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Fitted Numerical Scheme for Singularly Perturbed Differential Equations Having Small Delays

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Abstract. In this paper, singularly perturbed differential equations having delay on the convection and reaction terms are considered. The highest order derivative term in the equation is multiplied by a perturbation parameter ε taking arbitrary values in the interval $(0, 1]$. For small ε, the problem involves a boundary layer on the left or right side of the domain depending on the sign of the coefficient of the convective term. The terms involving the delay are approximated using Taylor series approximation. The resulting singularly perturbed boundary value problem is treated using exponentially fitted upwind finite difference method. The stability of the proposed scheme is analysed and investigated using maximum principle and barrier functions for solution bound. The formulated scheme converges independent of the perturbation parameter with rate of convergence $O(N^{-1})$. Richardson extrapolation technique is applied to accelerate the rate of convergence of the scheme to order $O(N^{-2})$. To validate the theoretical finding, three model examples having boundary layer behaviour are considered. The maximum absolute error and rate of convergence of the scheme are computed. The proposed scheme gives accurate and parameter uniformly convergent result.

Keywords: Delay differential equation, exponentially fitted method, singularly perturbed problem, uniform convergence.

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1. Introduction

In differential equations the evolution of the system depend on the present state of the system and the past did not influence the system. Differential equations play a prominent role in many disciplines including engineering, physics, economics, and biology. Currently different authors are working on analytical and numerical solution of fractional order differential equations using different techniques. To list few of them [2], [3, 4], [6], [12, 13, 14, 15, 16, 17, 18],[22], [25, 26], [30].

Differential difference equations or delay differential equations (DDEs) are differential equations where the evolution of the system does not only depend on the present state of the system but also depends on the past history. DDEs are called retarded type if the delay argument does not occur in the highest order derivative term, otherwise it is known as neutral type. DDEs arise in the mathematical modeling of various physical phenomena, for example in micro scale heat transfer [29], fluid dynamics [8], diffusion in polymers [19], reaction-diffusion equations [5], a lot of model in diseases or physiological processes [20] etc.

Singularly perturbed differential difference equations are differential equations in which its highest order derivative is multiplied by a small perturbation parameter and having delay parameters on the terms different from the highest derivative. The presence of singular perturbation parameter ε leads to bad approximation or oscillation in the computed solution while using standard numerical methods [7]. To avoid this oscillations unacceptably large number of mesh points are required when ε is very small. This is not practical and leads to rounding error. So, to overcome the drawbacks associated with the standard numerical methods, it is necessary to developed a numerical scheme which converges independent of the perturbation parameter.

Since developing higher order uniformly convergent numerical scheme is an active research area. Different authors developed a numerical scheme which converges independent of the perturbation parameter for the considered problem. Kadelbajoo and Ramesh in [9] consider the problem and used Taylor series approximation for the delay terms. The authors develop uniformly convergent schemes using upwind, midpoint upwind and hybrid of midpoint upwind and central on piecewise uniform mesh. Kumar and Kadalbajoo in [11] used Taylor series approximation for the delay terms and computed the numerical solution using B-spline collocation method on shishkin mesh. Their scheme converges uniformly with almost second order rate of convergence. Adilaxmi et.al in [1] first approximate the problem using Taylor series approximation and solved using non standard FDM with exponential fitting factor. In [24] Phaneendra and Lula apply the Gaussian quadrature two-point formula for treating the problem. In this paper, we developed second order uniformly convergent numerical scheme using exponentially fitted FDM. In addition to that we analyse the uniform convergence of the scheme. The proposed scheme gives accurate and oscillation free solution on uniform mesh.

2. Statement of the problem

Consider a class of singularly perturbed differential equations having delay on the convection and reaction terms of the form

$$-\varepsilon u''(x)+a(x)u'(x-\delta)+\beta(x)u(x)+\omega(x)u(x-\delta) = f(x),\ x \in \Omega = (0,1), \tag{2.1}$$

with interval-boundary conditions

$$u(x) = \phi(x),\ \ x \in \Omega_L = [-\delta, 0],\ \ u(1) = \gamma, \tag{2.2}$$

where $\varepsilon, (0 < \varepsilon \ll 1)$ is singular perturbation parameter and δ is delay parameter satisfying $\delta < \varepsilon$. The functions $a(x), \beta(x), \omega(x)$ and $f(x)$ are assumed to be smooth, bounded and not a function of ε. The values of $\phi(x)$ and γ are assumed finite constants. We assume also the coefficients of non-derivative terms β and ω satisfy

$$\beta(x) + \omega(x) \geq q^* > 0,\ \forall x \in \bar{\Omega}$$

for some constant q^*. This condition ensures that the solution of (2.1)-(2.2) exhibits boundary layer in the neighborhood of $x = 0$ or $x = 1$ depending on the sign of the convective term $a(x)$.

When the delay parameter is zero (i.e.,$\delta = 0$) the problem reduces to singularly perturbed BVPs, for small ε the problem exhibits boundary layer depending upon the value of the convective term coefficient $a(x)$. When $a(x) < 0$ regular boundary layer appears in the neighbourhood of $x = 0$ and $a(x) > 0$ corresponds to existence of a boundary layer in the neighbourhood of $x = 1$. If $a(x)$ change sign, shock layer will appear on the middle of the domain [31]. The layer is maintained for $\delta \neq 0$ but sufficiently small.

Our objective is to developed numerical scheme using exponentially fitted FDM for treating the problem in (2.1)-(2.2) accurately.

2.1. **Estimate for terms with the delay.** For $\delta < \varepsilon$, using Taylor's series approximation for the terms with the shifts is valid [27]. Accordingly the terms $u'(x-\delta)$ and $u(x-\delta)$ approximated as

$$\begin{aligned} u'(x-\delta) &\approx u'(x) - \delta u''(x) + O(\delta^2), \\ u(x-\delta) &\approx u(x) - \delta u'(x) + \frac{\delta^2}{2} u''(x) + O(\delta^3). \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.1) gives a singularly perturbed BVP

$$-c_\varepsilon(x)u''(x) + p(x)u'(x) + q(x)u(x) = f(x),\ x \in \Omega = (0,1), \tag{2.4}$$

with the boundary conditions

$$u(0) = \phi(0), \quad u(1) = \gamma, \tag{2.5}$$

where $c_\varepsilon(x) = \varepsilon + \delta a(x) - \frac{\delta^2}{2}\omega(x)$, $p(x) = a(x) - \delta\omega(x)$ and $d(x) = \beta(x)+\omega(x)$. For small values of ε, (2.4)-(2.5) is asymptotically equivalent to (2.1)-(2.2).

We assume, $0 < c_\varepsilon(x) \le \varepsilon - \delta M_1 - \delta^2 M_2 = c_\varepsilon$, where $a(x) \ge M_1$ and $\omega(x) \ge 2M_2$ for M_1 and M_2 are constants. Let us consider first the case $p(x) \le p^* < 0$ which imply occurrence of boundary layer on the left side of the domain, the other case $p(x) \ge p^* > 0$ imply the occurrence of boundary layer on the right side of the domain.

The problem obtained by setting $c_\varepsilon = 0$ in (2.4)-(2.5) is called reduced problem and given as

$$\begin{aligned} p(x)u_0'(x) + q(x)u_0(x) &= f(x),\ x \in \Omega, \\ u_0(0) &= \phi(0). \end{aligned} \tag{2.6}$$

It is a first order initial value problem. For small values of c_ε the solution of (2.6) is very close to the solution of (2.4)-(2.5).

Let L be denoted for the differential operator $Lz = -c_\varepsilon z''(x) + p(x)z'(x) + q(x)z(x)$ in equation (2.4)-(2.5).

2.2. **Properties of the analytical solution.**

Lemma 2.1. *(The Maximum Principle) Let z be a sufficiently smooth function defined on Ω which satisfies $z(0) \ge 0$ and $z(1) \ge 0$. Then $Lz(x) > 0,\ \forall x \in \Omega$ implies that $z(x) \ge 0,\ \forall x \in \bar{\Omega}$*

Proof. Let x^* be such that $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ and suppose that $z(x^*) < 0$. It is clear that $x^* \notin \{0,1\}$. Since $z(x^*) = \min_{(x)\in\bar{\Omega}} z(x)$ we have $z'(x^*) = 0$ and $z''(x^*) \ge 0$ implies that $Lz(x^*) < 0$ which is contradiction to the assumption that made above $Lz(x^*) > 0,\ \forall x \in \Omega$. Therefore $z(x) \ge 0,\ \forall x \in \bar{\Omega}$. □

Lemma 2.2. *(Stability estimate) Let $u(x)$ be the solution of (2.4)-(2.5). Then we obtain the bound*

$$|u(x)| \leq \frac{\|Lu\|}{q^*} + \max\{\phi(0), \gamma\} \tag{2.7}$$

where $q^* > 0$ is lower bound of $q(x)$.

Proof. By defining barrier functions $\vartheta^{\pm}(x)$ as $\vartheta^{\pm}(x) = \frac{\|Lu\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(x)$. At the boundary points we obtain

$$\begin{aligned} \vartheta^{\pm}(0) =& \frac{\|Lu\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(0) \geq 0. \\ \vartheta^{\pm}(1) =& \frac{\|Lu\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(1) \geq 0. \end{aligned}$$

and on the differential operator

$$\begin{aligned} L\vartheta^{\pm}(x) =& -c_\varepsilon \vartheta''_{\pm}(x) + p(x)\vartheta'_{\pm}(x) + q(x)\vartheta_{\pm}(x) \\ =& \mp c_\varepsilon u''(x) \pm p(x)u'(x) + q(x)\big(\frac{\|Lu\|}{q^*} + \max\{\phi(0), \gamma\} \pm u(x)\big) \\ =& q(x)\big(\frac{\|Lu\|}{q^*} + \max\{\phi(0), \gamma\}\big) \pm f(x) \\ \geq& 0, \text{ since } q^* > 0 \text{ is lower bound of } q(x). \end{aligned}$$

which implies $L\vartheta^{\pm}(x) \geq 0$. Hence using maximum principle in Lemma 2.1 we obtain $\vartheta^{\pm}(x) \geq 0, \ \forall x \in \bar{\Omega}$. □

Lemma 2.3. *The derivatives of the solution $u(x)$ of the problem in (2.4)-(2.5) is bounded as*

$$\begin{aligned} \left| u^{(k)}(x) \right| \leq& C\big(1 + c_\varepsilon^{-k} \exp(-\frac{p^* x}{c_\varepsilon})\big), x \in \bar{\Omega}, \ for\ left\ leyer, \\ \left| u^{(k)}(x) \right| \leq& C\big(1 + c_\varepsilon^{-k} \exp(-\frac{p^*(1-x)}{c_\varepsilon})\big), x \in \bar{\Omega}, \ for\ right\ layer. \end{aligned} \tag{2.8}$$

for $0 \leq k \leq 4$, where $p(x) \geq p^* > 0$ for right boundary layer case and $p(x) \leq p^* < 0$ for left boundary layer case.

Proof. See on [10] or [21]. □

3. Formulation of Numerical Scheme

The domain $\bar{\Omega} = [0, 1]$ is discretized into N equal number of subintervals each of length $h = \frac{1}{N}$. Let $\bar{\Omega}^N = \{x_i = ih\}_0^N$ be the discretized domain satisfying $x_0 = 0, x_i = ih, i = 1, 2, ..., N-1$ and $x_N = 1$. Using the theory developed in asymptotic method for solving singularly perturbed BVPs. We apply exponentially fitted operator finite difference method (FOFDM) for treating numerically the problem in (2.4)-(2.5). We consider and treat separately the left and the right boundary layer cases.

1. Left boundary layer problem

In this case, the sign of the coefficient function $p(x)$ is negative and the boundary layer occurs near $x = 0$. For left boundary layer problem from the theory of singular perturbation in [23], the asymptotic solution of the zeros order approximation for the problem in (2.4)-(2.5) is given as

$$\begin{aligned} u(x) =& u_0(x) + \frac{p(0)}{p(x)}(\phi(0) - u_0(0)) \exp\big(-\int_0^x \big(\frac{p(x)}{c_\varepsilon} - \frac{q(x)}{p(x)}\big)dx\big) \\ &+ O(c_\varepsilon), \end{aligned} \tag{3.1}$$

where u_0 is the solution of the reduced problem. Using Taylors series approximation for $u_0(x), p(x)$ and $q(x)$ centring at $x_i = ih$ up to first order and considering $c_\varepsilon \to 0$, the discretized form of (3.1) becomes

$$u(ih) = u_0(ih) + (\phi(0) - u_0(0)) \exp(-p(x_i)(i\rho)), \tag{3.2}$$

where $\rho = h/c_\varepsilon, \;\; h = 1/N$. Similarly, we write

$$\begin{aligned} u((i+1)h) = u_0(ih) + (\phi(0) - u_0(0)) \exp(-p(x_i)(i+1)\rho), \\ u((i-1)h) = u_0(ih) + (\phi(0) - u_0(0)) \exp(-p(x_i)(i-1)\rho). \end{aligned} \tag{3.3}$$

Next, on uniform points $\bar{\Omega}^N = \{x_i\}_{i=0}^N$ with $h = x_{i+1} - x_i$. Let U_i denote the approximation for $u(x_i)$ in discretizing the problem. Using upwind finite difference we write the numerical scheme as

$$-c_\varepsilon \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + p(x_i)\frac{U_{i+1} - U_i}{h} + q(x_i)U_i = f(x_i), \tag{3.4}$$

for $i = 1, 2, ..., N-1$.

To handle the effect of the singular perturbation parameter exponentially fitting factor $\sigma_1(\rho)$ is multiplied on the term containing the singular perturbation parameter as

$$-c_\varepsilon \sigma_1(\rho)\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + p(x_i)\frac{U_{i+1} - U_i}{h} + q(x_i)U_i = f(x_i). \tag{3.5}$$

Since h is small, multiplying both sides of (3.5) by h and truncating the term $(f(x_i) - q(x_i)U_i)h$ (because $f(x_i) - q(x_i)U_i$ is bounded) results to

$$-\frac{\sigma_1(\rho)}{\rho}\big(U_{i-1} - 2U_i + U_{i+1}\big) + p(x_i)\big(U_{i+1} - U_i\big) = 0. \tag{3.6}$$

since $\rho = h/c_\varepsilon$. Substituting the results in (3.2) and (3.3) into (3.6) and simplifying the exponential fitting factor is obtained as

$$\sigma_1(\rho) = \rho p(x_i)\left[\frac{\exp(-\rho p(x_i)) - 1}{\exp(\rho p(x_i)) - 2 + \exp(-\rho p(x_i))}\right]. \tag{3.7}$$

Hence, the required finite difference scheme becomes

$$L_L^h U_i = f(x_i),\ i = 1, 2, ..., N-1 \tag{3.8}$$

with the boundary values $U_0 = \phi(0)$ and $U_N = \gamma$ where

$$L_L^h U_i = -c_\varepsilon \sigma_1(\rho) \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + p(x_i) \frac{U_{i+1} - U_i}{h} + q(x_i) U_i.$$

2. Right boundary layer problem

In this case the sign of the coefficient function $p(x)$ is positive and the boundary layer occurs near $x = 1$. From [23], the asymptotic solution of zeros order approximation of the problem in (2.4)-(2.5) is given as

$$\begin{aligned} u(x) = & u_0(x) + \frac{p(1)}{p(x)} (\gamma - u_0(1)) \exp\big(- \int_x^1 \big(\frac{p(x)}{c_\varepsilon} - \frac{q(x)}{p(x)} \big) dx \big) \\ & + O(c_\varepsilon). \end{aligned} \tag{3.9}$$

In this case, using the backward difference for first derivative term the scheme is written as

$$-c_\varepsilon \sigma_2(\rho) \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + p(x_i) \frac{U_i - U_{i-1}}{h} + q(x_i) U_i = f(x_i). \tag{3.10}$$

for $i = 1, 2, ..., N-1$. Using the same procedure as the left boundary layer case the exponential fitting factor is obtained as

$$\sigma_2(\rho) = \rho p(x_i) \left[\frac{1 - \exp(\rho p(x_i))}{\exp(\rho p(x_i)) - 2 + \exp(-\rho(p(x_i))} \right]. \tag{3.11}$$

The required exponentially fitted finite difference scheme becomes

$$L_R^h U_i = f(x_i),\ i = 1, 2, ..., N-1 \tag{3.12}$$

with data on the boundary $U_0 = \phi(0)$ and $U_N = \gamma$, where

$$L_R^h U_i = -c_\varepsilon \sigma_2(\rho) \frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + p(x_i) \frac{U_i - U_{i-1}}{h} + q(x_i) U_i.$$

3.1. **Parameter uniform convergence analysis.** Here, we give the convergence analysis for the right boundary layer case and in similar manner the left boundary layer case follows. First, we need to prove the discrete maximum principle for the proposed scheme in (3.8) for guaranteeing existence of unique discrete solution.

Lemma 3.1 (The Discrete Maximum Principle)**.** *Assume that, the mesh function $z(x_i)$ satisfies $z(x_0) \geq 0$ and $z(x_N) \geq 0$. If $L_R^h z(x_i) \geq 0$, for $1 \leq i \leq N-1$, then $z(x_i) \geq 0$, for $0 \leq i \leq N$.*

Proof. Let choose k such that $z(x_k) = \min_{x_i} z(x_i)$, $1 \leq i \leq N-1$. If $z(x_k) \geq 0$, the proof completed. We can see that $z(x_{k+1}) - z(x_k) \geq 0$ and $z(x_k) - z(x_{k-1}) \leq 0$. Now from (3.8), we obtain $L_R^h z(x_k) < 0$, which contradicts $L_R^h z(x_k) \geq 0$. Hence, the assumption is wrong. We conclude that $z(x_i) \geq 0$, $\forall i$, $0 \leq i \leq N$. □

Lemma 3.2 (Discrete Stability Estimate)**.** *The solution U_i of the discrete scheme in (3.8) satisfy the following bound*

$$|U_i| \leq \frac{\|L_R^h U_i\|}{q^*} + \max\{U_0, U_N\}. \tag{3.13}$$

Proof. Let $p = \frac{\|L_R^h U_i\|}{q^*} + \max\{U_0, U_N\}$ and define the barrier function $\vartheta_i^{\pm}$ by $\vartheta_i^{\pm} = p \pm U_i$.
On the boundary points, we obtain

$$\vartheta_0^{\pm} = p \pm U_0 = \frac{\|L_R^h U_i\|}{q^*} + \max\{U_0, U_N\} \pm \phi(0) \geq 0.$$

$$\vartheta_N^{\pm} = p \pm U_N = \frac{\|L_R^h U_i\|}{q^*} + \max\{U_0, U_N\} \pm \gamma \geq 0.$$

On the discretized spatial domain x_i, $0 < i < N$, we obtain

$$\begin{aligned} L_R^h \vartheta_i^{\pm} =& - c_\varepsilon \sigma(\rho)\Big(\frac{p \pm U_{i+1} - 2(p \pm U_i) + p \pm U_{i-1}}{h^2}\Big) \\ &+ p(x_i)\Big(\frac{p \pm U_i - (p \pm U_{i-1})}{h}\Big) + q(x_i)(p \pm U_i) \\ =& q(x_i) p \pm L_R^h U_i \\ =& q(x_i)\Big(\frac{\|L_R^h U_i\|}{q^*} + \max\{U_0, U_N\}\Big) \pm f(x_i) \geq 0, \quad \textit{since} \;\; q(x_i) \geq q^*. \end{aligned}$$

By discrete maximum principle in Lemma 3.1, we obtain $\vartheta_i^{\pm} \geq 0$, $\forall x_i \in \bar{\Omega}^N$. Hence the required bound is obtained. □

Let us denote the difference operators for approximating the first and second derivatives as

$$\begin{aligned} D^- u(x_i) =& \frac{U_i - U_{i-1}}{h}, \quad D^+ u(x_i) = \frac{U_{i+1} - U_i}{h} \text{ and} \\ D^+ D^- u(x_i) =& \frac{U_{i-1} - U_i + U_{i+1}}{h^2} \end{aligned} \tag{3.14}$$

The following theorem gives truncation error bound of the proposed scheme.

Theorem 3.3. *Let $u(x_i)$ and U_i be respectively the exact and approximate solution of* (2.4)-(2.5). *Then the truncation error satisfies the following bound*

$$\left\| L_R^h(u(x_i)-U_i) \right\| \leq CN^{-1}\big(1+c_\varepsilon^{-4} \max_{1\leq i\leq N-1} \exp\big(-\frac{p^*(1-x_i)}{c_\varepsilon}\big)\big). \quad (3.15)$$

Proof. Let us consider the truncation error defined as

$$\begin{aligned} L^h u(x_i) - L_R^h U_i = &- c_\varepsilon\big(u''(x_i) - \sigma(\rho)D^+D^-u(x_i)\big) \\ &+ p(x_i)\big(u'(x_i) - D^-u(x_i)\big), \end{aligned} \quad (3.16)$$

where $c_\varepsilon\sigma(\rho) = p(x_i)N^{-1}\big[\frac{1-\exp(\rho p(x_i))}{\exp(\rho p(x_i))-2+\exp(-\rho(p(x_i))}\big]$ since $\rho = \frac{N^{-1}}{c_\varepsilon}$.
In our assumption $c_\varepsilon \leq h = N^{-1}$. By considering N is fixed and taking the limit for $c_\varepsilon \to 0$, we obtain

$$\lim_{c_\varepsilon\to 0} c_\varepsilon\sigma(\rho) = \lim_{c_\varepsilon\to 0} p(x_i)N^{-1}\left[\frac{1-\exp(\rho p(x_i))}{\exp(\rho p(x_i))-2+\exp(-\rho(p(x_i))}\right] = CN^{-1}.$$

where C is constant independent of N and c_ε. Using Taylor's series approximation the differences of the derivatives is bounded as

$$\begin{aligned} \left\| u''(x_i) - D^+D^-u(x_i) \right\| &\leq CN^{-2}\left\| u^{(4)}(x_i) \right\|, \\ \left\| u'(x_i) - D^-u(x_i) \right\| &\leq CN^{-1}\left\| u''(x_i) \right\|, \end{aligned} \quad (3.17)$$

where $\left\| u^{(k)}(x_i) \right\| = \max_{0\leq i\leq N} \left| u^{(k)}(x_i) \right|, k = 2, 4.$

Now using the bounds in (3.17) we write (3.16) as

$$\left\| L_R^h\big(u(x_i) - U_i\big) \right\| \leq CN^{-3}\left\| u^{(4)}(x_i) \right\| + CN^{-1}\left\| u''(x_i) \right\|. \quad (3.18)$$

The aim is to show that the scheme convergence independent of the perturbation parameter c_ε. Using the bounds for the derivatives of the solution in Lemma 2.3, the truncation error in (3.18) is bounded as

$$\begin{aligned} \left\| L_R^h(u(x_i) - U_i) \right\| \leq &CN^{-3}\big(1 + c_\varepsilon^{-4}\exp\big(-\frac{p^*(1-x_i)}{c_\varepsilon}\big)\big) \\ &+ CN^{-1}\big(1 + c_\varepsilon^{-2}\exp\big(-\frac{p^*(1-x_i)}{c_\varepsilon}\big)\big). \end{aligned}$$

Since $c_\varepsilon^{-4} \geq c_\varepsilon^{-2}$,

$$\left\| L_R^h(u(x_i)-U_i) \right\| \leq CN^{-1}\big(1+c_\varepsilon^{-4} \max_{1\leq i\leq N-1} \exp\big(-\frac{p^*(1-x_i)}{c_\varepsilon}\big)\big). \quad (3.19)$$

□

Lemma 3.4. *For $c_\varepsilon \to 0$ and for given fixed N, we obtain*

$$\lim_{c_\varepsilon \to 0} \max_j \frac{\exp\left(-\frac{p^* x_j}{c_\varepsilon}\right)}{c_\varepsilon^m} = 0, \ m = 1, 2, 3, ...$$
$$\lim_{c_\varepsilon \to 0} \max_j \frac{\exp\left(-\frac{p^*(1-x_j)}{c_\varepsilon}\right)}{c_\varepsilon^m} = 0, \ m = 1, 2, 3, ... \quad (3.20)$$

where $x_j = jh, h = 1/N, \forall j = 1, 2, ..., N-1$.

Proof. See in [28] or [32]. □

Theorem 3.5. *Under the hypothesis of boundedness of discrete solution, the solution of the discrete schemes in* (3.8) *satisfy the following parameter uniform bound*

$$\sup_{0 < c_\varepsilon \ll 1} \|u(x_i) - U_i\| \leq CN^{-1}. \quad (3.21)$$

Proof. Substituting the results in Lemma 3.4 into Theorem 3.3 then applying the discrete maximum principle gives the required bound. □

3.2. **Richardson Extrapolation.** Here, we apply the Richardson extrapolation technique to accelerate the rate of convergence of the proposed scheme. Richardson Extrapolation is a convergence acceleration technique which involves combination of two computed approximations of solution. From (3.21) we have

$$u(x_i) - U_i \leq CN^{-1}, \quad (3.22)$$

where $u(x_i)$ and U_i are exact and approximate solutions respectively and C is constant independent of ε and N. Let U_i^{2N} denoted for an approximate solution on $2N$ number of mesh points by including the mid points. From (3.22) we have

$$u(x_i) - U_i \leq CN^{-1} + R_N, \quad (3.23)$$

So, this works for any $h/2 \neq 0$ gives

$$u(x_i) - U_i^{2N} \leq CN^{-1}/2 + R_{2N}, \quad (3.24)$$

where the remainders R_N and R_{2N} are order of N^{-1} and N^{-2} . Combining (3.23) and (3.24) leads to

$$u(x_i) - (2U_i^{2N} - U_i) \leq CN^{-2}$$

which gives that

$$U_i^{ext} = 2U_i^{2N} - U_i \quad (3.25)$$

is also an approximate solution. The total truncation error for the approximate solution in (3.25) becomes

$$\sup_{0 < c_\varepsilon \ll 1} \|u(x_i) - U_i^{ext}\| \leq CN^{-2}. \quad (3.26)$$

4. Examples and Numerical Results

To demonstrate the efficiency of the proposed scheme, we solved two examples having boundary layer behavior.

Example 4.1. We consider the problem,

$$\varepsilon u''(x) + (1+x)u'(x-\delta) + \sin(2x)u(x-\delta) - e^{-x}u(x) = \sin(2x) + 3e^{-x}$$

with interval-boundary conditions $u(x) = -1,\ -\delta \le x < 0$ and $u(1) = 1$.

Example 4.2. We consider the problem,

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - e^{-2x}u(x-\delta) + e^{-x}u(x) = 0$$

with interval-boundary conditions $u(x) = 1,\ -\delta \le x < 0$ and $u(1) = -1$.

Example 4.3. We consider the problem,

$$-\varepsilon u''(x) + (1+x)u'(x-\delta) - e^{-2x}u(x-\delta) + e^{-x}u(x) = e^{x-1}$$

with interval-boundary conditions $u(x) = 1,\ -\delta \le x < 0$ and $u(1) = -1$.

Since the exact solution of the considered problems are not known, the maximum absolute errors are estimated by using the double mesh principle [31] and it is defined by

$$E_\varepsilon^N = \max_{0\le i\le N} |U_i^N - U_i^{2N}|,$$

where U_i^N stands for the numerical solution of the problem on N number of mesh points and U_i^{2N} stands for the numerical solution of the problem on $2N$ number of mesh points by including the mid-points $x_{i+h/2}$ into the mesh numbers. The parameter uniform error estimate is defined as

$$E^N = \max_{\varepsilon} |E_\varepsilon^N|.$$

The rate of convergence of the scheme is given by

$$R_\varepsilon^N = \frac{\log(E_\varepsilon^N) - \log(E_\varepsilon^{2N})}{\log(2)}.$$

and the parameter uniform rate of convergence is given as

$$R^N = \frac{\log(E^N) - \log(E^{2N})}{\log(2)}.$$

The maximum absolute error of Example 4.1-4.3 is given in Table 1 - 3 respectively for different values of perturbation parameter $\varepsilon = 10^0 \to 10^{-10}$. As we observe on these tables for each N as $\varepsilon \to 0$, the maximum absolute error becomes stable and uniform, which indicate that the proposed scheme convergence independent of the perturbation parameter. In the last two rows of these tables, we observe the parameter uniform error and the parameter uniform rate of convergence. As one observes the scheme have second order uniform rate of convergence. The results in Table 4 indicates, the maximum absolute error of the proposed scheme for different values of the delay parameter while keeping $\varepsilon = 10^{-1}$. In Table 5, we compared the the maximum absolute error of the proposed scheme with the result given in [9] and [11]. As one see the results in this table, the proposed scheme gives more accurate result.

In Figure 1, one observe the influence of the delay parameter on the behaviour of the solution of Example 4.1 and 4.2 respectively on (a) and (b). From this figure one can observe that for left boundary layer problem as the values of the delay parameter increases the thickness of the boundary layer decreases and vis versa for right boundary layer problem. From Figure 2 one observe that, the problem in Example 4.1 has left boundary layer and Example 4.2 has right boundary layer. As the perturbation parameter goes small the graphs of the solution forms strong boundary layer.

TABLE 1. Example 4.1, maximum absolute error for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	$N \rightarrow 32$	64	128	256
10^{0}	5.7255e-05	3.5794e-06	2.2370e-07	1.3415e-08
10^{-01}	6.0572e-04	3.7992e-05	2.3750e-06	1.4616e-07
10^{-02}	3.7191e-03	3.8015e-04	2.4258e-05	1.5194e-06
10^{-03}	2.7289e-03	7.2061e-04	2.1252e-04	1.5113e-05
10^{-04}	2.7284e-03	7.0334e-04	1.7716e-04	5.3123e-05
10^{-05}	2.7284e-03	7.0333e-04	1.7718e-04	4.4379e-05
10^{-06}	2.7284e-03	7.0333e-04	1.7718e-04	4.4379e-05
10^{-07}	2.7284e-03	7.0333e-04	1.7718e-04	4.4379e-05
10^{-08}	2.7284e-03	7.0333e-04	1.7718e-04	4.4379e-05
10^{-09}	2.7284e-03	7.0333e-04	1.7718e-04	4.4379e-05
10^{-10}	2.7284e-03	7.0333e-04	1.7718e-04	4.4379e-05
E^N	2.7284e-03	7.0333e-04	1.7718e-04	4.4379e-05
R^N	1.9558	1.9890	1.9973	-

5. Conclusion

In this paper, singularly perturbed differential equation having small delay parameter on the convection and reaction terms of the problem is considered. The solution of the problem have boundary layer on left or right side of the domain depending on the sign of the convective term. The analytical properties of the solution is discussed. Uniformly convergent numerical scheme is developed using exponentially fitted upwind finite difference method. The developed scheme satisfies the discrete

TABLE 2. Example 4.2, maximum absolute error for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	$N \rightarrow 32$	64	128	256
10^{0}	3.2328e-06	2.0210e-07	1.2629e-08	8.2101e-10
10^{-01}	4.9452e-05	3.0937e-06	1.9337e-07	1.2100e-08
10^{-02}	3.6521e-04	2.9106e-05	1.8364e-06	1.1480e-07
10^{-03}	5.5227e-04	9.5288e-05	1.7674e-05	1.1326e-06
10^{-04}	5.5397e-04	1.4140e-04	3.4324e-05	5.3005e-06
10^{-05}	5.5398e-04	1.4140e-04	3.5532e-05	8.8936e-06
10^{-06}	5.5398e-04	1.4140e-04	3.5532e-05	8.8944e-06
10^{-07}	5.5398e-04	1.4140e-04	3.5532e-05	8.8944e-06
10^{-08}	5.5398e-04	1.4140e-04	3.5532e-05	8.8944e-06
10^{-09}	5.5398e-04	1.4140e-04	3.5532e-05	8.8944e-06
10^{-10}	5.5398e-04	1.4140e-04	3.5532e-05	8.8944e-06
E^N	5.5398e-04	1.4140e-04	3.5532e-05	8.8944e-06
R^N	1.9701	1.9926	1.9981	-

TABLE 3. Example 4.3, maximum absolute error for $\delta = 0.3\varepsilon$.

$\varepsilon \downarrow$	$N \rightarrow 32$	64	128	256
10^{0}	3.5812e-06	2.2390e-07	1.3994e-08	8.9445e-10
10^{-01}	5.1145e-05	3.2051e-06	2.0037e-07	1.2537e-08
10^{-02}	3.1057e-04	2.8811e-05	1.8087e-06	1.1308e-07
10^{-03}	5.5106e-04	9.4857e-05	1.6853e-05	1.1171e-06
10^{-04}	5.5304e-04	1.4116e-04	3.4252e-05	4.1609e-06
10^{-05}	5.5307e-04	1.4117e-04	3.5473e-05	8.8789e-06
10^{-06}	5.5307e-04	1.4117e-04	3.5473e-05	8.8797e-06
10^{-07}	5.5307e-04	1.4117e-04	3.5473e-05	8.8797e-06
10^{-08}	5.5307e-04	1.4117e-04	3.5473e-05	8.8797e-06
10^{-09}	5.5307e-04	1.4117e-04	3.5473e-05	8.8797e-06
10^{-10}	5.5307e-04	1.4117e-04	3.5473e-05	8.8797e-06
E^N	5.5398e-04	1.4140e-04	3.5532e-05	8.8797e-06
R^N	1.9701	1.9926	2.0005	-

maximum principle and uniform stability estimate. The stability and the parameter uniform convergence of the scheme theoretically investigated. The scheme converges independent of the perturbation parameter with order of convergence one. Richardson extrapolation technique is applied to accelerate the rate of convergence of the scheme to order two. Three test examples having boundary layers are considered to validate the theoretical finding. The results in the tables and figures indicate the

TABLE 4. Maximum absolute error for different valued of delay parameters for $\varepsilon = 0.1$.

$\delta \downarrow$	$N \rightarrow 32$	64	128	256
	Example 4.1			
$\delta = 0$	2.0687e-04	1.2942e-05	8.0876e-07	5.4466e-08
$\delta = 0.1\varepsilon$	2.5964e-04	1.6268e-05	1.0167e-06	6.2153e-08
$\delta = 0.2\varepsilon$	3.3324e-04	1.6268e-05	1.3035e-06	7.6211e-08
$\delta = 0.3\varepsilon$	4.3724e-04	2.7498e-05	1.7188e-06	1.1024e-07
$\delta = 0.4\varepsilon$	6.0572e-04	3.7992e-05	2.3750e-06	1.4616e-07
	Example 4.2			
$\delta = 0$	1.1432e-04	7.2280e-06	4.5181e-07	2.8239e-08
$\delta = 0.1\varepsilon$	8.2408e-05	5.1686e-06	3.2307e-07	2.0188e-08
$\delta = 0.2\varepsilon$	6.2556e-05	3.9144e-06	2.4473e-07	1.5301e-08
$\delta = 0.3\varepsilon$	4.9452e-05	3.0937e-06	1.9337e-07	1.2100e-08
$\delta = 0.4\varepsilon$	4.0353e-05	2.5242e-06	1.5777e-07	9.8743e-09
	Example 4.3			
$\delta = 0$	1.2643e-04	7.9552e-06	4.9737e-07	3.1085e-08
$\delta = 0.1\varepsilon$	8.9042e-05	5.5739e-06	3.4843e-07	2.1773e-08
$\delta = 0.2\varepsilon$	6.6088e-05	4.1379e-06	2.5866e-07	1.6171e-08
$\delta = 0.3\varepsilon$	5.1145e-05	3.2051e-06	2.0037e-07	1.2537e-08
$\delta = 0.4\varepsilon$	4.0912e-05	2.5653e-06	1.6034e-07	1.0038e-08

TABLE 5. Comparison of maximum absolute error

ε	Prop	Scheme	Result	in [9]	Result	in [11]
$\downarrow$	$N \rightarrow 64$	128	$N \rightarrow 64$	128	$N \rightarrow 64$	128
	Example	4.2				
2^{-4}	4.8333e-06	3.0224e-07	2.15e-3	5.20e-4	2.79e-03	6.94e-04
2^{-8}	7.2180e-05	4.6372e-06	5.17e-3	2.73e-3	1.26e-02	4.23e-03
2^{-12}	1.4103e-04	2.4279e-05	2.00e-3	8.60e-4	1.25e-02	4.25e-03
2^{-16}	1.4140e-04	3.5532e-05	1.61e-3	4.81e-4	1.25e-02	4.17e-03
2^{-20}	1.4140e-04	3.5532e-05	1.59e-3	4.55e-4	1.25e-02	4.18e-03
2^{-24}	1.4140e-04	3.5532e-05	1.59e-3	4.54e-4	1.25e-02	4.18e-03
	Example	4.3				
2^{-4}	4.8921e-06	3.0588e-07	2.90e-3	9.41e-4		
2^{-8}	6.8712e-05	4.5914e-06	6.55e-3	3.40e-3		
2^{-12}	1.4077e-04	2.4181e-05	4.14e-3	1.99e-3		
2^{-16}	1.4117e-04	3.5473e-05	3.84e-3	1.68e-3		
2^{-20}	1.4117e-04	3.5473e-05	3.82e-3	1.66e-3		
2^{-24}	1.4117e-04	3.5473e-05	3.82e-3	1.66e-3		

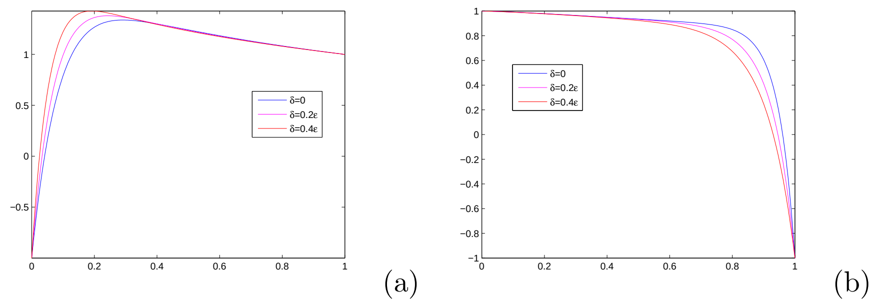

FIGURE 1. Effect of delay on the solution for $\varepsilon = 0.1$ (a) Example 4.1, (b) Example 4.2.

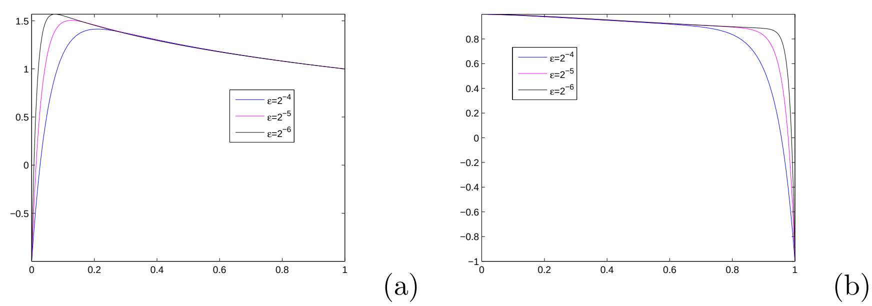

FIGURE 2. Effect of ε on the solution showing boundary layer formation (a) Example 4.1, (b) Example 4.2.

parameter uniform convergence of the scheme with rate of convergence two.

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