set $L(\tilde{f}_1;t)$ is a UP-subalgebra of A for all $t\in[0,1]$ with $L(\tilde{f}_1;t)\neq\emptyset$.*

Proof. Assume that $(A,\tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. Let $t\in[0,1]$ be such that $L(\tilde{f}_1;t)\neq\emptyset$ and let $x,y\in L(\tilde{f}_1;t)$. Then $\tilde{f}_1(x)\leq t$ and $\tilde{f}_1(y)\leq t$. Since $(A,\tilde{f})$ is a length 4-fuzzy UP-subalgebra of A, we have

$$\tilde{f}_1(x\cdot y)\leq\max\{\tilde{f}_1(x),\tilde{f}_1(y)\}\leq t.$$

Thus $x\cdot y\in L(\tilde{f}_1;t)$. Hence, $L(\tilde{f}_1;t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t\in[0,1]$, the set $L_1(\tilde{f};t)$ is a UP-subalgebra of A if $L(\tilde{f}_1;t)\neq\emptyset$. Let $x,y\in A$. Then $\tilde{f}_1(x),\tilde{f}_1(y)\in[0,1]$. Choose $t=\max\{\tilde{f}_1(x),\tilde{f}_1(y)\}$. Thus $\tilde{f}_1(x)\leq t$ and $\tilde{f}_1(y)\leq t$ and so $x,y\in L(\tilde{f}_1;t)\neq\emptyset$. By assumption, we have $L(\tilde{f}_1;t)$ is a UP-subalgebra of A and so $x\cdot y\in L(\tilde{f}_1;t)$. Thus

$$\tilde{f}_1(x\cdot y)\leq t=\max\{\tilde{f}_1(x),\tilde{f}_1(y)\}.$$

Hence, $(A,\tilde{f}_1)$ is a 4-fuzzy UP-subalgebra of A, that is, $(A,\tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. □

Corollary 3.16. *If $(A,\tilde{f})$ is a length 2-fuzzy UP-subalgebra of A, then the set $L(\tilde{f}_1;t)$ is a UP-subalgebra of A for all $t\in[0,1]$ with $L(\tilde{f}_1;t)\neq\emptyset$.*

Proof. It is straightforward by Theorems 4.7 and 3.15. □

The following example show that the converse of Corollary 3.16 is not true.

Example 3.17. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 3.14. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.6, 0.9) & (0.3, 0.8] & [0.4, 0.6) \cup (0.7, 0.8] & [0.1, 0.8] & (0.2, 0.9] \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_l = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.3 & 0.5 & 0.4 & 0.7 & 0.7 \end{pmatrix}.$$

We have

$$L(\tilde{f}_l; t) = \begin{cases} A & \text{if } t \in [0.7, 1], \\ \{0, 1, 2\} & \text{if } t \in [0.5, 0.7), \\ \{0, 2\} & \text{if } t \in [0.4, 0.5), \\ \{0\} & \text{if } t \in [0.3, 0.4), \\ \emptyset & \text{if } t \in [0, 0.3) \end{cases}$$

and so $L(\tilde{f}_l; t)$ is a UP-subalgebra of A for all $t \in [0, 1]$ with $L(\tilde{f}_l; t) \neq \emptyset$. We see that

$$\tilde{f}_l(0 \cdot 3) = 0.7 \not\leq 0.3 = \min\{\tilde{f}_l(0), \tilde{f}_l(3)\}.$$

Thus $(A, \tilde{f}_l)$ is not a 2-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is not a length 2-fuzzy UP-subalgebra of A.

Theorem 3.18. *A hyper structure $(A, \tilde{f})$ over A is a length 2(3)-fuzzy UP-subalgebra of A if and only if the set $E(\tilde{f}_l; \tilde{f}_l(0)) = A$.*

Proof. Assume that $(A, \tilde{f})$ is a length 2-fuzzy UP-subalgebra of A. Then $\tilde{f}_l$ is a 2-fuzzy UP-subalgebra of A. By Theorem 2.17, we have $\tilde{f}_l$ is constant and so $\tilde{f}_l(x) = \tilde{f}_l(0)$ for all $x \in A$. Thus $x \in E(\tilde{f}_l; \tilde{f}_l(0))$ for all $x \in A$. Therefore, $E(\tilde{f}_l; \tilde{f}_l(0)) = A$.

Conversely, assume that $E(\tilde{f}_l; \tilde{f}_l(0)) = A$. Then $\tilde{f}_l(x) = \tilde{f}_l(0)$, for all $x \in A$. Thus $\tilde{f}_l$ is constant. By Theorem 2.17, we have $\tilde{f}_l$ is a 2-fuzzy UP-subalgebra$A$. Therefore, $(A, \tilde{f})$ is a length 2-fuzzy UP-subalgebra of A. □

Theorem 3.19. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 1-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 1-fuzzy UP-subalgebra of A. Let $x, y \in A$. Since $(A, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in A$.

Since $(A, \tilde{f}_{\sup})$ is a 1-fuzzy UP-subalgebra of A, we have $\tilde{f}_{\sup}(x \cdot y) \geq \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}$. Thus

$$\begin{aligned}
\tilde{f}_{\mathrm{l}}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\
&= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(0) \\
&\geq \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\} - \tilde{f}_{\inf}(0) \\
&= \min\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(0), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(0)\} \\
&= \min\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\} \\
&= \min\{\tilde{f}_{\mathrm{l}}(x), \tilde{f}_{\mathrm{l}}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\mathrm{l}})$ is a 1-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. □

Corollary 3.20. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 3-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.13 and 3.19. □

Corollary 3.21. *For $j \in \{1, 3\}$, every $(2(3), j)$-hyperfuzzy UP-subalgebra of A is a length 1-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 3.19 and Corollary 3.20. □

The following example show that the converse of Corollary 3.21 is not true.

Example 3.22. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given in as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	1	0	0	4
3	0	1	2	0	4
4	0	4	2	3	0

Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.1, 1) & (0.3, 0.8] & [0, 0.8] & [0.1, 0.3) & [0.1, 0.3) \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{l}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.9 & 0.2 & 0.8 & 0.2 & 0.2 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.1 & 0.3 & 0 & 0.1 & 0.1 \end{pmatrix},$$

we have $\tilde{f}_{\inf}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\inf}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(2(3), j)$-hyperfuzzy UP-subalgebra of A for $j \in \{1, 3\}$.

Theorem 3.23. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A. Let $x, y \in A$. Since $(A, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in A$. Since $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A, we have $\tilde{f}_{\sup}(x \cdot y) \leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}$. Thus

$$\begin{aligned}
\tilde{f}_{l}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\
&= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(0) \\
&\leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\} - \tilde{f}_{\inf}(0) \\
&= \max\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(0), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(0)\} \\
&= \max\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\} \\
&= \max\{\tilde{f}_{l}(x), \tilde{f}_{l}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{l})$ is a 4-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. □

Corollary 3.24. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 2-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.15 and 3.23. □

Corollary 3.25. *For $j \in \{2, 4\}$, every $(2(3), j)$-hyperfuzzy UP-subalgebra of A is a length 4-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 3.23 and Corollary 3.24. □

The following example show that the converse of Corollary 3.25 is not true.

Example 3.26. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 3.22. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.1, 0.4) & (0.2, 0.5] & [0.2, 0.7] & [0.3, 0.9) & [0.1, 1) \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{l}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.3 & 0.3 & 0.5 & 0.6 & 0.9 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.1 & 0.2 & 0.2 & 0.3 & 0.1 \end{pmatrix},$$

we have $\tilde{f}_{\inf}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\inf}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(2(3), j)$-hyperfuzzy UP-subalgebra of A for $j \in \{2, 4\}$.

Theorem 3.27. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A. Let $x, y \in A$. Since $(A, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in A$. Since $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A, we have $\tilde{f}_{\inf}(x \cdot y) \leq \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}$. Thus

$$\begin{aligned} \tilde{f}_{\mathrm{l}}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\ &= \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x \cdot y) \\ &\geq \tilde{f}_{\sup}(0) - \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\mathrm{l}}(x), \tilde{f}_{\mathrm{l}}(y)\}. \end{aligned}$$

Hence, $(A, \tilde{f}_{\mathrm{l}})$ is a 1-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. □

Corollary 3.28. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 2-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.15 and 3.27. □

Corollary 3.29. *For $i \in \{2, 4\}$, every $(i, 2(3))$-hyperfuzzy UP-subalgebra of A is a length 1-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 3.27 and Corollary 3.28. □

The following example show that the converse of Corollary 3.29 is not true.

Example 3.30. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 3.22. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.1, 1) & (0.2, 0.8] & [0.3, 0.8] & [0.4, 0.7) & [0.5, 0.7) \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{l}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.9 & 0.6 & 0.5 & 0.3 & 0.2 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.1 & 0.8 & 0.8 & 0.7 & 0.7 \end{pmatrix},$$

we have $\tilde{f}_{\sup}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\sup}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(j, 2(3))$-hyperfuzzy UP-subalgebra of A for $j \in \{2, 4\}$.

Theorem 3.31. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A. Let $x, y \in A$. Since $(A, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in A$. Since $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A, we have $\tilde{f}_{\inf}(x \cdot y) \geq \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}$. Thus

$$\begin{aligned}
\tilde{f}_{\mathrm{l}}(x \cdot y) &= \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\
&= \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x \cdot y) \\
&\leq \tilde{f}_{\sup}(0) - \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\} \\
&= \max\{\tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(y)\} \\
&= \max\{\tilde{f}_{\sup}(x) - \tilde{f}_{\inf}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{\inf}(y)\} \\
&= \max\{\tilde{f}_{\mathrm{l}}(x), \tilde{f}_{\mathrm{l}}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\mathrm{l}})$ is a 4-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. □

Corollary 3.32. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 3-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.13 and 3.31. □

Corollary 3.33. *For $i \in \{1, 3\}$, every $(i, 2(3))$-hyperfuzzy UP-subalgebra of A is a length 4-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 3.31 and Corollary 3.32. □

The following example show that the converse of Corollary 3.33 is not true.

Example 3.34. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 3.22. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.5, 0.6) & (0.4, 0.75] & [0.3, 0.8] & [0.2, 0.8) & [0.1, 1) \end{pmatrix}.$$

Then the length of $\tilde{f}$ is given as follows:

$$\tilde{f}_{l} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.1 & 0.35 & 0.5 & 0.7 & 0.9 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.6 & 0.75 & 0.8 & 0.8 & 1 \end{pmatrix},$$

we have $\tilde{f}_{\sup}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\sup}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(j, 2(3))$-hyperfuzzy UP-subalgebra of A for $j \in \{1, 3\}$.

Theorem 3.35. *If $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A in which $\tilde{f}_{\inf}$ is constant, then $(A, \tilde{f})$ is a $(k, 1)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A in which $\tilde{f}_{\inf}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\inf}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\inf}$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in A$. Let $x, y \in A$. Then

$\tilde{f}_l(x \cdot y) = \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(0)$. Thus

$$\begin{aligned}\tilde{f}_{\sup}(x \cdot y) &= \tilde{f}_l(x \cdot y) + \tilde{f}_{\inf}(0) \\ &\geq \min\{\tilde{f}_l(x), \tilde{f}_l(y)\} + \tilde{f}_{\inf}(0) \\ &= \min\{\tilde{f}_l(x) + \tilde{f}_{\inf}(0), \tilde{f}_l(y) + \tilde{f}_{\inf}(0)\} \\ &= \min\{\tilde{f}_l(x) + \tilde{f}_{\inf}(x), \tilde{f}_l(y) + \tilde{f}_{\inf}(y)\} \\ &= \min\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.\end{aligned}$$

Hence, $(A, \tilde{f}_{\sup})$ is a 1-fuzzy UP-subalgebra of A. Therefore, $(A, \tilde{f})$ is a $(k, 1)$-hyperfuzzy UP-subalgebra of A. □

Corollary 3.36. *If $(A, \tilde{f})$ is a length 3-fuzzy UP-subalgebra of A in which $\tilde{f}_{\inf}$ is constant, then $(A, \tilde{f})$ is a $(k, 1)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. It is straightforward by Theorems 4.6 and 3.35. □

Theorem 3.37. *If $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\inf}$ is constant, then $(A, \tilde{f})$ is a $(k, 4)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\inf}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\inf}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\inf}$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in A$. Let $x, y \in A$. Then $\tilde{f}_l(x \cdot y) = \tilde{f}_{\sup}(x \cdot y) - \tilde{f}_{\inf}(0)$. Thus

$$\begin{aligned}\tilde{f}_{\sup}(x \cdot y) &= \tilde{f}_l(x \cdot y) + \tilde{f}_{\inf}(0) \\ &\leq \max\{\tilde{f}_l(x), \tilde{f}_l(y)\} + \tilde{f}_{\inf}(0) \\ &= \max\{\tilde{f}_l(x) + \tilde{f}_{\inf}(0), \tilde{f}_l(y) + \tilde{f}_{\inf}(0)\} \\ &= \max\{\tilde{f}_l(x) + \tilde{f}_{\inf}(x), \tilde{f}_l(y) + \tilde{f}_{\inf}(y)\} \\ &= \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.\end{aligned}$$

Hence, $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A. Therefore, $(A, \tilde{f})$ is a $(k, 4)$-hyperfuzzy UP-subalgebra of A. □

Corollary 3.38. *If $(A, \tilde{f})$ is a length 2-fuzzy UP-subalgebra of A in which $\tilde{f}_{\inf}$ is constant, then $(A, \tilde{f})$ is a $(k, 4)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. It is straightforward by Theorems 4.7 and 3.37. □

Theorem 3.39. *If $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant, then $(A, \tilde{f})$ is a $(4, k)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(A, \tilde{f})$ is a length 1-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\sup}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\sup}$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in A$. Let $x, y \in A$. Then $\tilde{f}_{l}(x \cdot y) = \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x \cdot y)$. Thus

$$\begin{aligned}
\tilde{f}_{\inf}(x \cdot y) &= \tilde{f}_{\sup}(0) - \tilde{f}_{l}(x \cdot y) \\
&\leq \tilde{f}_{\sup}(0) - \min\{\tilde{f}_{l}(x), \tilde{f}_{l}(y)\} \\
&= \max\{\tilde{f}_{\sup}(0) - \tilde{f}_{l}(x), \tilde{f}_{\sup}(0) - \tilde{f}_{l}(y)\} \\
&= \max\{\tilde{f}_{\sup}(x) - \tilde{f}_{l}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{l}(y)\} \\
&= \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A. Therefore, $(A, \tilde{f})$ is a $(4, k)$-hyperfuzzy UP-subalgebra of A. □

Corollary 3.40. *If $(A, \tilde{f})$ is a length 3-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant, then $(A, \tilde{f})$ is a $(4, k)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. It is straightforward by Theorems 4.6 and 3.39. □

Theorem 3.41. *If $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant, then $(A, \tilde{f})$ is a $(1, k)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(A, \tilde{f})$ is a length 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\sup}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\sup}$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in A$. Let $x, y \in A$. Then $\tilde{f}_{l}(x \cdot y) = \tilde{f}_{\sup}(0) - \tilde{f}_{\inf}(x \cdot y)$. Thus

$$\begin{aligned}
\tilde{f}_{\inf}(x \cdot y) &= \tilde{f}_{\sup}(0) - \tilde{f}_{l}(x \cdot y) \\
&\geq \tilde{f}_{\sup}(0) - \max\{\tilde{f}_{l}(x), \tilde{f}_{l}(y)\} \\
&= \min\{\tilde{f}_{\sup}(0) - \tilde{f}_{l}(x), \tilde{f}_{\sup}(0) - \tilde{f}_{l}(y)\} \\
&= \min\{\tilde{f}_{\sup}(x) - \tilde{f}_{l}(x), \tilde{f}_{\sup}(y) - \tilde{f}_{l}(y)\} \\
&= \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A. Therefore, $(A, \tilde{f})$ is a $(1, k)$-hyperfuzzy UP-subalgebra of A. □

Corollary 3.42. *If $(A, \tilde{f})$ is a length 2-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant, then $(A, \tilde{f})$ is a $(1, k)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. It is straightforward by Theorems 4.7 and 3.41. □

4. Mean of a hyper structure in UP-algebras

In this section, we introduce the notion of the mean of a hyper structure in UP-algebras. The notions of mean fuzzy UP-subalgebras of UP-algebras are introduced, and related properties are investigated. Relations between mean fuzzy UP-subalgebras and hyperfuzzy UP-subalgebras are established. Moreover, we discuss the relationships among mean fuzzy UP-subalgebras and upper level subsets, lower level subsets, and equal level subsets of the length of a hyper structure in UP-algebras.

Definition 4.1. Given a hyper structure $(A, \tilde{f})$ over A, we define a fuzzy structures $(A, \tilde{f}_{\mathrm{m}})$ in A as follows:

$$\tilde{f}_{\mathrm{m}} : A \to [0,1], x \mapsto \frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}$$

which is called the *mean* of $\tilde{f}$.

Definition 4.2. A hyper structure $(A, \tilde{f})$ over A is called a *mean 1-fuzzy* (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) *UP-subalgebra* of A if a fuzzy structures $(A, \tilde{f}_{\mathrm{m}})$ is a 1-fuzzy (resp., 2-fuzzy, 3-fuzzy and 4-fuzzy) UP-subalgebra of A.

Example 4.3. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	0
3	0	0	2	0	4
4	0	0	1	3	0

Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.6, 0.9) & (0.5, 0.9] & [0.2, 0.4) \cup [0.5, 0.8) & [0.3, 0.5] & [0.1, 0.3] \cup (0.4, 0.6] \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.75 & 0.7 & 0.5 & 0.4 & 0.35 \end{pmatrix}.$$

Thus $(A, \tilde{f}_{\mathrm{m}})$ is a 1-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A.

Proposition 4.4. *If $(A, \tilde{f})$ is a mean k-fuzzy UP-subalgebra of A for $k = 1, 3$, then*

$$(\forall x \in A)(\tilde{f}_{\mathrm{m}}(0) \geq \tilde{f}_{\mathrm{m}}(x)). \tag{4.1}$$

Proof. It is straightforward by Proposition 2.11. □

Proposition 4.5. *If $(A, \tilde{f})$ is a mean k-fuzzy UP-subalgebra of A for $k = 2, 4$, then*

$$(\forall x \in A)(\tilde{f}_{\mathrm{m}}(0) \leq \tilde{f}_{\mathrm{m}}(x)). \tag{4.2}$$

Proof. It is straightforward by Proposition 2.12. □

Theorem 4.6. *Every mean 3-fuzzy UP-subalgebra of A is a mean 1-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 2.13. □

Theorem 4.7. *Every mean 2-fuzzy UP-subalgebra of A is a mean 4-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorem 2.15. □

Theorem 4.8. *Mean 2-fuzzy UP-subalgebra and mean 3-fuzzy UP-subalgebra of A coincide.*

Proof. It is straightforward by Theorems 2.17 and 2.18. □

Theorem 4.9. *Given a UP-subalgebra S of A and $B_1, B_2 \in \tilde{P}([0,1])$, let $(A, \tilde{f})$ be a hyper structure over A given by*

$$\tilde{f} : A \to \tilde{P}([0,1]), x \mapsto \begin{cases} B_2 & \text{if } x \in S, \\ B_1 & \text{otherwise.} \end{cases}$$

(i) If $\sup B_2 \geq \sup B_1$ and $\inf B_2 \geq \inf B_1$, then $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A.
(ii) If $\sup B_2 \leq \sup B_1$ and $\inf B_2 \leq \inf B_1$, then $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A.

Proof. If $x \in S$, then $\tilde{f}(x) = B_2$ and so

$$\tilde{f}_{\mathrm{m}}(x) = \frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2} = \frac{\sup \tilde{f}(x) + \inf \tilde{f}(x)}{2} = \frac{\sup B_2 + \inf B_2}{2}.$$

If $x \notin S$, then $\tilde{f}(x) = B_1$ and so

$$\tilde{f}_{\mathrm{m}}(x) = \frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2} = \frac{\sup \tilde{f}(x) + \inf \tilde{f}(x)}{2} = \frac{\sup B_1 + \inf B_1}{2}.$$

Assume that $\sup B_2 \geq \sup B_1$ and $\inf B_2 \geq \inf B_1$. Then $\frac{\sup B_2 + \inf B_2}{2} \geq \frac{\sup B_1 + \inf B_1}{2}$.

Case 1: Let $x, y \in S$. Then $\tilde{f}_{\mathrm{m}}(x) = \dfrac{\sup B_2 + \inf B_2}{2}$ and $\tilde{f}_{\mathrm{m}}(y) = \dfrac{\sup B_2 + \inf B_2}{2}$. Thus $\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} = \dfrac{\sup B_2 + \inf B_2}{2}$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so

$$\tilde{f}_{\mathrm{m}}(x \cdot y) = \frac{\sup B_2 + \inf B_2}{2}.$$

Thus

$$\tilde{f}_{\mathrm{m}}(x \cdot y) = \frac{\sup B_2 + \inf B_2}{2} = (\geq) \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Case 2: Let $x, y \notin S$. Then $\tilde{f}_{\mathrm{m}}(x) = \dfrac{\sup B_1 + \inf B_1}{2}$ and $\tilde{f}_{\mathrm{m}}(y) = \dfrac{\sup B_1 + \inf B_1}{2}$, so $\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} = \dfrac{\sup B_1 + \inf B_1}{2}$. Thus

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\tilde{f}_{\mathrm{m}}(x) = \dfrac{\sup B_1 + \inf B_1}{2}$ and $\tilde{f}_{\mathrm{m}}(y) = \dfrac{\sup B_2 + \inf B_2}{2}$, so $\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} = \dfrac{\sup B_1 + \inf B_1}{2}$. Thus

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\tilde{f}_{\mathrm{m}}(x) = \dfrac{\sup B_2 + \inf B_2}{2}$ and $\tilde{f}_{\mathrm{m}}(y) = \dfrac{\sup B_1 + \inf B_1}{2}$, so $\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} = \dfrac{\sup B_1 + \inf B_1}{2}$. Thus

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq \frac{\sup B_1 + \inf B_1}{2} = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Hence, $\tilde{f}_{\mathrm{m}}$ is a 1-fuzzy UP-subalgebra of A and so $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A.

Assume that $\sup B_2 \leq \sup B_1$ and $\inf B_2 \leq \inf B_1$. Then $\dfrac{\sup B_2 + \inf B_2}{2} \leq \dfrac{\sup B_1 + \inf B_1}{2}$.

Case 1: Let $x, y \in S$. Then $\tilde{f}_{\mathrm{m}}(x) = \dfrac{\sup B_2 + \inf B_2}{2}$ and $\tilde{f}_{\mathrm{m}}(y) = \dfrac{\sup B_2 + \inf B_2}{2}$, so $\max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} = \dfrac{\sup B_2 + \inf B_2}{2}$. Since S is a UP-subalgebra of A, we have $x \cdot y \in S$ and so

$$\tilde{f}_{\mathrm{m}}(x \cdot y) = \frac{\sup B_2 + \inf B_2}{2}.$$

Thus

$$\tilde{f}_{\rm m}(x\cdot y)=\frac{\sup B_2+\inf B_2}{2}=(\leq)\max\{\tilde{f}_{\rm m}(x),\tilde{f}_{\rm m}(y)\}$$

Case 2: Let $x, y \notin S$. Then $\tilde{f}_{\rm m}(x)=\dfrac{\sup B_1+\inf B_1}{2}$ and $\tilde{f}_{\rm m}(y)=\dfrac{\sup B_1+\inf B_1}{2}$, so $\max\{\tilde{f}_{\rm m}(x),\tilde{f}_{\rm m}(y)\}=\dfrac{\sup B_1+\inf B_1}{2}$. Thus

$$\tilde{f}_{\rm m}(x\cdot y)\leq\frac{\sup B_1+\inf B_1}{2}=\max\{\tilde{f}_{\rm m}(x),\tilde{f}_{\rm m}(y)\}.$$

Case 3: Let $x \notin S$ and $y \in S$. Then $\tilde{f}_{\rm m}(x)=\dfrac{\sup B_1+\inf B_1}{2}$ and $\tilde{f}_{\rm m}(y)=\dfrac{\sup B_2+\inf B_2}{2}$. so $\max\{\tilde{f}_{\rm m}(x),\tilde{f}_{\rm m}(y)\}=\dfrac{\sup B_1+\inf B_1}{2}$. Thus

$$\tilde{f}_{\rm m}(x\cdot y)\leq\frac{\sup B_1+\inf B_1}{2}=\max\{\tilde{f}_{\rm m}(x),\tilde{f}_{\rm m}(y)\}.$$

Case 4: Let $x \in S$ and $y \notin S$. Then $\tilde{f}_{\rm m}(x)=\dfrac{\sup B_2+\inf B_2}{2}$ and $\tilde{f}_{\rm m}(y)=\dfrac{\sup B_1+\inf B_1}{2}$, so $\max\{\tilde{f}_{\rm m}(x),\tilde{f}_{\rm m}(y)\}=\dfrac{\sup B_1+\inf B_1}{2}$. Thus

$$\tilde{f}_{\rm m}(x\cdot y)\leq\frac{\sup B_1+\inf B_1}{2}=\max\{\tilde{f}_{\rm m}(x),\tilde{f}_{\rm m}(y)\}.$$

Hence, $\tilde{f}_{\rm m}$ is a 4-fuzzy UP-subalgebra of A and so $(A,\tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. □

Example 4.10. Consider a UP-algebra $A=\{0,1,2,3,4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	0
2	0	1	0	3	4
3	0	1	2	0	4
4	0	3	2	3	0

Then $S=\{0,1,2\}$ is a UP-subalgebra of A. Let $(A,\tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f}=\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.04,0.18) & [0.04,0.18) & [0.04,0.18) & (0.03,0.11] & (0.03,0.11] \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\rm m}=\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.11 & 0.11 & 0.11 & 0.07 & 0.07 \end{pmatrix}.$$

By Theorem 4.9, we have $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A. We see that

$$\tilde{f}_{\mathrm{m}}(2 \cdot 3) = 0.07 \ngeq 0.11 = \max\{0.11, 0.07\} = \max\{\tilde{f}_{\mathrm{m}}(2), \tilde{f}_{\mathrm{m}}(3)\}.$$

Thus $(A, \tilde{f}_{\mathrm{m}})$ is not a 3-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is not a mean 3-fuzzy UP-subalgebra of A. Give a UP-subalgebra $S = \{0, 1, 2, 3\}$ of A, let $(A, \tilde{f})$ be a hyper structure over A given by

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ (0.4, 0.7) & (0.4, 0.7) & (0.4, 0.7) & (0.4, 0.7) & [0.5, 0.9) \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.55 & 0.55 & 0.55 & 0.55 & 0.7 \end{pmatrix}.$$

By Theorem 4.9, we have $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. We see that

$$\tilde{f}_{\mathrm{m}}(2 \cdot 4) = \tilde{f}_{\mathrm{m}}(4) = 0.7 \nleq 0.55 = \max\{0.11, 0.07\} = \min\{\tilde{f}_{\mathrm{m}}(2), \tilde{f}_{\mathrm{m}}(4)\}.$$

Thus $(A, \tilde{f}_{\mathrm{m}})$ is not a 2-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is not a mean 2-fuzzy UP-subalgebra of A.

Theorem 4.11. *A hyper structure $(A, \tilde{f})$ over A is a mean 1-fuzzy UP-subalgebra of A if and only if the set $U(\tilde{f}_{\mathrm{m}}; t)$ is a UP-subalgebra of A for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$.*

Proof. Assume that $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$ and let $x, y \in U(\tilde{f}_{\mathrm{m}}; t)$. Then $\tilde{f}_{\mathrm{m}}(x) \geq t$ and $\tilde{f}_{\mathrm{m}}(y) \geq t$. Since $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A, we have

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} \geq t.$$

Thus $x \cdot y \in U(\tilde{f}_{\mathrm{m}}; t)$. Hence, $U(\tilde{f}_{\mathrm{m}}; t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [0, 1]$, the set $U(\tilde{f}_{\mathrm{m}}; t)$ is a UP-subalgebra of A if $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$. Let $x, y \in A$. Then $\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y) \in [0, 1]$. Choose $t = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}$. Thus $\tilde{f}_{\mathrm{m}}(x) \geq t$ and $\tilde{f}_{\mathrm{m}}(y) \geq t$ and so $x, y \in U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$. By assumption, we have $U(\tilde{f}_{\mathrm{m}}; t)$ is a UP-subalgebra of A and so $x \cdot y \in U(\tilde{f}_{\mathrm{m}}; t)$. Thus

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \geq t = \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.$$

Hence, $(A, \tilde{f}_{\mathrm{m}})$ is a 1-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A. □

Corollary 4.12. *If $(A, \tilde{f})$ is a mean 3-fuzzy UP-subalgebra of A, then $U(\tilde{f}_{\mathrm{m}}; t)$ is a UP-subalgebra of A for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$.*

Proof. It is straightforward by Theorems 4.6 and 4.11. □

The following example show that the converse of Corollary 4.12 is not true.

Example 4.13. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	1	0	0	0
3	0	1	2	0	0
4	0	1	2	3	0

Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.6, 0.9) & (0.5, 0.7] & [0.2, 0.4] \cup (0.5, 0.9] & (0.3, 0.5] & [0.1, 0.7] \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.75 & 0.6 & 0.55 & 0.4 & 0.4 \end{pmatrix}.$$

We have

$$U(\tilde{f}_{\mathrm{m}}; t) = \begin{cases} \emptyset & \text{if } t \in (0.75, 1], \\ \{0\} & \text{if } t \in (0.6, 0.75], \\ \{0, 1\} & \text{if } t \in (0.55, 0.6], \\ \{0, 1, 2\} & \text{if } t \in (0.4, 0.55], \\ A & \text{if } t \in [0, 0.4] \end{cases}$$

and so $U(\tilde{f}_{\mathrm{m}}; t)$ is a UP-subalgebra of A for all $t \in [0, 1]$ with $U(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$. We see that

$$\tilde{f}_{\mathrm{m}}(0 \cdot 2) = \tilde{f}_{\mathrm{m}}(2) = 0.55 \not\geq 0.75 = \max\{\tilde{f}_{\mathrm{m}}(0), \tilde{f}_{\mathrm{m}}(4)\}.$$

Thus $(A, \tilde{f}_{\mathrm{m}})$ is not a 3-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is not a mean 3-fuzzy UP-subalgebra of A.

Theorem 4.14. *A hyper structure $(A, \tilde{f})$ over A is a mean 4-fuzzy UP-subalgebra of A if and only if the set $L(\tilde{f}_{\mathrm{m}}; t)$ is a UP-subalgebra of A for all $t \in [0, 1]$ with $L(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$.*

Proof. Assume that $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $L(\tilde{f}_{\mathrm{m}}; t) \neq \emptyset$ and let $x, y \in L(\tilde{f}_{\mathrm{m}}; t)$. Then $\tilde{f}_{\mathrm{m}}(x) \leq t$ and $\tilde{f}_{\mathrm{m}}(y) \leq t$. Since $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A, we have

$$\tilde{f}_{\mathrm{m}}(x \cdot y) \leq \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} \leq t.$$

Thus $x \cdot y \in L(\tilde{f}_{\rm m}; t)$. Hence, $L(\tilde{f}_{\rm m}; t)$ is a UP-subalgebra of A.

Conversely, assume that for all $t \in [0, 1]$, the set $L(\tilde{f}_{\rm m}; t)$ is a UP-subalgebra of A if $L(\tilde{f}_{\rm m}; t) \neq \emptyset$. Let $x, y \in A$. Then $\tilde{f}_{\rm m}(x), \tilde{f}_{\rm m}(y) \in [0, 1]$. Choose $t = \max\{\tilde{f}_{\rm m}(x), \tilde{f}_{\rm m}(y)\}$. Thus $\tilde{f}_{\rm m}(x) \leq t$ and $\tilde{f}_{\rm m}(y) \leq t$, and so $x, y \in L(\tilde{f}_{\rm m}; t) \neq \emptyset$. By assumption, we have $L(\tilde{f}_{\rm m}; t)$ is a UP-subalgebra of A and so $x \cdot y \in L(\tilde{f}_{\rm m}; t)$. Thus

$$\tilde{f}_{\rm m}(x \cdot y) \leq t = \max\{\tilde{f}_{\rm m}(x), \tilde{f}_{\rm m}(y)\}.$$

Hence, $(A, \tilde{f}_{\rm m})$ is a 4-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. □

Corollary 4.15. *If $(A, \tilde{f})$ is a mean 2-fuzzy UP-subalgebra of A, then $L(\tilde{f}_{\rm m}; t)$ is a UP-subalgebra of A for all $t \in [0, 1]$ with $L(\tilde{f}_{\rm m}; t) \neq \emptyset$.*

Proof. It is straightforward by Theorems 4.7 and 4.14. □

The following example show that the converse of Corollary 4.15 is not true.

Example 4.16. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	4
3	0	1	1	0	4
4	0	1	2	3	0

Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.3, 0.5) & [0.3, 0.4] \cup (0.5, 0.7] & [0.1, 0.9] & [0.5, 0.6] \cup (0.8, 0.9] & [0.7, 0.8] \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\rm m} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.4 & 0.5 & 0.5 & 0.7 & 0.75 \end{pmatrix}.$$

We have

$$L(\tilde{f}_{\rm m}; t) = \begin{cases} A & \text{if } t \in [0.75, 1], \\ \{0, 1, 2, 3\} & \text{if } t \in [0.7, 0.75), \\ \{0, 1, 2\} & \text{if } t \in [0.5, 0.7), \\ \{0\} & \text{if } t \in [0.4, 0.5), \\ \emptyset & \text{if } t \in [0, 0.4) \end{cases}$$

and so $L(\tilde{f}_{\mathrm{m}};t)$ is a UP-subalgebra of A for all $t\in[0,1]$ with $L(\tilde{f}_{\mathrm{m}};t)\neq\emptyset$. We see that

$$\tilde{f}_{\mathrm{m}}(0\cdot 2)=0.5\nleq 0.4=\min\{\tilde{f}_{\mathrm{m}}(0),\tilde{f}_{\mathrm{m}}(2)\}.$$

Thus $(A,\tilde{f}_{\mathrm{m}})$ is not a 2-fuzzy UP-subalgebra of A, that is, $(A,\tilde{f})$ is not a mean 2-fuzzy UP-subalgebra of A.

Theorem 4.17. *A hyper structure $(A,\tilde{f})$ over A is a mean 2(3)-fuzzy UP-subalgebra of A if and only if the set $E(\tilde{f}_{\mathrm{m}};\tilde{f}_{\mathrm{m}}(0))=A$.*

Proof. Assume that $(A,\tilde{f})$ is a mean 2-fuzzy UP-subalgebra of A. Then $\tilde{f}_{\mathrm{m}}$ is a 2-fuzzy UP-subalgebra of A. By Theorem 2.17, we have is constant and so $\tilde{f}_{\mathrm{m}}(x)=\tilde{f}_{\mathrm{m}}(0)$ for all $x\in A$. Thus $x\in E(\tilde{f}_{\mathrm{m}};\tilde{f}_{\mathrm{m}}(0))$ for all $x\in A$. Therefore, $E(\tilde{f}_{\mathrm{m}};\tilde{f}_{\mathrm{m}}(0))=A$.

Conversely, assume that $E(\tilde{f}_{\mathrm{m}};\tilde{f}_{\mathrm{m}}(0))=A$. Then $\tilde{f}_{\mathrm{m}}(x)=\tilde{f}_{\mathrm{m}}(0)$ for all $x\in A$. Thus $\tilde{f}_{\mathrm{m}}$ is constant. By Theorem 2.17, we have $\tilde{f}_{\mathrm{m}}$ is a 2-fuzzy UP-subalgebra$A$. Therefore, $(A,\tilde{f})$ is a mean 2-fuzzy UP-subalgebra of A. □

Theorem 4.18. *If $(A,\tilde{f})$ is a hyper structure over A in which $(A,\tilde{f}_{\mathrm{inf}})$ is constant and $(A,\tilde{f}_{\mathrm{sup}})$ is a 1-fuzzy UP-subalgebra of A, then $(A,\tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A,\tilde{f})$ is a hyper structure over A in which $(A,\tilde{f}_{\mathrm{inf}})$ is constant and $(A,\tilde{f}_{\mathrm{sup}})$ is a 1-fuzzy UP-subalgebra of A. Let $x,y\in A$. Since $(A,\tilde{f}_{\mathrm{inf}})$ is constant, we have $\tilde{f}_{\mathrm{inf}}(x)=\tilde{f}_{\mathrm{inf}}(0)$ for all $x\in A$. Since $(A,\tilde{f}_{\mathrm{sup}})$ is a 1-fuzzy UP-subalgebra of A, we have $\tilde{f}_{\mathrm{sup}}(x\cdot y)\geq\min\{\tilde{f}_{\mathrm{sup}}(x),\tilde{f}_{\mathrm{sup}}(y)\}$. Thus

$$\begin{aligned}
\tilde{f}_{\mathrm{m}}(x\cdot y)&=\frac{\tilde{f}_{\mathrm{sup}}(x\cdot y)+\tilde{f}_{\mathrm{inf}}(x\cdot y)}{2}\\
&=\frac{\tilde{f}_{\mathrm{sup}}(x\cdot y)}{2}+\frac{\tilde{f}_{\mathrm{inf}}(0)}{2}\\
&\geq\min\{\frac{\tilde{f}_{\mathrm{sup}}(x)}{2},\frac{\tilde{f}_{\mathrm{sup}}(y)}{2}\}+\frac{\tilde{f}_{\mathrm{inf}}(0)}{2}\\
&=\min\{\frac{\tilde{f}_{\mathrm{sup}}(x)}{2}+\frac{\tilde{f}_{\mathrm{inf}}(0)}{2},\frac{\tilde{f}_{\mathrm{sup}}(y)}{2}+\frac{\tilde{f}_{\mathrm{inf}}(0)}{2}\}\\
&=\min\{\frac{\tilde{f}_{\mathrm{sup}}(x)+\tilde{f}_{\mathrm{inf}}(x)}{2},\frac{\tilde{f}_{\mathrm{sup}}(y)+\tilde{f}_{\mathrm{inf}}(y)}{2}\}\\
&=\min\{\tilde{f}_{\mathrm{m}}(x),\tilde{f}_{\mathrm{m}}(y)\}.
\end{aligned}$$

Hence $(A,\tilde{f}_{\mathrm{m}})$ is a 1-fuzzy UP-subalgebra of A, that is, $(A,\tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A. □

Corollary 4.19. *If $(A, \tilde{f})$ be a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 3-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.13 and 4.18. □

Corollary 4.20. *For $j \in \{1, 3\}$, every $(2(3), j)$-hyperfuzzy UP-subalgebra is a mean 1-fuzzy UP-subalgebra.*

Proof. It is straightforward by Theorems 4.18 and 4.19. □

The following example show that the converse of Corollary 4.20 is not true.

Example 4.21. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ with the binary operation $\cdot$ which is given as follows:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	2	0	3	4
3	0	2	2	0	4
4	0	2	2	3	0

Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.6, 0.9) & (0.5, 0.8] & [0.1, 0.9] & [0.3, 0.6) & [0.3, 0.6) \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.75 & 0.65 & 0.5 & 0.45 & 0.45 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.6 & 0.5 & 0.1 & 0.3 & 0.3 \end{pmatrix},$$

we have $\tilde{f}_{\inf}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\inf}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(2(3), j)$-hyperfuzzy UP-subalgebra of A for $j \in \{1, 3\}$.

Theorem 4.22. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A. Let $x, y \in A$. Since $(A, \tilde{f}_{\inf})$ is constant, we have $\tilde{f}_{\inf}(x) = \tilde{f}_{\inf}(0)$ for all $x \in A$.

Since $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A, we have $\tilde{f}_{\sup}(x \cdot y) \leq \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}$. Thus

$$\begin{aligned}
\tilde{f}_{\mathrm{m}}(x \cdot y) &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2} \\
&= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(0)}{2} \\
&= \frac{\tilde{f}_{\sup}(x \cdot y)}{2} + \frac{\tilde{f}_{\sup}(0)}{2} \\
&\leq \max\{\frac{\tilde{f}_{\sup}(x)}{2}, \frac{\tilde{f}_{\sup}(y)}{2}\} + \frac{\tilde{f}_{\sup}(0)}{2} \\
&= \max\{\frac{\tilde{f}_{\sup}(x)}{2} + \frac{\tilde{f}_{\sup}(0)}{2}, \frac{\tilde{f}_{\sup}(y)}{2} + \frac{\tilde{f}_{\sup}(0)}{2}\} \\
&= \max\{\frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(y) + \tilde{f}_{\inf}(y)}{2}\} \\
&= \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\mathrm{m}})$ is a 4-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. □

Corollary 4.23. *If $(A, \tilde{f})$ be a hyper structure over A in which $(A, \tilde{f}_{\inf})$ is constant and $(A, \tilde{f}_{\sup})$ is a 2-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.15 and 4.22. □

Corollary 4.24. *For $j \in \{2, 4\}$, every $(2(3), j)$-hyperfuzzy UP-subalgebra is a mean 4-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 4.22 and 4.23. □

The following example show that the converse of Corollary 4.32 is not true.

Example 4.25. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 4.21. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.3, 0.6) & (0.3, 0.6] & [0.1, 0.9] & [0.5, 0.8) & [0.6, 0.9) \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.45 & 0.45 & 0.5 & 0.65 & 0.75 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\inf} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.3 & 0.3 & 0.1 & 0.5 & 0.6 \end{pmatrix},$$

we have $\tilde{f}_{\inf}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\inf}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(2(3), j)$-hyperfuzzy UP-subalgebra of A for $j \in \{2, 4\}$.

Theorem 4.26. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A. Let $x, y \in A$. Since $(A, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for some $x \in A$. Since $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A, we have $\tilde{f}_{\inf}(x \cdot y) \leq \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}$. Thus

$$\begin{aligned} \tilde{f}_{\mathrm{m}}(x \cdot y) &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2} \\ &= \frac{\tilde{f}_{\sup}(0) + \tilde{f}_{\inf}(x \cdot y)}{2} \\ &= \frac{\tilde{f}_{\sup}(0)}{2} + \frac{\tilde{f}_{\inf}(x \cdot y)}{2} \\ &\leq \frac{\tilde{f}_{\sup}(0)}{2} + \max\{\frac{\tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\inf}(y)}{2}\} \\ &= \max\{\frac{\tilde{f}_{\sup}(0)}{2} + \frac{\tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(0)}{2} + \frac{\tilde{f}_{\inf}(y)}{2}\} \\ &= \max\{\frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(y) + \tilde{f}_{\inf}(y)}{2}\} \\ &= \max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}. \end{aligned}$$

Hence, $(A, \tilde{f}_{\mathrm{m}})$ is a 4-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. □

Corollary 4.27. *If $(A, \tilde{f})$ be a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 2-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.15 and 4.26. □

Corollary 4.28. *For $i \in \{2, 4\}$, every $(i, 2(3))$-hyperfuzzy UP-subalgebra is a mean 4-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 4.26 and Corollary 4.27. □

The following example show that the converse of Corollary 4.28 is not true.

Example 4.29. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 4.21. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.6, 0.9) & (0.5, 0.8] & [0.4, 0.9] & [0.3, 0.6) & [0.3, 0.6) \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.75 & 0.65 & 0.65 & 0.45 & 0.45 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.6 & 0.5 & 0.4 & 0.6 & 0.6 \end{pmatrix},$$

we have $\tilde{f}_{\sup}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\sup}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(j, 2(3))$-hyperfuzzy UP-subalgebra of A for $j \in \{2, 4\}$.

Theorem 4.30. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A.*

Proof. Assume that $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A. Let $x, y \in A$. Since $(A, \tilde{f}_{\sup})$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all$x \in A$. Since $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A, we obtain $\tilde{f}_{\inf}(x \cdot y) \geq$

$\min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}$. Thus

$$\begin{aligned}
\tilde{f}_{\mathrm{m}}(x \cdot y) &= \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2} \\
&= \frac{\tilde{f}_{\sup}(0) + \tilde{f}_{\inf}(x \cdot y)}{2} \\
&= \frac{\tilde{f}_{\sup}(0)}{2} + \frac{\tilde{f}_{\inf}(x \cdot y)}{2} \\
&\geq \frac{\tilde{f}_{\sup}(0)}{2} + \min\{\frac{\tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\inf}(y)}{2}\} \\
&= \min\{\frac{\tilde{f}_{\sup}(0)}{2} + \frac{\tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(0)}{2} + \frac{\tilde{f}_{\inf}(y)}{2}\} \\
&= \min\{\frac{\tilde{f}_{\sup}(x) + \tilde{f}_{\inf}(x)}{2}, \frac{\tilde{f}_{\sup}(y) + \tilde{f}_{\inf}(y)}{2}\} \\
&= \min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\mathrm{m}})$ is a 1-fuzzy UP-subalgebra of A, that is, $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A. □

Corollary 4.31. *If $(A, \tilde{f})$ is a hyper structure over A in which $(A, \tilde{f}_{\sup})$ is constant and $(A, \tilde{f}_{\inf})$ is a 3-fuzzy UP-subalgebra of A, then $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 2.13 and 4.22. □

Corollary 4.32. *For $i \in \{1, 3\}$, every $(i, 2(3))$-hyperfuzzy UP-subalgebra is a mean 1-fuzzy UP-subalgebra of A.*

Proof. It is straightforward by Theorems 4.30 and Corollary 4.31. □

The following example show that the converse of Corollary 4.32 is not true.

Example 4.33. Consider a UP-algebra $A = \{0, 1, 2, 3, 4\}$ in Example 4.21. Let $(A, \tilde{f})$ be a hyper structure over A in which $\tilde{f}$ is given as follows:

$$\tilde{f} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ [0.3, 0.6) & (0.3, 0.] & [0.4, 0.9] & [0.5, 0.8) & [0.6, 0.9) \end{pmatrix}.$$

Then the mean of $\tilde{f}$ is given as follows:

$$\tilde{f}_{\mathrm{m}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.45 & 0.45 & 0.65 & 0.65 & 0.75 \end{pmatrix}.$$

Thus $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A. Since

$$\tilde{f}_{\sup} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0.6 & 0.6 & 0.9 & 0.8 & 0.9 \end{pmatrix},$$

we have $\tilde{f}_{\text{sup}}$ is not constant. By Theorems 2.17 and 2.18, we have $\tilde{f}_{\text{sup}}$ is not 2(3)-fuzzy UP-subalgebra of A. Hence, $(A, \tilde{f})$ is not a $(j, 2(3))$-hyperfuzzy UP-subalgebra of A for $j \in \{1, 3\}$.

Theorem 4.34. *If $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A in which $\tilde{f}_{\text{inf}}$ is constant, then $(A, \tilde{f})$ is a $(k, 1)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(A, \tilde{f})$ is a mean 1-fuzzy UP-subalgebra of A in which $\tilde{f}_{\text{inf}}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\text{inf}}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\text{inf}}$ is constant, we have $\tilde{f}_{\text{inf}}(x) = \tilde{f}_{\text{inf}}(0)$ for all $x \in A$. Let $x, y \in A$. Then

$$\tilde{f}_{\text{m}}(x \cdot y) = \frac{\tilde{f}_{\text{sup}}(x \cdot y) + \tilde{f}_{\text{inf}}(x \cdot y)}{2}.$$

Thus

$$\begin{aligned}
\tilde{f}_{\text{sup}}(x \cdot y) &= 2\tilde{f}_{\text{m}}(x \cdot y) - \tilde{f}_{\text{inf}}(x \cdot y) \\
&= 2\tilde{f}_{\text{m}}(x \cdot y) - \tilde{f}_{\text{inf}}(0) \\
&\geq 2\min\{\tilde{f}_{\text{m}}(x), \tilde{f}_{\text{m}}(y)\} - \tilde{f}_{\text{inf}}(0) \\
&= \min\{2\tilde{f}_{\text{m}}(x), 2\tilde{f}_{\text{m}}(y)\} - \tilde{f}_{\text{inf}}(0) \\
&= \min\{2\tilde{f}_{\text{m}}(x) - \tilde{f}_{\text{inf}}(0), 2\tilde{f}_{\text{m}}(y) - \tilde{f}_{\text{inf}}(0)\} \\
&= \min\{2\tilde{f}_{\text{m}}(x) - \tilde{f}_{\text{inf}}(x), 2\tilde{f}_{\text{m}}(y) - \tilde{f}_{\text{inf}}(y)\} \\
&= \min\{\tilde{f}_{\text{sup}}(x), \tilde{f}_{\text{sup}}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\text{sup}})$ is a 1-fuzzy UP-subalgebra of A. Therefore, $(A, \tilde{f})$ is a $(k,1)$-hyperfuzzy UP-subalgebra of A. □

Corollary 4.35. *If $(A, \tilde{f})$ is a mean 3-fuzzy subalgebra of A in which $\tilde{f}_{\text{inf}}$ is constant, then $(A, \tilde{f})$ is a $(k, 1)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. It is straightforward by Theorems 2.13 and 4.34. □

Theorem 4.36. *If $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\text{inf}}$ is constant, then $(A, \tilde{f})$ is a $(k, 4)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\text{inf}}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\text{inf}}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\text{inf}}$ is constant, we have $\tilde{f}_{\text{inf}}(x) = \tilde{f}_{\text{inf}}(0)$ for all $x \in A$. Let $x, y \in A$. Then

$$\tilde{f}_{\text{m}}(x \cdot y) = \frac{\tilde{f}_{\text{sup}}(x \cdot y) + \tilde{f}_{\text{inf}}(x \cdot y)}{2}.$$

Thus

$$\begin{aligned}
\tilde{f}_{\sup}(x \cdot y) &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\inf}(x \cdot y) \\
&= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\inf}(0) \\
&\leq 2\max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\inf}(0) \\
&= \max\{2\tilde{f}_{\mathrm{m}}(x), 2\tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\inf}(0) \\
&= \max\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\inf}(0), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\inf}(0)\} \\
&= \max\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\inf}(x), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\inf}(y)\} \\
&= \max\{\tilde{f}_{\sup}(x), \tilde{f}_{\sup}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\sup})$ is a 4-fuzzy UP-subalgebra of A. Therefore, $(A, \tilde{f})$ is a $(k, 4)$-hyperfuzzy UP-subalgebra of A. □

Corollary 4.37. *If $(A, \tilde{f})$ is a mean 2-fuzzy UP-subalgebra of A in which $\tilde{f}_{\inf}$ is constant, then $(A, \tilde{f})$ is a $(k, 4)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. It is straightforward by Theorems 2.15 and 4.36. □

Theorem 4.38. *If $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant, then $(A, \tilde{f})$ is a $(4, k)$-hyperfuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$.*

Proof. Assume that $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\sup}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\sup}$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in A$. Let $x, y \in A$. Then

$$\tilde{f}_{\mathrm{m}}(x \cdot y) = \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2}.$$

Thus

$$\begin{aligned}
\tilde{f}_{\inf}(x \cdot y) &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(x \cdot y) \\
&= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(0) \\
&\geq 2\max\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\sup}(0) \\
&= \max\{2\tilde{f}_{\mathrm{m}}(x), 2\tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\sup}(0) \\
&= \max\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\sup}(0), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\sup}(0)\} \\
&= \max\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\sup}(x), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\sup}(y)\} \\
&= \max\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\inf})$ is a 4-fuzzy UP-subalgebra of A. Therefore, $(A, \tilde{f})$ is a $(4, k)$-hyperfuzzy UP-subalgebra of A. □

Corollary 4.39. *If* $(A, \tilde{f})$ *is a mean* 2*-fuzzy UP-subalgebra of* A *in which* $\tilde{f}_{\sup}$ *is constant, then* $(A, \tilde{f})$ *is a* $(4, k)$*-hyperfuzzy UP-subalgebra of* A *for* $k \in \{1, 2, 3, 4\}$*.*

Proof. It is straightforward by Theorems 2.15 and 4.38. □

Theorem 4.40. *If* $(A, \tilde{f})$ *is a mean* 1*-fuzzy UP-subalgebra of* A *in which* $\tilde{f}_{\sup}$ *is constant, then* $(A, \tilde{f})$ *is a* $(1, k)$*-hyperfuzzy UP-subalgebra of* A *for* $k \in \{1, 2, 3, 4\}$*.*

Proof. Assume that $(A, \tilde{f})$ is a mean 4-fuzzy UP-subalgebra of A in which $\tilde{f}_{\sup}$ is constant. By Theorems 2.13, 2.15, 2.17, and 2.18, we have $\tilde{f}_{\sup}$ is a k-fuzzy UP-subalgebra of A for $k \in \{1, 2, 3, 4\}$. Since $\tilde{f}_{\sup}$ is constant, we have $\tilde{f}_{\sup}(x) = \tilde{f}_{\sup}(0)$ for all $x \in A$. Let $x, y \in A$. Then

$$\tilde{f}_{\mathrm{m}}(x \cdot y) = \frac{\tilde{f}_{\sup}(x \cdot y) + \tilde{f}_{\inf}(x \cdot y)}{2}.$$

Thus

$$\begin{aligned}
\tilde{f}_{\inf}(x \cdot y) &= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(x \cdot y) \\
&= 2\tilde{f}_{\mathrm{m}}(x \cdot y) - \tilde{f}_{\sup}(0) \\
&\geq 2\min\{\tilde{f}_{\mathrm{m}}(x), \tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\sup}(0) \\
&= \min\{2\tilde{f}_{\mathrm{m}}(x), 2\tilde{f}_{\mathrm{m}}(y)\} - \tilde{f}_{\sup}(0) \\
&= \min\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\sup}(0), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\sup}(0)\} \\
&= \min\{2\tilde{f}_{\mathrm{m}}(x) - \tilde{f}_{\sup}(x), 2\tilde{f}_{\mathrm{m}}(y) - \tilde{f}_{\sup}(y)\} \\
&= \min\{\tilde{f}_{\inf}(x), \tilde{f}_{\inf}(y)\}.
\end{aligned}$$

Hence, $(A, \tilde{f}_{\inf})$ is a 1-fuzzy UP-subalgebra of A Therefore, $(A, \tilde{f})$ is a $(1, k)$-hyperfuzzy UP-subalgebra of A. □

Corollary 4.41. *If* $(A, \tilde{f})$ *is a mean* 3*-fuzzy UP-subalgebra of* A *in which* $\tilde{f}_{\sup}$ *is constant, then* $(A, \tilde{f})$ *is a* $(1, k)$*-hyperfuzzy UP-subalgebra of* A *for* $k \in \{1, 2, 3, 4\}$*.*

Proof. It is straightforward by Theorems 2.13 and 4.40. □

5. Conclusions

In this paper, we have introduced notions of length fuzzy UP-subalgebras and mean fuzzy UP-subalgebras of UP-algebras and investigated some of their important properties. Relations between length fuzzy UP-subalgebras (resp., mean fuzzy UP-subalgebras) and hyperfuzzy UP-subalgebras are established. Then we have the table of some relations between length fuzzy UP-subalgebras and hyperfuzzy UP-subalgebras (see Figure 1), and mean fuzzy UP-subalgebras and hyperfuzzy UP-subalgebras (see Figure 2) below.

FIGURE 1. length fuzzy UP-subalgebras and hyperfuzzy UP-subalgebras

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Figure 2. mean fuzzy UP-subalgebras and hyperfuzzy UP-subalgebras

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