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Wreath product of permutation groups and their actions on a sets

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Abstract. The object of wreath product of permutation groups is defined the actions on cartesian product of two sets. In this paper we consider $S(\Gamma)$ and $S(\Delta)$ be permutation groups on Γ and Δ respectively, and $S(\Gamma)^{\Delta}$ be the set of all maps of Δ into the permutations group $S(\Gamma)$. That is $S(\Gamma)^{\Delta} = \{f : \Delta \longrightarrow S(\Gamma)\}$. $S(\Gamma)^{\Delta}$ is a group with respect to the multiplication defined by for all δ in Δ by $(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta)$. After that, we introduce the notion of $S(\Delta)$ actions on $S(\Gamma)^{\Delta}$: $S(\Delta) \times S(\Gamma)^{\Delta} \longrightarrow S(\Gamma)^{\Delta}, (s, f) \longmapsto s \cdot f = f^s$, where $f^s(\delta) = (f \circ s^{-1})(\delta) = (fs^{-1})(\delta)$ for all $\delta \in \Delta$. Finaly, we give the wreath product W of $S(\Gamma)$ by $S(\Delta)$, and the action of W on $\Gamma \times \Delta$

Keywords: group, acts of group in a set, morphism of groups, semi-direct product of groups, wreath product of groups.

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1. Introduction

The product of two groups can be generalized from semi-direct products even further to wrath products. In Mathematics, the wreath product in group theory is specialized product of two groups. Wreath product is an important tool in the classification of permutation groups and also provides a way of constructing interesting examples of groups. The wreath product and its generalisations play an important role in the

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algebraic theory. For example, the can be used to prove the theorem on the decomposition of every finite semi-group automation into a step wise combination of flip-flope and simple group automata.

The remainder of this paper is organized as follows. In Section 2, some mathematical preliminaries. In Section 3, we give the proposition in the concept of wreath product of groups.In Section 3, we introduce the wreat product of permutation groups and the notion of group actions on a set and its concepts like the orbit and the stabilizer. Finally, we draw our conclusions in Section 4.

2. Preliminaries

Let $S(X)$ the set of one to one and onto functions on the n-element set X, with multiplication to composition of functions. The elements of $S(X)$ are called permutations and $S(X)$ is called the symmetric group on X.

A group homomorphism is a well-defined map $\varphi : G \longrightarrow H$ between two groups G and H which preserves the multiplicative structure. In other words, $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$. A bijective homomorphisme is called an isomorphism. When there is an isomorphism between two groups G and H, we say G and H are isomorphic and we write $G \cong H$.

Let G and H be group and $\varphi : G \longrightarrow H$ be a homomorphism. Then $N = \ker\ \varphi$ is a normal subgroup of G and the induced map $\overline{\varphi} : G/N \longrightarrow \operatorname{Im}(\varphi) \leq H, Ng \longmapsto \varphi(g)$ is an isomorphism between the quotient group G/N and the image $\operatorname{Im}(\varphi)$.

Let G be a group and X be a non empty set. We say that G acts on the set X if to each g in G and each x in X, there corresponds a unique point gx in X such that, for all x in X and g_1, g_2 in G we have that

$$(g_1 g_2).x = g_1.(g_2.x) \text{ and } 1_G x = x.$$

To be explicit, we say under the condition that G acts on the set X on the left. The stabilizer of an element $x \in X$ under the action of G is defined by :

$$G_x = \{g \in G : g.x = x\}.$$

The kernel of an action $G \times X \longrightarrow X, (g, x) \longmapsto g.x$ is given by:

$$\ker = \{g \in G : g.x = x \text{ for all } x \in X\}.$$

We define the orbit containing $x \in X$ to be $G.x = \{g.x, g \in G\}$.

Let G be a group acting on a set X. Then, for all $x \in X$, $|G_x|\,|G.x| = |G|$.

Let G and K be two groups. We say that G acts on K as a group if to each k in K there corresponds a unique element k^g in K such that for g_1, g_2, g in G and k_1, k_2, k in K we have that

$$(k^{g_1})^{g_2} = k^{g_1 g_2}, k^{1_G} = k \text{ and } (k_1 k_2)^g = k_1^g k_2^g.$$

Given any groups G and H and a morphism $\theta : G \longrightarrow Aut\,(H)$, denote the automorphism $\theta\,(g)$ by θ_g, then $G \times H$ is a group with the multiplication $(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 \theta_{g_1}(h_2))$, where $g_1, g_2 \in G$ and $h_1, h_2 \in H$.
The group $(G \times H, \cdot)$ is called the semi-direct product of G and H with respect to θ.

3. The wreath product of groups

In this section, we introduce the concept of wreath product of groups.

Theorem 3.1. *Let G and H be two groups. Let H^G be the set of all functions defined on G with values in H.*

(1) The set H^G forms a group such that for any $\varphi, \psi \in H^G$, let $\varphi\psi \in H^G$ in H^G be defined for all $x \in G$ by:
$$(\varphi\psi)(x) = \varphi(x)\,\psi(x).$$
(2) The group G acts on H^Gas a group in the following was:
if $a \in G, \varphi \in H^G$, then
$$(a \cdot \varphi)(x) = \varphi^a(x) = \varphi\left(xa^{-1}\right) \text{ for } x \in G.$$
(3) The set of all pairs (a, φ) where $a \in G, \varphi \in H^G$, with multiplications operation given by:
$$(a, \varphi)(b, \psi) = \left(ab, \varphi^b\psi\right) \text{ where } a, b \in G \text{ and } \varphi, \psi \in H^G$$

The resulting groupe W is called the wreath product of G and H, and is denoted by GW_rH.

Proof.

(1) First we will prove that the set H^G froms a group shch that for any $\varphi, \psi \in H^G$, let $\varphi\psi \in H^G$ in H^G ,
be defined for all $x \in G$ by $(\varphi\psi)(x) = \varphi(x)\,\psi(x)$.
  (i) H^G is non-empty and is closed with respect to multiplication. If $\varphi, \psi \in H^G$, then $\varphi(x), \psi(x) \in H$, for all $x \in G$. Hence $\varphi(x)\,\psi(x) \in H$. This implies that $(\varphi\psi)(x) \in H$ and so $\varphi\psi \in H^G$.
  (ii) Since multuplication in H is associative, so also is the multuplication in H^G.
  (iii) The identity element in H^G is the map $e : G \longrightarrow H$ given by: $e(x) = 1_H$, for all $x \in G$, where 1_H is the identity element of H.
  (iv) For every element $\varphi \in H^G$ is defined for all $x \in G$ by $\varphi^{-1}(x) = (\varphi(x))^{-1}$. Thus H^G is a group with respect to the multiplication defined above.

(2) Second we will prove that G acts on H^G as group, assume that G acts on H^G as follows $G \times H^G \longrightarrow H^G; (a, \varphi) \longrightarrow \varphi^a$ such that for $x \in G$ we have $\varphi^a(x) = \varphi\left(xa^{-1}\right), a \in G, \varphi \in H^G$. Take $\varphi, \psi \in H^G$ and $a, b \in G$, then
   (i) $(\varphi^a)^b(x) = \varphi^a\left(xb^{-1}\right) = \varphi\left(\left(xb^{-1}\right)a^{-1}\right) = \varphi\left(x(ab)^{-1}\right) = \varphi^{ab}(x)$.
   (ii) $\varphi^{1_G}(x) = \varphi\left(x1_G^{-1}\right) = \varphi(x)$.
   (iii) $(\varphi\psi)^a(x) = \varphi\psi\left(xa^{-1}\right) = \varphi\left(xa^{-1}\right)\psi\left(xa^{-1}\right) = \varphi^a(x)\psi^a(x)$.

(3) Now we can construct the wreath product W of G and H, that is, the semidirect product of G and H^G, then we will prove that $G \times H^G$ is a group with multiplication $(a, \varphi)(b, \psi) = \left(ab, \varphi^b\psi\right)$. Then

(i) Closure property follows from the definition of multiplication.

(ii) Take $\varphi, \psi, \eta \in H^G$ and $a, b, c \in G$, then

$$\begin{aligned} ((a,\varphi)(b,\psi))(d,\eta) &= \left(ab, \varphi^b\psi\right)(d,\eta) \\ &= \left((ab)d, \left(\varphi^b\psi\right)^d\eta\right). \end{aligned}$$

Also we have:

$$\begin{aligned} (a,\varphi)((b,\psi)(d,\eta)) &= (a,\varphi)\left(bd, \psi^d\eta\right) \\ &= \left(a(bd), \varphi^{bd}\psi^d\eta\right) \\ &= \left((ab)d, \varphi^{bd}\psi^d\eta\right). \end{aligned}$$

Now if $x \in G$, then:

$$\begin{aligned} \left(\varphi^b\psi\right)^d\eta(x) &= \left(\varphi^b\psi\right)^d(x)\,\eta(x) \\ &= \left(\varphi^b\right)^d(x)\,\psi^d(x)\,\eta(x) \\ &= \varphi^b\left(xd^{-1}\right)\psi\left(xd^{-1}\right)\eta(x) \\ &= \varphi\left(xd^{-1}b^{-1}\right)\psi\left(xd^{-1}\right)\eta(x) \\ &= \varphi\left(x(bd)^{-1}\right)\psi\left(xd^{-1}\right)\eta(x) \\ &= \varphi^{bd}(x)\,\psi^d(x)\,\eta(x). \end{aligned}$$

And

$$\varphi^{bd}\psi^d\eta(x) = \varphi^{bd}(x)\,\psi^d(x)\,\eta(x).$$

And thus we have established the associativity of the multiplication on the set $G \times H^G$.

(iii) We know that for every $\varphi \in H^G, \varphi^{1_G} = \varphi$, now for every $g \in G$, the map $\varphi \longrightarrow \varphi^g$ is an automorphism of H^G. Also if e is the identity element in H^G, then $e^g = e$. We have:

$$\begin{aligned}(a,\varphi)(1_G,e) &= \left(a1_G,\varphi^{1_G}e\right)\\ &= (a,\varphi e)\\ &= (a,\varphi).\end{aligned}$$

Also

$$\begin{aligned}(1_G,e)(a,\varphi) &= (1_G a,e^a\varphi)\\ &= (a,e\varphi)\\ &= (a,\varphi).\end{aligned}$$

Thus identity element exists.

(iv) We have:

$$\begin{aligned}(a,\varphi)\left(a^{-1},\left(\varphi^{-1}\right)^{(a)^{-1}}\right) &= \left(a^{-1},\left(\varphi^{-1}\right)^{(a)^{-1}}\right)(a,\varphi)\\ &= (1_G,e).\end{aligned}$$

Thus the inverse element of (a,φ) is $\left(a^{-1},\left(\varphi^{-1}\right)^{(a)^{-1}}\right)$.

Hence $G \times H^G$ is a group with respect to the multiplication defined above. □

In following proposition, we show that the group H^G is a normal subgroup of W and G is a subgroup of W.

Proposition 3.2.

(1) *If G and H^G are finite groups, then the wreath product W is a finite group of order $|W| = |G|\,.\,|H|^{|G|}$.*
(2) *The group H^G is a normal subgroup of W and G is a subgroup of W.*
(3) $G \cap H^G = (1_G, e)$.
(4) $GW_rH^G = G \times H^G$.

Proof.

(1) It is clear.
(2) We have injective maps $\Phi : H^G \longrightarrow G \times H^G$ given by $f \longmapsto (1_G, f)$, and $\Psi : G \longrightarrow G \times H^G$ given by $a \longmapsto (a, e)$.

And both are homomorphisms since

$$\begin{aligned}\Phi(f_1f_2) &= (1_G, f_1f_2)\\ &= \left(1_G1_G, f_1^{(1_G)}f_2\right)\\ &= (1_G, f_1)\,W_r\,(1_G, f_2)\\ &= \Phi(f_1)\,W_r\Phi(f_2)\,.\end{aligned}$$

And

$$\begin{aligned}\Psi(ab) &= (ab, e)\\ &= \left(ab, e^be\right)\\ &= (a, e)\,W_r\,(b, e)\\ &= \Psi(a)\,W_r\Psi(b)\,.\end{aligned}$$

Then $H^G \cong \mathrm{Im}(\Phi) \leq G \times H^G$. And $G \cong \mathrm{Im}(\Psi) \leq G \times H^G$. These injective homomorphisms let us think of H^G and G as subgroups of $G \times H^G$.
Finally we must show that H^G is normal in $G \times H^G$, follow from the calculation,

$$\begin{aligned}(a, e)(1_G, f)(a, e)^{-1} &= (a, e)(1_G, f)\left(a^{-1}, \left(e^{-1}\right)^{a^{-1}}\right)\\ &= (a, e)(1_G, f)\left(a^{-1}, e\right)\\ &= \left(a1_G, e^{1_G}f\right)\left(a^{-1}, e\right)\\ &= \left(aa^{-1}, f\right)\\ &= (1_G, f)\,.\end{aligned}$$

(3) It is clear that $G \cap H^G = (1_G, e)$.
(4) We have $GW_rH^G = G \times H^G$, since

$$(a, e)\,W_r\,(1_G, f) = \left(a1_G, e^{1_G}f\right) = (a, f)\,,$$

for all $(a, f) \in G \times H^G$.

□

4. Wreath product of permutation groups

This section is issentially an upgrand of the results of Ibrahim A. A and Audu M. S (see [2]) on wreat product of permutation groups. After that, we introduce the notion of group actions on a set and its concepts like the orbit and the stabilizer.

Theorem 4.1. *Let $S(\Gamma)$ and $S(\Delta)$ be permutation groups on Γ and Δ respectively. Let $S(\Gamma)^{\Delta}$ be the set of all maps of Δ into the permutations group $S(\Gamma)$. That is $S(\Gamma)^{\Delta} = \{f : \Delta \longrightarrow S(\Gamma)\}$. For any f_1, f_2 in*

$S(\Gamma)^{\Delta}$, let $f_1 f_2$ in $S(\Gamma)^{\Delta}$ be defined for all δ in Δ by $(f_1 f_2)(\delta) = f_1(\delta) f_2(\delta)$. With respect to this operation of multiplication, $S(\Gamma)^{\Delta}$ acquires the structure of a group.

Proof.

(i) $S(\Gamma)^{\Delta}$ is non-empty and is cosed with respect to multiplication. If $f_1, f_2 \in S(\Gamma)^{\Delta}$, then $f_1(\delta), f_2(\delta) \in S(\Gamma)$. Hence $f_1(\delta) f_2(\delta) \in S(\Gamma)$. This implies that $(f_1 f_2)(\delta) \in S(\Gamma)$ and so $f_1 f_2 \in S(\Gamma)^{\Delta}$.

(ii) Since multiplication is associative so also is the multiplication in $S(\Gamma)^{\Delta}$.

(iii) The identity element in $S(\Gamma)^{\Delta}$ is the map $e : \Delta \longrightarrow S(\Gamma)$ given by:
$$e(\delta) = id_{\Gamma} \text{ for all } \delta \in \Delta$$
where id_{Γ} is the identity element of $S(\Gamma)$.

(iv) Every element $f \in S(\Gamma)^{\Delta}$ is defined for all $\delta \in \Delta$ by:
$$f^{-1}(\delta) = (f(\delta))^{-1}.$$

Thus $S(\Gamma)^{\Delta}$ is a group with respect to the multiplication defined above. We denote this group by P. □

Proposition 4.2. *Assume that $S(\Delta)$ acts on P as follows:*
$$\begin{array}{ccc} S(\Delta) \times S(\Gamma)^{\Delta} & \longrightarrow & S(\Gamma)^{\Delta} \\ (s, f) & \longmapsto & s \cdot f = f^s \end{array}$$
where $f^s(\delta) = \left(f \circ s^{-1}\right)(\delta) = \left(f s^{-1}\right)(\delta)$ for all $\delta \in \Delta$. Then $S(\Delta)$ acts on P as a group.

Proof. Take, $f, f_1, f_2 \in S(\Gamma)^{\Delta}$ and $s, s_1, s_2 \in S(\Delta)$ then

(i) $((s_1 s_2) \cdot f)(\delta) = f^{(s_1 s_2)}(\delta) = \left(f (s_1 s_2)^{-1}\right)(\delta) = \left(f \left(s_2^{-1} s_1^{-1}\right)\right)(\delta)$
$= \left(f s_2^{-1}\right)\left(s_1^{-1}(\delta)\right) = (s_1 \cdot (s_2 \cdot f))(\delta)$.

(ii) $f^{id_{\Delta}}(\delta) = \left(f id_{\Delta}^{-1}\right)(\delta) = (f id_{\Delta})(\delta) = (f)(\delta)$.

(iii) $(f_1 f_2)^s(\delta) = \left(f_1 f_2 \circ s^{-1}\right)(\delta) = f_1 f_2\left(s^{-1}(\delta)\right)$
$= f_1\left(s^{-1}(\delta)\right) f_2\left(s^{-1}(\delta)\right) = f_1^s(\delta) f_2^s(\delta)$.

□

Proposition 4.3. *The set of all ordered (f, s) with $f \in S(\Gamma)^{\Delta}$ and $s \in S(\Delta)$ acquires the structure of a group when we define for all $f_1, f_2 \in S(\Gamma)^{\Delta}$ and $s_1, s_2 \in S(\Delta)$*
$$(f_1, s_1)(f_2, s_2) = \left(f_1 f_2^{s_1^{-1}}, s_1 s_2\right).$$

Thus $S(\Gamma)^{\Delta} \times S(\Delta)$ is a group with respect to the multiplication defined above. We denote this group by W. The resulting groupe W is called the wreath product of $S(\Gamma)$ by $S(\Delta)$, and is denoted by $W = S(\Gamma)\, W_r S(\Delta)$.

Proof.

(i) Closure property follows from the definition of multiplication.

(ii) Take $f_1, f_2, f_3 \in S(\Gamma)^{\Delta}$ and $s_1, s_2, s_3 \in S(\Delta)$. Then,

$$\begin{aligned}[(f_1, s_1)(f_2, s_2)](f_3, s_3) &= \left(f_1 f_2^{s_1^{-1}}, s_1 s_2\right)(f_3, s_3)\\ &= \left(f_1 f_2^{s_1^{-1}} f_3^{(s_1 s_2)^{-1}}, s_1 s_2 s_3\right)\\ &= \left(f_1 f_2^{s_1^{-1}} f_3^{s_2^{-1} s_1^{-1}}, s_1 s_2 s_3\right).\end{aligned}$$

Also, we have in the same manner that,

$$\begin{aligned}(f_1, s_1)[(f_2, s_2)(f_3, s_3)] &= (f_1, s_1)\left(f_2 f_3^{s_2^{-1}}, s_2 s_3\right)\\ &= \left(f_1 \left(f_2 f_3^{s_2^{-1}}\right)^{s_1^{-1}}, s_1 s_2 s_3\right)\\ &= \left(f_1 f_2^{s_1^{-1}} f_3^{s_2^{-1} s_1^{-1}}, s_1 s_2 s_3\right).\end{aligned}$$

Hence multiplication is associative.

(iii) We know that for every $f \in S(\Gamma)^{\Delta}$, $f^{id_\Delta} = f$. Now for every $s \in S(\Delta)$, the map $f \longmapsto f^s$ is an automorphism of $S(\Gamma)^{\Delta}$. Also if e is the identity element in $S(\Gamma)^{\Delta}$, then $e^s = e$. Also, $\left(f^{-1}\right)^s = (f^s)^{-1}$. Now $(f, s)(e, id_\Delta) = \left(f e^{s^{-1}}, s \circ id_\Delta\right) = (f, s)$. Also, using the reverse order, we have that,

$$\begin{aligned}(e, id_\Delta)(f, s) &= \left(e f^{(id_\Delta)^{-1}}, id_\Delta \circ s\right)\\ &= (f, s).\end{aligned}$$

Thus identity element exists.

(iv) $(f, s)\left(\left(f^{-1}\right)^s, s^{-1}\right) = \left(\left(f^{-1}\right)^s, s^{-1}\right)(f, s) = (e, id_\Delta)$.

□

In following proposition, we show that the group $S(\Gamma)^{\Delta}$ is a normal subgroup of W and $S(\Delta)$ is a subgroup of W.

Proposition 4.4.

(1) *If $S(\Delta)$ and $S(\Gamma)$ are finite groups, then the wreath product W is a finite group of order $|W| = |S(\Gamma)|^{|\Delta|} . |S(\Delta)|$.*

(2) *The group* $S(\Gamma)^{\Delta}$ *is a normal subgroup of* W *and* $S(\Delta)$ *is a subgroup of* W.
(3) $S(\Gamma)^{\Delta} \cap S(\Delta) = (e, id_{\Delta})$.
(4) $S(\Gamma)^{\Delta} W_r S(\Delta) = S(\Gamma)^{\Delta} \times S(\Delta)$.
(5) *The action of* W *on* $\Gamma \times \Delta$ *is given by:*

$$(f, s)(\gamma, \delta) = (f(\delta)(\gamma), s(\delta))$$

for all $(f, s) \in S(\Gamma)^{\Delta} \times S(\Delta)$ *and* $(\gamma, \delta) \in \Gamma \times \Delta$.

Proof.

(1) It is clear.
(2) We have injective maps:

$$\begin{array}{rcl} \Phi : S(\Gamma)^{\Delta} & \longrightarrow & S(\Gamma)^{\Delta} \times S(\Delta) \\ f & \longmapsto & (f, id_{\Delta}) \end{array}, \text{ and}$$

$$\begin{array}{rcl} \Psi : S(\Delta) & \longrightarrow & S(\Gamma)^{\Delta} \times S(\Delta) \\ s & \longmapsto & (e, s) \end{array}$$

And both are homomorphisms since

$$\begin{aligned} \Phi(f_1 f_2) &= (f_1 f_2, id_{\Delta}) \\ &= \left(f_1 f_2^{(id_{\Delta})^{-1}}, id_{\Delta} \circ id_{\Delta}\right) \\ &= (f_1, id_{\Delta}) W_r (f_2, id_{\Delta}) \\ &= \Phi(f_1) W_r \Phi(f_2). \end{aligned}$$

And

$$\begin{aligned} \Psi(s_1 \circ s_2) &= (e, s_1 \circ s_2) \\ &= \left(e e^{(s_1)^{-1}}, s_1 \circ s_2\right) \\ &= (e, s_1) W_r (e, s_2) \\ &= \Psi(s_1) W_r \Psi(s_2). \end{aligned}$$

Then $S(\Gamma)^{\Delta} \cong \mathrm{Im}(\Phi) \leq S(\Gamma)^{\Delta} \times S(\Delta)$. And $S(\Delta) \cong \mathrm{Im}(\Psi) \leq S(\Gamma)^{\Delta} \times S(\Delta)$. These injective homomorphisms let us think of $S(\Gamma)^{\Delta}$ and $S(\Delta)$ as subgroups of $S(\Gamma)^{\Delta} \times S(\Delta)$.
Finally we must show that $S(\Gamma)^{\Delta}$ is normal in $S(\Gamma)^{\Delta} \times S(\Delta)$, follow from the calculation,

$$\begin{aligned} (e, s)(f, id_{\Delta})(e, s)^{-1} &= (e, s)(f, id_{\Delta})\left(\left(e^{-1}\right)^{s}, s^{-1}\right) \\ &= (e, s)(f, id_{\Delta})\left(e, s^{-1}\right) \\ &= \left(e f^{(s)^{-1}}, s \circ id_{\Delta}\right)\left(e, s^{-1}\right) \\ &= \left(f^{(s)^{-1}}, id_{\Delta}\right). \end{aligned}$$

(3) It is clear that $S(\Gamma)^{\Delta} \cap S(\Delta) = (e, id_{\Delta})$.
(4) We have $S(\Gamma)^{\Delta} W_r S(\Delta) = S(\Gamma)^{\Delta} \times S(\Delta)$ since

$$(f, id_{\Delta}) W_r (e, s) = \left(f e^{(id_{\Delta})^{-1}}, id_{\Delta} \circ s \right) = (f, s)$$

for all $(f, s) \in S(\Gamma)^{\Delta} \times S(\Delta)$
(5) Take, $(f_1, s_1), (f_2, s_2) \in S(\Gamma)^{\Delta} \times S(\Delta)$ and $(\gamma, \delta) \in \Gamma \times \Delta$, then
   (i) $(e, id_{\Delta})(\gamma, \delta) = (e(\delta)(\gamma), id_{\Delta}(\delta)) = (id_{\Gamma}(\gamma), \delta) = (\gamma, \delta)$.
   (ii) $[(f_1, s_1)(f_2, s_2)](\gamma, \delta) = \left(f_1 f_2^{s_1^{-1}}, s_1 s_2 \right)(\gamma, \delta)$
   $= \left(f_1 f_2^{s_1^{-1}}(\delta)(\gamma), s_1 s_2(\delta) \right) = \left(\left(f_1(\delta) f_2^{s_1^{-1}}(\delta) \right)(\gamma), s_1 s_2(\delta) \right)$
   $= ((f_1(\delta)(f_2 \circ s_1)(\delta))(\gamma), s_1 s_2(\delta))$.
   Also, we have in the same manner that,
   $(f_1, s_1)[(f_2, s_2)(\gamma, \delta)] = (f_1, s_1)(f_2(\delta)(\gamma), s_2(\delta))$
   $= (f_1(s_2(\delta))(f_2(\delta)(\gamma)), s_1 s_2(\delta))$.

□

Proposition 4.5. *Under the action of W on $\Gamma \times \Delta$, the stabilizer of any point (γ, δ) in $\Gamma \times \Delta$ denoted by $W_{(\gamma,\delta)}$ is given by:*

$$W_{(\gamma,\delta)} = S(\Gamma)^{\Delta}(\delta)_{\gamma} \times S(\Delta)_{\delta}.$$

Where $S(\Gamma)^{\Delta}(\delta)_{\gamma}$ is the set of all $f(\delta)$ that stabilize γ, and $S(\Delta)_{\delta}$ is the stabilizer of δ under the action of $S(\Delta)$ on Δ.

Proof. We have:

$$\begin{aligned} W_{(\gamma,\delta)} &= \left\{ (f, s) \in S(\Gamma)^{\Delta} \times S(\Delta) / (f, s)(\gamma, \delta) = (\gamma, \delta) \right\} \\ &= \left\{ (f, s) \in S(\Gamma)^{\Delta} \times S(\Delta) / (f(\delta)\gamma, s(\delta)) = (\gamma, \delta) \right\} \\ &= \left\{ (f, s) \in S(\Gamma)^{\Delta} \times S(\Delta) / f(\delta)\gamma = \gamma, s(\delta) = \delta \right\} \\ &= S(\Gamma)^{\Delta}(\delta)_{\gamma} \times S(\Delta)_{\delta}. \end{aligned}$$

□

Example 4.6. Consider the permutation groups $S(\Gamma) = \{(1), (12)\}$ and $S(\Delta) = \{(1), (12), (13), (23), (123), (132)\}$ on the sets $\Gamma = \{1, 2\}$ and $\Delta = \{1, 2, 3\}$ respectively. Let $S(\Gamma)^{\Delta} = \{f : \Delta \longrightarrow S(\Gamma)\}$, then $|S(\Gamma)|^{|\Delta|} = 2^3 = 8$. The mappings are follows:
$f_1 : 1 \longmapsto (1), 2 \longmapsto (1), 3 \longmapsto (1)$
$f_2 : 1 \longmapsto (1), 2 \longmapsto (1), 3 \longmapsto (12)$
$f_3 : 1 \longmapsto (1), 2 \longmapsto (12), 3 \longmapsto (1)$
$f_4 : 1 \longmapsto (1), 2 \longmapsto (12), 3 \longmapsto (12)$

$f_5 : 1 \longmapsto (12), 2 \longmapsto (1), 3 \longmapsto (1)$
$f_6 : 1 \longmapsto (12), 2 \longmapsto (1), 3 \longmapsto (12)$
$f_7 : 1 \longmapsto (12), 2 \longmapsto (12), 3 \longmapsto (1)$
$f_8 : 1 \longmapsto (12), 2 \longmapsto (12), 3 \longmapsto (12)$.
We can easily verify that $S(\Gamma)^{\Delta}$ is a group with respect to the operation

$$(\varphi\psi)(\delta) = (\varphi)(\delta)(\psi)(\delta) \text{ where } \delta \in \Delta.$$

We have:

$$\begin{aligned} S(\Gamma)^{\Delta} \times S(\Delta) &= \left\{ (f, s) / f \in S(\Gamma)^{\Delta}, s \in S(\Delta) \right\} \\ &= \{ (f_i, (1)), (f_i, (12)), (f_i, (12)), (f_i, (23)), \\ &\quad (f_i, (123)), (f_i, (132)), 1 \le i \le 8 \}. \end{aligned}$$

And $\left| S(\Gamma)^{\Delta} \times S(\Delta) \right| = \left| S(\Gamma)^{\Delta} \right| . |S(\Delta)| = 8.6 = 48.$
$S(\Gamma)^{\Delta} \times S(\Delta)$ is a group with respect to the operation

$$(\varphi, s_1)(\psi, s_2) = \left(\varphi \psi^{(s_1)^{-1}}, s_1 s_2 \right).$$

We have $\Gamma \times \Delta = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$.
The stabilizer of $(1,1)$ denoted by:

$$\begin{aligned} W_{(1,1)} &= S(\Gamma)^{\Delta}(1)_1 \times S(\Delta)_1 \\ &= \{f_1, f_2, f_3, f_4\} \times \{(1), (23)\} \\ &= \{(f_1, (1)), (f_2, (1)), (f_3, (1)), (f_4, (1)), (f_1, (23)), (f_2, (23)), \\ &\quad (f_3, (23)), (f_4, (23))\}. \end{aligned}$$

Then $W_{(1,1)}$ is a subgroup of $S(\Gamma)^{\Delta} \times S(\Delta)$ of order 8.
Also, we have is the same manner that,

$$\begin{aligned} W_{(1,2)} &= S(\Gamma)^{\Delta}(2)_1 \times S(\Delta)_2 \\ &= \{f_1, f_2, f_5, f_6\} \times \{(1), (13)\} \\ &= \{(f_1, (1)), (f_2, (1)), (f_5, (1)), (f_6, (1)), (f_1, (13)), (f_2, (13)), \\ &\quad (f_5, (13)), (f_6, (13))\} \end{aligned}$$

Then $W_{(1,2)}$ is a subgroup of $S(\Gamma)^{\Delta} \times S(\Delta)$ of order 8.

$$\begin{aligned} W_{(1,3)} &= S(\Gamma)^{\Delta}(3)_1 \times S(\Delta)_3 \\ &= \{f_1, f_3, f_5, f_7\} \times \{(1), (12)\} \\ &= \{(f_1, (1)), (f_3, (1)), (f_5, (1)), (f_7, (1)), (f_1, (12)), (f_3, (12)), \\ &\quad (f_5, (12)), (f_7, (12))\}. \end{aligned}$$

Then $W_{(1,3)}$ is a subgroup of $S(\Gamma)^{\Delta} \times S(\Delta)$ of order 8.

$W_{(2,1)} = S(\Gamma)^{\Delta}(1)_2 \times S(\Delta)_1$
$= \{f_1, f_2, f_3, f_4\} \times \{(1), (23)\}$
$= \{(f_1, (1)), (f_2, (1)), (f_3, (1)), (f_4, (1)), (f_1, (23)), (f_2, (23)),$
$(f_3, (23)), (f_4, (23))\}.$

Then $W_{(1,3)}$ is a subgroup of $S(\Gamma)^{\Delta} \times S(\Delta)$ of order 8.

$W_{(2,2)} = S(\Gamma)^{\Delta}(2)_2 \times S(\Delta)_2$
$= \{f_1, f_2, f_5, f_6\} \times \{(1), (13)\}$
$= \{(f_1, (1)), (f_2, (1)), (f_5, (1)), (f_6, (1)), (f_1, (13)), (f_2, (13)),$
$(f_5, (13)), (f_6, (13))\}.$

Then $W_{(2,2)}$ is a subgroup of $S(\Gamma)^{\Delta} \times S(\Delta)$ of order 8.

$W_{(2,3)} = S(\Gamma)^{\Delta}(3)_2 \times S(\Delta)_3$
$= \{f_1, f_3, f_5, f_7\} \times \{(1), (12)\}$
$= \{(f_1, (1)), (f_3, (1)), (f_5, (1)), (f_7, (1)), (f_1, (12)), (f_3, (12)),$
$(f_5, (12)), (f_7, (12))\}.$

Then $W_{(2,3)}$ is a subgroup of $S(\Gamma)^{\Delta} \times S(\Delta)$ of order 8.

Finally, we have:

$$W_{(1,1)} = W_{(2,1)}, W_{(1,2)} = W_{(2,2)}, W_{(1,3)} = W_{(2,3)}.$$

For $(\gamma, \delta) \in \Gamma \times \Delta$, we have $\left|W_{(\gamma,\delta)}\right| . |W(\gamma, \delta)| = |W|$, then

$$|W(\gamma, \delta)| = \frac{|W|}{\left|W_{(\gamma,\delta)}\right|} = \frac{48}{8} = 6.$$

In this example, we have:

$(f_1, (1))(1,1) = (f_2, (1))(1,1) = (f_3, (1))(1,1) = (f_4, (1))(1,1) = (1,1)$

$(f_1, (12))(1,1) = (f_2, (12))(1,1) = (f_3, (12))(1,1) = (f_4, (12))(1,1)$
$= (1,2)$

$(f_1, (13))(1,1) = (f_2, (13))(1,1) = (f_3, (13))(1,1) = (f_4, (13))(1,1)$
$= (1,1)$

$(f_1, (23))(1,1) = (f_2, (23))(1,1) = (f_3, (23))(1,1) = (f_4, (23))(1,1)$
$= (1,1)$

$(f_1, (123))(1,1) = (f_2, (123))(1,1) = (f_3, (123))(1,1) = (f_4, (123))(1,1)$
$= (1,2)$

$(f_1, (132))(1,1) = (f_2, (132))(1,1) = (f_3, (132))(1,1) = (f_4, (132))(1,1)$
$= (1,3)$

$(f_5, (1))(1,1) = (f_6, (1))(1,1) = (f_7, (1))(1,1) = (f_8, (1))(1,1)$
$= (2,1)$

$$\left(f_5,(12)\right)(1,1)=\left(f_6,(12)\right)(1,1)=\left(f_7,(12)\right)(1,1)=\left(f_8,(12)\right)(1,1)=(2,2)$$

$$\left(f_5,(13)\right)(1,1)=\left(f_6,(13)\right)(1,1)=\left(f_7,(13)\right)(1,1)=\left(f_8,(13)\right)(1,1)=(2,3)$$

$$\left(f_5,(23)\right)(1,1)=\left(f_6,(23)\right)(1,1)=\left(f_7,(23)\right)(1,1)=\left(f_8,(23)\right)(1,1)=(2,1)$$

$$\left(f_5,(123)\right)(1,1)=\left(f_6,(123)\right)(1,1)=\left(f_7,(123)\right)(1,1)=\left(f_8,(123)\right)(1,1)=(2,2)$$

$$\left(f_5,(132)\right)(1,1)=\left(f_6,(132)\right)(1,1)=\left(f_7,(132)\right)(1,1)=\left(f_8,(132)\right)(1,1)=(2,3)$$

Then the orbit of $(1,1)$ is

$$\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3)\}=\Gamma\times\Delta.$$

5. Conclusion

In this paper, we present some propositions on the wreath product of groups. And we give some examples.

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