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Weak topological centers and cohomological properties

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Abstract.

Let B be a Banach $A-bimodule$. We introduce the weak topological centers of left module action and we show it by $\tilde{Z}^{\ell}_{B^{**}}(A^{**})$. For a compact group, we show that $L^1(G) = \tilde{Z}^{\ell}_{M(G)^{**}}(L^1(G)^{**})$ and on the other hand we have $\tilde{Z}^{\ell}_1(c_0^{**}) \neq c_0^{**}$. Thus the weak topological centers are different with topological centers of left or right module actions. In this manuscript, we investigate the relationships between two concepts with some conclusions in Banach algebras. We also have some application of this new concept and topological centers of module actions in the cohomological properties of Banach algebras, spacial, in the weak amenability and n-weak amenability of Banach algebras.

Keywords: Arens regularity, Topological centers, Weak topological center, Amenability, Weak amenability, Cohomology groups.

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1. Introduction

Let B be a Banach $A-bimodule$. A derivation from A into B is a bounded linear mapping $D : A \to B$ such that

$$D(xy) = xD(y) + D(x)y \ for\ all\ x,\ y \in A.$$

The space of continuous derivations from A into B is denoted by $Z^1(A, B)$. Easy example of derivations are the inner derivations, which are given for each $b \in B$ by

$$\delta_b(a) = ab - ba \ for\ all\ a \in A.$$

The space of inner derivations from A into B is denoted by $N^1(A, B)$. The Banach algebra A is said to be a amenable, when for every Banach $A-bimodule$ B, the inner derivations are only derivations existing from A into B^*. It is clear that A is amenable if and only if $H^1(A, B^*) = Z^1(A, B^*)/N^1(A, B^*) = \{0\}$. The concept of amenability for a Banach algebra A, introduced by Johnson in 1972, has proved to be of enormous importance problems in Banach algebra theory, see [14]. For Banach $A-$ $bimodule$, B, the quotient space $H^1(A, B)$ is called the first cohomology group of A with coefficients in B.
Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} that he called m is Arens regular whenever $m^{***} = m^{t***t}$, for more information see [1, 12, 19].

Recently, the subject of regularity of bounded bilinear mappings and Banach module actions have been investigated in [6, 9, 12, 13]. In [8], Eshaghi Gordji and Fillali gave several significant results related to the topological centers of Banach module actions. In [19], the authors have obtained a criterion for the regularity of f, from which they gave several results related to the regularity of Banach module actions with some applications to the second adjoint of a derivation. For a good and rich source of information on this subject, we refer the reader to the Memoire in [7]. We also shall mostly follow [4] as a general reference on Banach algebras.

Regarding A as a Banach $A - bimodule$, the operation $\pi : A \times A \to A$ extends to π^{***} and π^{t***t} defined on $A^{**} \times A^{**}$. These extensions are known, respectively, as the first (left) and the second (right) Arens products, and with each of them, the second dual space A^{**} becomes a Banach algebra. The regularity of a normed algebra A is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Suppose that A is a Banach algebra and B is a Banach $A - bimodule$. Since B^{**} is a Banach $A^{**} - bimodule$, where A^{**} is equipped with the first Arens product, we define the topological center of the right module action of A^{**} on B^{**} as follows:

$$Z^{\ell}_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} : \text{ the map } a'' \to \pi_r^{***}(b'', a'') \ : \ A^{**} \to B^{**}$$
$$\text{is weak}^* - \text{weak}^* \text{ continuous}\}.$$

In this way, we write $Z^{\ell}_{B^{**}}(A^{**}) = Z(\pi_\ell)$, $Z^{r}_{A^{**}}(B^{**}) = Z(\pi_\ell^t)$ and $Z^{r}_{B^{**}}(A^{**}) = Z(\pi_r^t)$, where $\pi_\ell : \ A \times B \to B$ *and* $\pi_r : \ B \times A \to B$ are the left and right module actions of A on B, for more information, see [6, 9].

If we set $B = A$, we write $Z^{\ell}_{A^{**}}(A^{**}) = Z_1(A^{**}) = Z_1^{\ell}(A^{**})$ and $Z^{r}_{A^{**}}(A^{**}) = Z_2(A^{**}) = Z_2^{r}(A^{**})$, for more information see [17]. Let B be a Banach $A-bimodule$ and $n \geq 0$. Suppose that $B^{(n)}$ is an $n-th$ *dual* of B. Then $B^{(n)}$ is also Banach $A - bimodule$, that is, for every $a \in A$, $b^{(n)} \in B^{(n)}$ and $b^{(n-1)} \in B^{(n-1)}$, we define

$$\langle b^{(n)}a, b^{(n-1)}\rangle = \langle b^{(n)}, ab^{(n-1)}\rangle,$$
$$\langle ab^{(n)}, b^{(n-1)}\rangle = \langle b^{(n)}, b^{(n-1)}a\rangle.$$

Let $A^{(n)}$ and $B^{(n)}$ be $n - th$ *dual* of A and B, respectively. By [23], for an even number $n \geq 0$, $B^{(n)}$ is a Banach $A^{(n)} - bimodule$. Then for $n \geq 2$, we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n-1)}$, that is, for all $b^{(n)} \in B^{(n)}$, $b^{(n-1)} \in B^{(n-1)}$ and $a^{(n-2)} \in A^{(n-2)}$ we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-2)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-2)}\rangle.$$

If n is odd number, then for $n \geq 1$, we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $b^{(n)} \in B^{(n)}$, $b^{(n-1)} \in B^{(n-1)}$ and $a^{(n-1)} \in A^{(n-1)}$ we define

$$\langle b^{(n)}b^{(n-1)}, a^{(n-1)}\rangle = \langle b^{(n)}, b^{(n-1)}a^{(n-1)}\rangle.$$

and if $n = 0$, we take $A^{(0)} = A$ and $B^{(0)} = B$.
So we can define the topological centers of module actions of $A^{(n)}$ on $B^{(n)}$ similarly.

2. **Weak topological center of module actions**

In this section, we introduce a new concept as weak topological center of Banach algebras, module actions and we study their relationship with topological centers of module actions with some conclusions in the group algebras.

Definition 2.1. Let B be a Banach $A - bimodule$. We define the weak topological centers of left module action as follows:

$$\tilde{Z}^{\ell}_{B^{**}}(A^{**}) = \{a'' \in A^{**} : \text{ the maping } b'' \to a''b'' \text{ is weak}^*\text{−weak continuous } \},$$

$$\tilde{Z}^{\ell}_{A^{**}}(B^{**}) = \{b'' \in B^{**} : \text{ the maping } a'' \to b''a'' \text{ is weak}^*\text{−weak continuous } \}.$$

Definition of $\tilde{Z}^r_{B^{**}}(A^{**})$ and $\tilde{Z}^r_{A^{**}}(B^{**})$ are similar for right module action. If $B = A$, we write $\tilde{Z}^\ell_{A^{**}}(A^{**}) = \tilde{Z}^\ell_1(A^{**})$ and $\tilde{Z}^r_{A^{**}}(A^{**}) = \tilde{Z}^r_1(A^{**})$, and the spaces $\tilde{Z}^\ell_2(A^{**})$ and $\tilde{Z}^r_2(A^{**})$ have similar definitions with respect to the second Arens product. It is clear that $\tilde{Z}^\ell_i(A^{**})$ and $\tilde{Z}^r_i(A^{**})$ for each $i \in \{1, 2\}$, are subspaces of A^{**} with respect to the both Arens products. In general, by easy calculations, we have the following results:

(1) If $\tilde{Z}^\ell_1(A^{**}) = A^{**}$ or $\tilde{Z}^r_1(A^{**}) = A^{**}$, then A is Arens regular.
(2) Let $B \subseteq A^{**}$. Then $B\tilde{Z}^\ell_1(A^{**}) \subseteq \tilde{Z}^\ell_1(A^{**})$ and $\tilde{Z}^r_1(A^{**})B \subseteq \tilde{Z}^r_1(A^{**})$.
(3) If $\tilde{Z}^\ell_1(A^{**}) = A$, then A is a right ideal in A^{**}.
(4) If $\tilde{Z}^r_1(A^{**}) = A$, then A is a left ideal in A^{**}.
(5) If $\tilde{Z}^\ell_1(A^{**}) = Z^r_1(A^{**}) = A$, then A is an ideal in A^{**}.
(6) If $A^{***}A^{**} \subseteq A^*$, then $\tilde{Z}^\ell_1(A^{**}) = Z_1(A^{**}) = A^{**}$.
(7) If $A^{**}A^{***} \subseteq A^*$, then $\tilde{Z}^r_1(A^{**}) = A^{**}$.
(8) Suppose that $A^{***}A \subseteq A^*$. If A is left strongly Arens irregular, then
$\tilde{Z}^\ell_1(A^{**}) = A$

In the following example, we show that if a Banach algebra A is Arens regular or strongly Arens irregular on the left, in general, $\tilde{Z}^\ell_1(A^{**})$ is not A^{**} or A, respectively.

Example 2.2. (1) Let A be nonreflexive Arens regular Banach algebra and $e'' \in A^{**}$ be a left unite element of A^{**}. Then $\tilde{Z}^\ell_1(A^{**}) \neq A^{**}$. Since $c_0^{**} = \ell^\infty$ and Arens product in c_0^{**} coincide with the given natural product in ℓ^∞, c_0^{**} is unital. Thus $\tilde{Z}^\ell_1(c_0^{**}) \neq c_0^{**}$

(2) Suppose that G is a locally compact group. Then by notice to [21], we know that in spacial case, $M(G)$ is left strong Arens irregular, but $\tilde{Z}^\ell_1(M(G)^{**}) \neq M(G)^{**}$.
(3) By ([4], Example 3.6.22(i)), we know that c_0 is Arens regular, and so $Z_1(c_0^{**}) = c_0^{**}$, but we claim that $\tilde{Z}^\ell_1(c_0^{**}) = c_0$. Indeed $c_0^{**} = \ell^\infty$ with the point-wise product. We can identity ℓ^∞ with $C(\beta\mathbb{N})$ the continuous functions on the Stone-Cech compactification, and then we find that $(\ell^\infty)^* = M(\beta\mathbb{N})$ the measure space, and so the weak topology center is those $f \in C(\beta\mathbb{N})$ with $f\mu \in \ell^1$ for all $\mu \in M(\beta\mathbb{N})$. By considering point masses in $M(\beta\mathbb{N})$ (ultra-filter limits along $\mathbb{N}$), it is easy to show that $f \in c_0$.

Definition 2.3. Let A be a Banach algebra. The subspace of A^{***} annihilating A will be denoted by $A^\perp = \{a''' \in A^{***} :\ a''' |_A = 0\}$.

Theorem 2.4. *Let A be a Banach algebra. Then we have the following assertions.*

1. *If A is a left ideal in A^{**}, then $A \subseteq \tilde{Z}_1^{\ell}(A^{**}) \cap \tilde{Z}_2^{\ell}(A^{**})$.*
2. *If A is a right ideal in A^{**}, then $A \subseteq \tilde{Z}_1^{r}(A^{**}) \cap \tilde{Z}_2^{r}(A^{**})$.*
3. *If A is an ideal in A^{**}, then $A \subseteq \tilde{Z}_1^{\ell}(A^{**}) \cap \tilde{Z}_2^{\ell}(A^{**}) \cap \tilde{Z}_1^{r}(A^{**}) \cap \tilde{Z}_2^{r}(A^{**})$.*
4. *If A is a left (resp. right) ideal in A^{**} and A^{**} has a left (resp. right) unit in $\tilde{Z}_1^{\ell}(A^{**})$ (resp. $\tilde{Z}_1^{r}(A^{**})$), then A is reflexive.*

Proof. (1) Assume that $(a''_\alpha)_\alpha \subseteq A^{**}$ and $a''_\alpha \xrightarrow{w^*} a''$. Let $a''' \in A^{***}$. Since $A^{***} = A^* \oplus A^{\perp}$, there are $a' \in A^*$ and $t \in A^{\perp}$ such that $a''' = (a', t)$. Then for every $a \in A$, we have

$$\begin{aligned}\langle a''', aa''_\alpha\rangle &= \langle (a', t), aa''_\alpha\rangle = \langle aa''_\alpha, a'\rangle \\ &\to \langle aa'', a'\rangle = \langle a''', aa''\rangle.\end{aligned}$$

It follows that $A \subseteq \tilde{Z}_1^{\ell}(A^{**})$. Since for every $a \in A$ and $a'' \in A^{**}$, we have $aa'' = aoa''$, similarly it follows that $A \subseteq \tilde{Z}_2^{\ell}(A^{**})$. Thus the result holds.

(2) The proof similar to (1).

(3) Obvious.

(4) Let $e \in \tilde{Z}_1^{\ell}(A^{**})$ be an unit element for A^{**}. Set $a''' \in A^{***}$ and $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$. Since $A^{***} = A^* \oplus A^{\perp}$, there are $a' \in A^*$ and $t \in A^{\perp}$ such that $a''' = (a', t)$. Thus

$$\begin{aligned}\langle a''', a''_\alpha\rangle = \langle a''', ea''_\alpha\rangle &= \langle (a', t), ea''_\alpha\rangle = \langle ea''_\alpha, a'\rangle \\ &\to \langle ea'', a'\rangle = \langle a''', a''\rangle.\end{aligned}$$

It follows that $a''_\alpha \xrightarrow{w} a''$. Hence A is reflexive.

□

Corollary 2.5. *Let A be a Banach algebra. Then we have the following assertions.*

1. *If A is a left ideal in A^{**} and left strongly Arens irregular, then $\tilde{Z}_1^{\ell}(A^{**}) = A$, and so A is an ideal in A^{**}.*
2. *If A is a right ideal in A^{**} and right strongly Arens irregular, then $\tilde{Z}_1^{r}(A^{**}) = A$, and so A is an ideal in A^{**}.*
3. *If A is an ideal in A^{**} and strongly Arens irregular, then $\tilde{Z}_1^{\ell}(A^{**}) = \tilde{Z}_1^{r}(A^{**}) = A$.*
4. *If A is a left (resp. right) ideal in A^{**} and A^{**} has a left (resp. right) unit in $\tilde{Z}_1^{\ell}(A^{**})$ (resp. $\tilde{Z}_1^{r}(A^{**})$), then A is reflexive.*

Proof. By using Theorem 2.4, the proof holds. □

Example 2.6. (1) Let G be a compact group. By using [16], we know that $L^1(G)$ is a strongly Arens irregular and $L^1(G)$ is an ideal in its second dual. Then by using the preceding corollary, we have $L^1(G)=\tilde{Z}_1^\ell(L^1(G)^{**})=\tilde{Z}_1^r(L^1(G)^{**})$.

(2) Let G be a locally compact group. Then, in general, by the preceding corollary, $M(G)$ is not a left or right ideal in its second dual.

Theorem 2.7. *Let B be a Banach $A-bimodule$. Then for every even number $n\geq 2$, we have the following assertions.*

(1) $Z^\ell_{A^{(n)}}(B^{(n+1)})=B^{(n+1)}$ *if and only if* $\tilde{Z}^r_{A^{(n)}}(B^{(n)})=B^{(n)}$.
(2) $Z^\ell_{A^{(n)}}(A^{(n+1)})=A^{(n+1)}$ *if and only if* $\tilde{Z}^r_1(A^{(n)})=A^{(n)}$.
(3) $Z^r_{A^{(n)}}(A^{(n+1)})=A^{(n+1)}$ *if and only if* $\tilde{Z}^\ell_1(A^{(n)})=A^{(n)}$.
(4) $Z^r_{B^{(n)}}(A^{(n+1)})=A^{(n+1)}$ *if and only if* $\tilde{Z}^\ell_{A^{(n)}}(B^{(n)})=B^{(n)}$.

Proof. 1) Suppose that ${Z^\ell}_{A^{(n)}}(B^{(n+1)})=B^{(n+1)}$ and $b^{(n)}\in B^{(n)}$. We show that the mapping $a^{(n)}\to a^{(n)}b^{(n)}$ is $weak^*-weak$ continuous. Assume that $(a^{(n)}_\alpha)_\alpha\subseteq A^{(n)}$ such that $a^{(n)}_\alpha\overset{w^*}{\to}a^{(n)}$. Then for all $b^{(n+1)}\in B^{(n+1)}$, we have $b^{(n+1)}a^{(n)}_\alpha\overset{w^*}{\to}b^{(n+1)}a^{(n)}$. It follows that

$$\begin{aligned}\langle b^{(n+1)},a^{(n)}_\alpha b^{(n)}\rangle&=\langle b^{(n+1)}a^{(n)}_\alpha,b^{(n)}\rangle\\&\to\langle b^{(n+1)}a^{(n)},b^{(n)}\rangle\\&=\langle b^{(n+1)},a^{(n)}b^{(n)}\rangle.\end{aligned}$$

Thus we conclude that $b^{(n)}\in\tilde{Z}^r_{A^{(n)}}(B^{(n)})$.
The converse is the same.
Proofs of (2), (3) and (4) similar to (1). □

Example 2.8. Let A be a non-reflexive Banach space and let $\langle f,x\rangle=1$ and $\|f\|\leq 1$ for some $f\in A^*$ and $x\in A$. We define the product on A by $ab=\langle f,b\rangle a$. It is clear that A is a Banach algebra with this product and it has right identity x. By easy calculation, for all $a'\in A^*$, $a''\in A^{**}$ and $a'''\in A^{***}$, we have

$$\begin{aligned}a'a&=\langle a',a\rangle f,\\a''a'&=\langle a'',f\rangle a',\\a'''a''&=\langle a'',a''\rangle\langle .,f\rangle.\end{aligned}$$

Therefore we have ${Z^\ell}_{A^{**}}(A^{***})\neq A^{***}$. So by Theorem 2.7, we have $\tilde{Z}^r_{A^{**}}(A^{**})\neq A^{**}$.
Similarly, if we define the product on A as $ab=\langle f,a\rangle b$ for all $a,\ b\in A$, then we have ${Z^\ell}_{A^{**}}(A^{***})=A^{***}$. By using Theorem 2.7, it follows that $\tilde{Z}^r_{A^{**}}(A^{**})=A^{**}$.

Theorem 2.9. *Let $n > 0$ be an even number and let B be a left (resp. right) Banach $A-module$ such that $A^{(n-2)}B^{(n)} \subseteq B^{(n-2)}$ (resp. $B^{(n)}A^{(n-2)} \subseteq B^{(n-2)}$).*

(1) *Then $A^{(n-2)} \subseteq \tilde{Z}^{\ell}_{B^{(n)}}(A^{(n)})$ (resp. $A^{(n-2)} \subseteq \tilde{Z}^{r}_{B^{(n)}}(A^{(n)})$).*
(2) *If $B^{(n)}$ has a left (resp. right) unit element in $\tilde{Z}^{\ell}_{B^{(n)}}(A^{(n)})$ (resp. $\tilde{Z}^{r}_{B^{(n)}}(A^{(n)})$), then A is reflexive.*
(3) *If $A^{(n-2)} \subset B^{(n-2)}$ and $A^{(n-2)}$ is left (resp. right) Arens irregular, then*

$$A^{(n-2)} = \tilde{Z}^{\ell}_{B^{(n)}}(A^{(n)}) \ (resp.\ A^{(n-2)} = \tilde{Z}^{r}_{B^{(n)}}(A^{(n)})).$$

Proof. (1) Assume that $(b^{(n)}_{\alpha})_{\alpha} \subseteq B^{(n)}$ such that $b^{(n)}_{\alpha} \xrightarrow{w^*} b^{(n)}$ in $B^{(n)}$. Let $b^{(n+1)} \in B^{(n+1)}$. Since $B^{(n+1)} = B^{(n-1)} \oplus B^{\perp}$, there are $b^{(n-1)} \in B^{(n-1)}$ and $t \in B^{\perp}$ such that $b^{(n+1)} = (b^{(n-1)}, t)$. Then for every $a^{(n-2)} \in A^{(n-2)}$, we have

$$\begin{aligned}\langle b^{(n+1)}, a^{(n-2)}b^{(n)}_{\alpha}\rangle &= \langle (b^{(n-1)}, t), a^{(n-2)}b^{(n)}_{\alpha}\rangle \\ &= \langle a^{(n-2)}b^{(n)}_{\alpha}, b^{(n-1)}\rangle \\ &\to \langle a^{(n-2)}b^{(n)}, b^{(n-1)}\rangle \\ &= \langle b^{(n+1)}, a^{(n-2)}b^{(n)}\rangle.\end{aligned}$$

It follows that $A^{(n-2)} \subset \tilde{Z}^{\ell}_{B^{(n)}}(A^{(n)})$.

(2) The proof is clear.

(3) Since $A^{(n-2)} \subset B^{(n-2)}$, $\tilde{Z}^{\ell}_{B^{(n)}}(A^{(n)}) \subset Z_1(A^{(n)}) = A^{(n-2)}$. Thus by using part (1), since $\tilde{Z}^{\ell}_{B^{(n)}}(A^{(n)})) \subseteq Z^{\ell}_{B^{(n)}}(A^{(n)}))$, we are done.

□

Example 2.10. Let G be a compact group. We know that $L^1(G) \subseteq M(G)$ and $L^1(G)$ is an ideal in $M(G)^{**}$. Since $L^1(G)$ is strongly Arens irregular, by preceding theorem, we conclude that

$$\tilde{Z}^{\ell}_{M(G)^{**}}(L^1(G)^{**}) \subseteq Z^{\ell}_{M(G)^{**}}(L^1(G)^{**}) \subseteq Z^{\ell}_1(L^1(G)^{**}) = L^1(G).$$

By Theorem 2.9, we have $L^1(G) \subseteq \tilde{Z}^{\ell}_{M(G)^{**}}(L^1(G)^{**})$. Thus we conclude that

$$L^1(G) = \tilde{Z}^{\ell}_{M(G)^{**}}(L^1(G)^{**}).$$

It is similar that

$$L^1(G) = \tilde{Z}^{r}_{M(G)^{**}}(L^1(G)^{**}).$$

3. **Weak amenability of Banach algebras**

A Banach algebra A is said to be a weakly amenable, if every derivation from A into A^* is inner. Similarly, A is weakly amenable if and only if $H^1(A,A^*) = Z^1(A,A^*)/N^1(A,A^*) = \{0\}$. The concept of weak amenability was first introduced by Bade, Curtis and Dales in [2] for commutative Banach algebras, and was extended to the noncommutative case by Johnson in [15]. In every parts of this section, $n \geq 0$ is an even number.

Theorem 3.1. *Assume that A is a Banach algebra and $\tilde{Z}_1^\ell(A^{(n)}) = A^{(n)}$ where $n \geq 2$. If $A^{(n)}$ is weakly amenable, then $A^{(n-2)}$ is weakly amenable.*

Proof. Suppose that $D \in Z^1(A^{(n-2)}, A^{(n-1)})$. First we show that

$$D'' \in Z^1(A^{(n)}, A^{(n+1)}).$$

Let $a^{(n)},\ b^{(n)} \in A^{(n)}$ and let $(a_\alpha^{(n-2)})_\alpha,\ (b_\beta^{(n-2)})_\beta \subseteq A^{(n-2)}$ such that $a_\alpha^{(n-2)} \xrightarrow{w^*} a^{(n)}$ and $b_\beta^{(n-2)} \xrightarrow{w^*} b^{(n)}$. Since $Z_1^\ell(A^{(n)}) = A^{(n)}$ we have

$$\lim_\alpha \lim_\beta a_\alpha^{(n-2)} D(b_\beta^{(n-2)}) = a^{(n)} D''(b^{(n)}).$$

On the other hand, we have

$$\lim_\alpha \lim_\beta D(a_\alpha^{(n-2)}) b_\beta^{(n-2)} = D''(a^{(n)}) b^{(n)}.$$

Since D is continuous, we conclude that

$$\begin{aligned} D''(a^{(n)}b^{(n)}) &= \lim_\alpha \lim_\beta D(a_\alpha^{(n-2)} b_\beta^{(n-2)}) \\ &= \lim_\alpha \lim_\beta a_\alpha^{(n-2)} D(b_\beta^{(n-2)}) + \lim_\alpha \lim_\beta D(a_\alpha^{(n-2)}) b_\beta^{(n-2)} \\ &= a^{(n)} D''(b^{(n)}) + D''(a^{(n)}) b^{(n)}. \end{aligned}$$

In the above equalities, the convergence are with respect to *weak** topology. Since $A^{(n)}$ is weakly amenable, D'' is inner. It follows that $D''(a^{(n)}) = a^{(n)}a^{(n+1)} - a^{(n+1)}a^{(n)}$ for some $a^{(n+1)} \in A^{(n+1)}$. Set $a^{(n-1)} = a^{(n+1)} \mid_{A^{(n-2)}}$ and
$a^{(n-2)} \in A^{(n-2)}$. Then

$$D(a^{(n-2)}) = D''(a^{(n-2)}) = a^{(n-2)}a^{(n-1)} - a^{(n-1)}a^{(n-2)} = \delta_{a^{(n-1)}}(a^{(n-2)}).$$

Consequently, we have $H^1(A^{(n-2)}, A^{(n-1)}) = 0$, and so $A^{(n-2)}$ is weakly amenable. □

Corollary 3.2. *Let A be a Banach algebra and let $\widetilde{wap}_\ell(A^{(n-1)}) \subseteq A^{(n)}$ whenever $n \geq 1$. If $A^{(n)}$ is weakly amenable, then $A^{(n-2)}$ is weakly amenable.*

Proof. Since $\widetilde{wap_\ell}(A^{(n-1)}) \subseteq A^{(n)}$, $\tilde{Z}_1^\ell(A^{(n)}) = A^{(n)}$. Then, by using Theorem 3.1, proof holds. $\square$

Corollary 3.3. *Let A be a Banach algebra and ${Z^\ell}_{A^{(n)}}(A^{(n+1)}) = A^{(n+1)}$, where $n \geq 2$. If $A^{(n)}$ is weakly amenable, then $A^{(n-2)}$ is weakly amenable.*

Corollary 3.4. *Let A be a Banach algebra and let $D : A^{(n-2)} \to A^{(n-1)}$ be a derivation where $n \geq 2$. Then $D'' : A^{(n)} \to A^{(n+1)}$ is a derivation when $\tilde{Z}_1^\ell(A^{(n)}) = A^{(n)}$.*

Theorem 3.5. *Let A be a Banach algebra and let B be a closed subalgebra of $A^{(n)}$ that is consisting of $A^{(n-2)}$ where $n \geq 2$. If $\tilde{Z}_1^\ell(B) = B$ and B is weakly amenable, then $A^{(n-2)}$ is weakly amenable.*

Proof. Suppose that $D : A^{(n-2)} \to A^{(n-1)}$ is a derivation and $p : A^{(n+1)} \to B^*$ is the restriction map, defined by $P(a^{(n+1)}) = a^{(n+1)} \mid_B$ for every $a^{(n+1)} \in A^{(n+1)}$. Since $\tilde{Z}_1^\ell(B) = B$, $\bar{D} = PoD'' \mid_B : B \to B'$ is a derivation. Since B is weakly amenable, there is $b' \in B^*$ such that $\bar{D} = \delta_{b'}$. We take $a^{(n-1)} = b' \mid_{A^{(n-2)}}$, then $D = \bar{D}$ on $A^{(n-2)}$. Consequently, we have $D = \delta_{a^{(n-1)}}$. $\square$

Corollary 3.6. *Let A be a Banach algebra. If $A^{***}A^{**} \subseteq A^*$ and A is weakly amenable, then A^{**} is weakly amenable.*

Proof. By using Corollary 2.4 and Theorem 3.2, proof holds. $\square$

Corollary 3.7. *Let A be a Banach algebra and let $\tilde{Z}_1^\ell(A^{(n)})$ be weakly amenable whenever $n \geq 2$. Then $A^{(n-2)}$ is weakly amenable.*

Theorem 3.8. *Let B be a Banach $A-$bimodule and $D : A^{(n)} \to B^{(n+1)}$ be a derivation for $n \geq 0$. If ${\tilde{Z}^\ell}_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$, then $D'' : A^{(n+2)} \to B^{(n+3)}$ is a derivation.*

Proof. Let $x^{(n+2)}$, $y^{(n+2)} \in A^{(n+2)}$ and let $(x_\alpha^{(n)})_\alpha$, $(y_\beta^{(n)})_\beta \subseteq A^{(n)}$ such that $x_\alpha^{(n)} \xrightarrow{w^*} x^{(n+2)}$ and $y_\beta^{(n)} \xrightarrow{w^*} y^{(n+2)}$ in $A^{(n+2)}$. Then for all $b^{(n+2)} \in B^{(n+2)}$, we have $b^{(n+2)}x_\alpha^{(n)} \xrightarrow{w} b^{(n+2)}x^{(n+2)}$. Consequently, since ${\tilde{Z}^\ell}_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$, we have

$$\begin{aligned}
\langle x_\alpha^{(n)} D''(y^{(n+2)}), b^{(n+2)} \rangle &= \langle D''(y^{(n+2)}), b^{(n+2)} x_\alpha^{(n)} \rangle \\
&\to \langle D''(y^{(n+2)}), b^{(n+2)} x^{(n+2)} \rangle \\
&= \langle x^{(n+2)} D''(y^{(n+2)}), b^{(n+2)} \rangle.
\end{aligned}$$

Also we have the following equality

$$\begin{aligned}
\langle D''(x^{(n+2)}) y_\beta^{(n)}, b^{(n+2)} \rangle &= \langle D''(x^{(n+2)}), y_\beta^{(n)} b^{(n+2)} \rangle \\
&\to \langle D''(x^{(n+2)}), y^{(n+2)} b^{(n+2)} \rangle \\
&= \langle D''(x^{(n+2)}) y^{(n+2)}, b^{(n+2)} \rangle.
\end{aligned}$$

Since D is continuous, it follows that

$$\begin{aligned} D''(x^{(n+2)}y^{(n+2)}) &= \lim_\alpha \lim_\beta D(x_\alpha^{(n)}y_\beta^{(n)}) \\ &= \lim_\alpha \lim_\beta x_\alpha^{(n)}D(y_\beta^{(n)}) + \lim_\alpha \lim_\beta D(x_\alpha^{(n)})y_\beta^{(n)} \\ &= x^{(n+2)}D''(y^{(n+2)}) + D''(x^{(n+2)})y^{(n+2)}. \end{aligned}$$

□

Corollary 3.9. *Let B be a Banach $A-$bimodule and $\tilde{Z}^\ell_{A^{**}}(B^{**}) = B^{**}$. If $H^1(A, B^*) = 0$, then $H^1(A^{**}, B^{***}) = 0$.*

Corollary 3.10. *Let B be a Banach $A-bimodule$ and $\tilde{Z}^\ell{}_{A^{**}}(B^{**}) = B^{**}$. If $D : A \to B^*$ is a derivation, then $D''(A^{**})B^{**} \subseteq A^*$.*

Proof. By using Theorem 3.8, Corollary 3.3 and [19] proof holds. □

4. Cohomological properties of Banach algebras

Let A be a Banach algebra and $n \geq 0$. Then A is called $n-weakly$ amenable if $H^1(A, A^{(n)}) = 0$, and is called permanently weakly amenable when A is $n-weakly$ amenable for each $n \geq 0$. In [5] Dales, Ghahramani, and Gronbaek introduced the concept of n-weak amenability for Banach algebras for each natural number n. They determined some relations between m- and n-weak amenability for general Banach algebras and for Banach algebras in various classes, and proved that, for every n, (n + 2)- weak amenability always implies n-weak amenability.

Theorem 4.1. *Let B be a Banach $A-bimodule$ and let $n \geq 1$. If $H^1(A, B^{(n+2)}) = 0$, then $H^1(A, B^{(n)}) = 0$.*

Proof. Let $D \in Z^1(A, B^{(n)})$ and $i : B^{(n)} \to B^{(n+2)}$ be the canonical linear mapping as $A-bimodule$ homomorphism. Take $\widetilde{D} = ioD$. Then we can be viewed $\widetilde{D}$ as an element of $Z^1(A, B^{(n+2)})$. Since $H^1(A, B^{(n+2)}) = 0$, there exist $b^{(n+2)} \in B^{(n+2)}$ such that

$$\widetilde{D}(a) = ab^{(n+2)} - b^{(n+2)}a,$$

for all $a \in A$. Set a $A-linear$ mapping P from $B^{(n+2)}$ into $B^{(n)}$ such that $Poi = I_{B^{(n)}}$. Then we have $Po\widetilde{D} = (Poi)oD = D$, and so $D(a) = Po\widetilde{D}(a) = aP(b^{(n+2)}) - P(b^{(n+2)})a$ for all $a \in A$. It follows that $D \in N^1(A, B^{(n)})$. Consequently $H^1(A, B^{(n)}) = 0$. □

Theorem 4.2. *Let B be a Banach $A-bimodule$ and $D : A \to B^{(2n)}$ be a continuous derivation. Assume that $Z^\ell_{A^{(2n)}}(B^{(2n)}) = B^{(2n)}$. Then there is a continuous derivation $\widetilde{D} : A^{(2n)} \to B^{(2n)}$ such that $\widetilde{D}(a) = D(a)$ for all $a \in A$.*

Proof. By using Proposition 1.7 from [5], the linear mapping $D'' : A^{**} \to B^{(2n+2)}$ is a continuous derivation. Take $X = B^{(2n-2)}$. Since $Z_{A^{(2n)}}(X^{**}) = Z_{A^{(2n)}}(B^{(2n)}) = B^{(2n)} = X^{**}$, by Proposition 1.8 from [5] the canonical projection $P : X^{(4)} \to X^{**}$ is a $A^{**} - bimodule$ morphism. Set $\widetilde{D} = PoD''$. Then $\widetilde{D}$ is a continuous derivation from A^{**} into $B^{(2n)}$. Now by replacing A^{**} by A and repeating of the proof, result holds. □

Corollary 4.3. *Let B be a Banach $A - bimodule$ and $n \geq 0$. If $Z^{\ell}_{A^{(2n)}}(B^{(2n)}) = B^{(2n)}$ and $H^1(A^{(2n+2)}, B^{(2n+2)}) = 0$, then $H^1(A, B^{(2n)}) = 0$.*

Proof. By using Proposition 1.7 from [5] and preceding theorem the result holds. □

Corollary 4.4. [5]. *Let A be a Banach algebra such that $A^{(2n)}$ is Arens regular and $H^1(A^{(2n+2)}), A^{(2n+2)}) = 0$ for each $n \geq 0$. Then A is $2n-$ weakly amenable.*

Assume that A is Banach algebra and $n \geq 0$. We define $A^{[n]}$ as a subset of A as follows

$$A^{[n]} = \{a_1 a_2 ... a_n : \ a_1, a_2, ... a_n \in A\}.$$

We write A^n the linear span of $A^{[n]}$ as a subalgebra of A.

Theorem 4.5. *Let A be a Banach algebra and $n \geq 0$. Let $A^{[2n]}$ dense in A and suppose that B is a Banach $A - bimodule$. Assume that AB^{**} and $B^{**}A$ are subsets of B. If $H^1(A, B^*) = 0$, then $H^1(A, B^{(2n+1)}) = 0$.*

Proof. For $n = 0$ the result is clear. Let $B^{\perp}$ be the space of functionals in $B^{(2n+1)}$ which annihilate $i(B)$ where $i : B \to B^{(2n)}$ is a natural canonical mapping. Then, by using lemma 1 [23], we have the following equality

$$B^{(2n+1)} = i(B^*) \oplus B^{\perp},$$

it follows that

$$H^1(A, B^{(2n+1)}) = H^1(A, i(B^*)) \oplus H^1(A, B^{\perp}).$$

Without lose generality, we replace $i(B^*)$ by B^*. Since $H^1(A, B^*) = 0$, it is enough to show that $H^1(A, B^{\perp}) = 0$.
Now, take the linear mappings L_a and R_a from B into itself by $L_a(b) = ab$ and $R_a(b) = ba$ for all $a \in A$. Since $AB^{**} \subseteq B$ and $B^{**}A \subseteq B$, $L^{**}_a(b'') = ab''$ and $R^{**}_a(b'') = b''a$ for every $a \in A$, respectively. Consequently, L_a and R_a from B into itself are weakly compact. It follows that for each $a \in A$ the linear mappings $L^{(2n)}_a$ and $R^{(2n)}_a$ from $B^{(n)}$ into $B^{(n)}$ are weakly compact and for every $b^{(2n)} \in B^{(2n)}$, we have $L^{(2n)}_a(b^{(2n)}) = ab^{(2n)} \in B^{(2n-2)}$ and $R^{(2n)}_a(b^{(2n)}) = b^{(2n)}a \in B^{(2n-2)}$. Set $a_1, a_2, ..., a_n \in$

A and $b^{(2n)} \in B^{(2n)}$. Then $a_1a_2...a_nb^{(2n)}$ and $b^{(2n)}a_1a_2...a_n$ are belong to B. Suppose that $D \in Z^1(A, B^\perp)$ and let $a, x \in A^{[n]}$. Then for every $b^{(2n)} \in B^{(2n)}$, since $xb^{(2n)}, b^{(2n)}a \in B$, we have the following equality

$$\langle D(ax), b^{(2n)}\rangle = \langle aD(x), b^{(2n)}\rangle + \langle D(a)x, b^{(2n)}\rangle$$
$$= \langle D(x), b^{(2n)}a\rangle + \langle D(a), xb^{(2n)}\rangle = 0.$$

It follows that $D \mid_{A^{[2n]}} = 0$. Since $A^{[2n]}$ dense in A, $D = 0$. Hence $H^1(A, B^\perp) = 0$ and result follows. $\square$

Corollary 4.6. (1) *Let A be a Banach algebra with left bounded approximate identity, and let B be a Banach $A-$bimodule. Suppose that AB^{**} and $B^{**}A$ are subset of B. Then $H^1(A, B^{(2n+1)}) = 0$ for all $n \geq 0$, whenever $H^1(A, B^*) = 0$.*

(2) *Let A be an amenable Banach algebra and B be a Banach $A-$bimodule. If AB^{**} and $B^{**}A$ are subset of B, then $H^1(A, B^{(2n+1)}) = 0$.*

Example 4.7. Assume that G is a compact group.

(1) We know that $L^1(G)$ is $M(G)-$*bimodule* and $L^1(G)$ is an ideal in the second dual of $M(G)$, $M(G)^{**}$. By using corollary 1.2 from [18], we have $H^1(L^1(G), M(G)^*) = 0$. Then for every $n \geq 1$, by using preceding corollary, we conclude that
$$H^1(L^1(G), M(G)^{(2n+1)}) = 0.$$

(2) Since $L^1(G)$ is an ideal in its second dual , $L^1(G)^{**}$, by using [15], $L^1(G)$ is a weakly amenable. Then by preceding corollary, $L^1(G)$ is $(2n + 1) - weakly$ amenable.

Corollary 4.8. *Let A be a Banach algebra and let $A^{[2n]}$ be dense in A. Suppose that AB^{**} and $B^{**}A$ are subset of B. Then the following are equivalent.*

(1) $H^1(A, B^*) = 0$.
(2) $H^1(A, B^{(2n+1)}) = 0$ *for some* $n \geq 0$.
(3) $H^1(A, B^{(2n+1)}) = 0$ *for each* $n \geq 0$.

Corollary 4.9. [5]. *Let A be a weakly amenable Banach algebra such that A is an ideal in A^{**}. Then A is $(2n + 1) - weakly$ amenable for each $n \geq 0$.*

Proof. By using Proposition 1.3 from [5] and preceding theorem, result holds. $\square$

Assume that A and B are Banach algebras. Then $A \oplus B$, with norm
$$\| (a, b) \| = \| a \| + \| b \|,$$

and product $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ is a Banach algebra. It is clear that if X is a Banach A *and* $B-$*bimodule*, then X is a Banach $A \oplus B-$*bimodule*.
In the following, we investigated the relationships between the cohomological property of $A \oplus B$ with A and B.

Theorem 4.10. *Suppose that A and B are Banach algebras. Let X be a Banach A and $B-$bimodule. Then, $H^1(A \oplus B, X) = 0$ if and only if $H^1(A, X) = H^1(B, X) = 0$.*

Proof. Suppose that $H^1(A \oplus B, X) = 0$. Assume that $D_1 \in Z^1(A, X)$ and $D_2 \in Z^1(B, X)$. Take $D = (D_1, D_2)$. Then for every $a_1, a_2 \in A$ and $b_1, b_2 \in B$, we have

$$\begin{aligned} D((a_1, b_1)(a_2, b_2)) &= D(a_1a_2, b_1b_2) = (D_1(a_1a_2), D_2(b_1b_2)) \\ &= (a_1D_1(a_2) + D_1(a_1)a_2, b_1D_2(b_2) + D_2(b_1)b_2) \\ &= (a_1D_1(a_2), b_1D_2(b_2)) + (D_1(a_1)a_2 + D_2(b_1)b_2) \\ &= (a_1, b_1)(D_1(a_2), D_2(b_2)) + (D_1(a_1), D_2(b_1))(a_2, b_2) \\ &= (a_1, b_1)D(a_2, b_2) + D(a_1, b_1)(a_2, b_2). \end{aligned}$$

It follows that $D \in Z^1(A \oplus B, X)$. Since $H^1(A \oplus B, X) = 0$, there is $x \in X$ such that $D = \delta_x$ where $\delta_x \in N^1(A \oplus B, X)$. Since $\delta_x = (\delta_x^1, \delta_x^2)$ where $\delta_x^1 \in N^1(A, X)$ and $\delta_x^2 \in N^1(B, X)$, we have $D_1 = \delta_x^1$ and $D_2 = \delta_x^2$. Thus $H^1(A, X) = H^1(B, X) = 0$.
For the converse, take A as an ideal in $A \oplus B$, and so by using Proposition 2.8.66 from [4], proof holds. □

Example 4.11. Let G be a locally compact group and X be a Banach $L^1(G)-$*bimodule*. Then by [5], pp.27 and 28, X^{**} is a Banach $L^1(G)^{**}-$*bimodule*. Since $L^1(G)^{**} = LUC(G)^* \oplus LUC(G)^{\perp}$, by using preceding theorem, we have

$$H^1(L^1(G)^{**}, X^{**}) = 0$$

if and only if $H^1(LUC(G)^*, X^{**}) = H^1(LUC(G)^{\perp}, X^{**}) = 0$.
On the other hand, we know that $L^1(G)^{**} = L^1(G) \oplus C_0(G)^{\perp}$. By [15], we know that, $H^1(L^1(G), L^{\infty}(G)) = 0$. Then by using preceding theorem, $H^1(L^1(G)^{**}, L^{\infty}(G)) = 0$, if and only if $H^1(C_0(G)^{\perp}, L^{\infty}(G)) = 0$.

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