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A computational method for nonlinear mixed Volterra-Fredholm integral equations

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Abstract. In this article the nonlinear mixed Volterra-Fredholm integral equations are investigated by means of the modified three-dimensional block-pulse functions (**M3D-BFs**). This method converts the nonlinear mixed Volterra-Fredholm integral equations into a nonlinear system of algebraic equations. The illustrative examples are provided to demonstrate the applicability and simplicity of our scheme.

Keywords: Nonlinear mixed Volterra-Fredholm integral equations; Block-pulse functions; Operational matrix; Orthogonal functions.

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1. INTRODUCTION

Let us consider the general nonlinear mixed Volterra-Fredholm integral equations of the second kind in the following form:

$$u(x,y,z) = f(x,y,z) + \int_0^x \left(\int\int_\Omega H(x,y,z,s,t,r,u(s,t,r))drdt \right) ds; \quad (x,y,z) \in [0,1) \times \Omega \quad , \tag{1.1}$$

where $u(x,y,z)$ is an unknown function, $f(x,y,z)$ and $H(x,y,z,s,t,r,u(s,t,r))$ are analytical functions on $[0,1) \times \Omega$ and $[0,1) \times \Omega^3$, respectively, where Ω is

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a close subset on $\mathbb{R}^2$. The existence and uniqueness of the solution for the Eq. (1.1) is discussed in [9,11]. Equations of this type arise in the theory of nonlinear parabolic boundary value problems, the mathematical model of the spatiotemporal development of an epidemic and an various physical, mechanical, and biological problems [5,16]. Significant progress has been made in numerical analysis linear and nonlinear version of the Eq. (1.1). For the linear case, some methods for numerical treatment are given in [8,9,11]. For nonlinear case, the literature of integral equations contains few numerical methods [6,12] for handling the Eq. (1.1). In recent years, there has been renewed interest in Eq. (1.1), such as the time collocation and time discretization method [2,11], the particular trapezoidal Nyström method [7], the Adomian decomposition method [1,4,12] and so on.
The present paper has been organized as follows. In Section 2, we will introduce M3D-BFs and its properties. In Section 3, theorems are proved for convergence analysis. In Section 4, we will apply these sets of M3D-BFs for approximating the solution of nonlinear mixed Volterra-Fredholm integral equations. Numerical results are reported in Section 5. Finally, Section 6 concludes the paper.

2. M3D-BFs and their properties

2.1. **Definition.** An $(m+1)^3$-set of M3D-BFs consists of $(m+1)^3$ functions which are defined over district $D=[0,1)\times[0,1)\times[0,1)$ as follows :

$$\phi_{i_1,i_2,i_3}(x,y,z)=\begin{cases}1, & (x,y,z)\in D_{i_1,i_2,i_3}\\ 0, & otherwise\end{cases} \quad ; \quad i_1,i_2,i_3=0(1)m, \tag{2.1}$$

where $D_{i_1,i_2,i_3}=\{(x,y,z)|x\in I_{i_1,\varepsilon}, y\in I_{i_2,\varepsilon}, z\in I_{i_3,\varepsilon}\}$, and

$$I_{\alpha,\varepsilon}=\begin{cases}[0,h-\varepsilon), & \alpha=0\\ [\alpha h-\varepsilon,(\alpha+1)h-\varepsilon), & \alpha=1(1)m\\ [1-\varepsilon,1), & \alpha=m\end{cases}, \tag{2.2}$$

where m is an arbitrary positive integer, and $h=\frac{1}{m}$.
Since each M3D-BF takes only one value in its subregion, the M3D-BFs can be expressed by the three modified one-dimensional block-pulse functions (M1D-BFs) :

$$\phi_{i_1,i_2,i_3}(x,y,z)=\phi_{i_1}(x)\phi_{i_2}(y)\phi_{i_3}(z), \tag{2.3}$$

where $\phi_{i_1}(x)$, $\phi_{i_2}(y)$ and $\phi_{i_3}(z)$ are the M1D-BFs related to variables x, y and z, respectively [13]. The M3D-BFs are disjointed with each other:

$$\phi_{i_1,i_2,i_3}(x,y,z)\phi_{j_1,j_2,j_3}(x,y,z)=\begin{cases}\phi_{i_1,i_2,i_3}(x,y,z), & i_1=j_1\ ,\ i_2=j_2\ ,\ i_3=j_3\\ 0, & \text{otherwise}\end{cases}, \tag{2.4}$$

and are orthogonal with each other:

$$\int_0^1\int_0^1\int_0^1 \phi_{i_1,i_2,i_3}(x,y,z)\phi_{j_1,j_2,j_3}(x,y,z)dzdydx = \begin{cases} \triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon})\triangle(I_{i_3,\varepsilon}), & \begin{matrix} i_1=j_1 \\ i_2=j_2 \\ i_3=j_3 \end{matrix} \\ 0, & \text{otherwise} \end{cases}, \quad (2.5)$$

where $(x,y,z)\in D$, $i_1,i_2,i_3,j_1,j_2,j_3=0(1)m$ and $\triangle(I_{i_1,\varepsilon})$, $\triangle(I_{i_2,\varepsilon})$ and $\triangle(I_{i_3,\varepsilon})$ are the length of intervals $I_{i_1,\varepsilon}$, $I_{i_2,\varepsilon}$ and $I_{i_3,\varepsilon}$, respectively.

2.2. **Vector forms.** Now, consider the first $(m+1)^3$ terms of M3D-BFs and write them concisely as $(m+1)^3$-vector:

$$\Phi_{m,\varepsilon}(x,y,z)=[\phi_{0,0,0}(x,y,z),\dots,\phi_{0,0,m}(x,y,z),\dots,\phi_{0,m,m}(x,y,z),\dots,\phi_{m,m,m}(x,y,z)]^T; \quad (2.6)$$

where $(x,y,z)\in D$.
From the above representation and Eq. (2.4) we have:

$$\Phi_{m,\varepsilon}(x,y,z)\Phi^T_{m,\varepsilon}(x,y,z)$$

$$=\begin{pmatrix} \phi_{0,0,0}(x,y,z) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \phi_{0,0,m}(x,y,z) & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \phi_{m,m,m}(x,y,z) \end{pmatrix}_{(m+1)^3\times(m+1)^3}. \quad (2.7)$$

Let X be a $(m+1)^3$-vector by using Eq. (2.7) we will have:

$$\Phi_{m,\varepsilon}(x,y,z)\Phi^T_{m,\varepsilon}(x,y,z)X=\widetilde{X}\Phi_{m,\varepsilon}(x,y,z), \quad (2.8)$$

where $\widetilde{X}=diag(X)$ is a $(m+1)^3\times(m+1)^3$ diagonal matrix.

2.3. **M3D-BFs expansions.** A function $f(x,y,z)$ defined over district $L^2(D)$ may be expanded by the M3D-BFs as:
$f(x,y,z)\simeq f_{m,\varepsilon}(x,y,z)$

$$=\sum_{i_1=0}^{m}\sum_{i_2=0}^{m}\sum_{i_3=0}^{m} f_{i_1,i_2,i_3}\phi_{i_1,i_2,i_3}(x,y,z)=F^T_{m,\varepsilon}\Phi_{m,\varepsilon}(x,y,z)=\Phi^T_{m,\varepsilon}(x,y,z)F_{m,\varepsilon}, \quad (2.9)$$

where $F_{m,\varepsilon}$ is an $(m+1)^3\times 1$ vector given by:

$$F_{m,\varepsilon}=[f_{0,0,0},\dots,f_{0,0,m},\dots,f_{0,m,m},\dots,f_{m,m,m}]^T, \quad (2.10)$$

and $\Phi_{m,\varepsilon}(x,y,z)$ is defined in Eq. (2.6), and f_{i_1,i_2,i_3}, are obtained as:

$$f_{i_1,i_2,i_3}=\frac{1}{\triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon})\triangle(I_{i_3,\varepsilon})}\int_{I_{i_1,\varepsilon}}\int_{I_{i_2,\varepsilon}}\int_{I_{i_3,\varepsilon}} f(x,y,z)dzdydx. \quad (2.11)$$

Similarly a function of six variables, $k(x,y,z,s,t,r)$, on district $L^2(D\times D)$ may be approximated with respect to M3D-BFs such as:

$$k(x,y,z,s,t,r)\simeq\Phi^T_{m,\varepsilon}(x,y,z)K_{m,\varepsilon}\Phi_{m,\varepsilon}(s,t,r), \quad (2.12)$$

where $\Phi_{m,\varepsilon}(x,y,z)$ and $\Phi_{m,\varepsilon}(s,t,r)$ are M3D-BFs vector of dimension $(m+1)^3$, and $K_{m,\varepsilon}$ is the $(m+1)^3 \times (m+1)^3$ M3D-BFs coefficients matrix.

3. **Convergence analysis**

In this sections, we show that the given method in the previous sections is convergent and its order of convergence is $O(\frac{1}{km})$. For our purposes, we will need the following theorems.

Theorem 2.1. Let

$$f_{m,\varepsilon}(x,y,z) = \sum_{i_1=0}^{m}\sum_{i_2=0}^{m}\sum_{i_3=0}^{m} f_{i_1,i_2,i_3}\phi_{i_1,i_2,i_3}(x,y,z) \ ,$$

and

$f_{i_1,i_2,i_3} = \frac{1}{\triangle(I_{i_1,\varepsilon})\triangle(I_{i_2,\varepsilon})\triangle(I_{i_3,\varepsilon})}\int_0^1\int_0^1\int_0^1 f(x,y,z)\phi_{i_1,i_2,i_3}(x,y,z)dzdydx \quad ;$
$i_1,i_2,i_3 = 0(1)(m)$.

Then

$$\int_0^1\int_0^1\int_0^1 (f(x,y,z) - f_{m,\varepsilon}(x,y,z))^2 dxdydz \ , \tag{3.1}$$

achieves its minimum value. Moreover, we have:

$$\int_0^1\int_0^1\int_0^1 f^2(x,y,z)dxdydz = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty} f^2_{i_1,i_2,i_3}||\phi_{i_1,i_2,i_3}(x,y,z)||^2 \ . \tag{3.2}$$

Proof. Proof is like similar theorem in [10].

Theorem 2.2. Assume $f(x,y,z)$ is continuous and is differentiable over district $[-h,1+h]\times[-h,1+h]\times[-h,1+h]$, and $f_{m,\varepsilon_i}(x,y,z)$; $\varepsilon_i = \frac{ih}{k}$, for $i = 0(1)(k-1)$, are correspondingly M3D-BFs(ε_0)=3D-BFs, M3D-BFs(ε_1), ..., M3D-BFs(ε_{k-1}) expansions of $f(x,y,z)$ based on $(m+1)^3$ M3D-BFs over district D and

$$\bar{f}_{m,k}(x,y,z) = \frac{1}{k}\sum_{i=0}^{k-1} f_{m,\varepsilon_i}(x,y,z),$$

then for sufficient large m we have:

$$||f(x,y,z) - \bar{f}_{m,k}(x,y,z)||_\infty \lesssim \frac{1}{k}\max_{\varepsilon_i}||f(x,y,z) - f_{m,\varepsilon_i}(x,y,z)||_\infty.$$

Proof. Proof is like similar theorem in [15]

Theorem 2.3. Let the representation error between $f(x,y,z)$ and its three-dimensional block-pulse functions, $f_m(x,y,z) = f_{m,\varepsilon_0}(x,y,z)$ (M3D-BFs(ε_0)=3D-BFs), over the district D, as follows :

$$e(x,y,z) = f(x,y,z) - f_m(x,y,z) \ .$$

Then $||e(x,y,z)|| = O(\frac{1}{m})$ and

$$\lim_{m\to+\infty} f_m(x,y,z) = \lim_{m\to+\infty} f_{m,\varepsilon_0}(x,y,z) = f(x,y,z) \ .$$

Proof. Proof is to similar theorem in [14].

Theorem 2.2 and 2.3 conclude that error estimation for M3D-BFs is

$||e(x,y,z)|| = O(\frac{1}{km})$.
Also, similarly [15], we can show that:

$$\lim_{m\to+\infty} f_{m,\varepsilon_i}(x,y,z) = f(x,y,z). \tag{3.3}$$

4. **Method of solution**

In this section, we solve nonlinear mixed Volterra-Fredholm integral equations of the second kind of the form Eq. (1.1) with

$$H(x,y,z,s,t,r,u(s,t,r)) = k(x,y,z,s,t,r)[u(s,t,r)]^p; \qquad p \text{ is positive integer }, \tag{4.1}$$

by using M3D-BFs.
We now approximate functions $u(x,y,z), f(x,y,z), [u(x,y,z)]^p$ and $k(x,y,z,s,t,r)$ with respect to M3D-BFs by manipulation as Section 2:

$$\begin{aligned} u(x,y,z) &\simeq \Phi^T_{m,\varepsilon}(x,y,z)U_{m,\varepsilon}, \\ f(x,y,z) &\simeq \Phi^T_{m,\varepsilon}(x,y,z)F_{m,\varepsilon}, \\ [u(x,y,z)]^p &\simeq \Phi^T_{m,\varepsilon}(x,y,z)U_{m,\varepsilon,p}, \\ k(x,y,z,s,t,r) &\simeq \Phi^T_{m,\varepsilon}(x,y,z)K_{m,\varepsilon}\Phi_{m,\varepsilon}(s,t,r), \end{aligned} \tag{4.2}$$

where $\Phi_{m,\varepsilon}(x,y,z)$ is defined in Eq. (2.6), the vectors $U_{m,\varepsilon}, F_{m,\varepsilon}, U_{m,\varepsilon,p}$, and matrix $K_{m,\varepsilon}$ are M3D-BFs coefficients of $u(x,y,z), f(x,y,z), [u(x,y,z)]^p$ and $k(x,y,z,s,t,r)$ respectively.

Lemma 2.1. Let $(m+1)^3$-vectors $U_{m,\varepsilon}$ and $U_{m,\varepsilon,p}$ be M3D-BFs coefficients of $u(x,y,z)$ and $[u(x,y,z)]^p$, respectively. If

$$U_{m,\varepsilon} = [u_{0,0,0},\ldots,u_{0,0,m},\ldots,u_{0,m,m},\ldots,u_{m,m,m}]^T, \tag{4.3}$$

then we have:

$$U_{m,\varepsilon,p} = [u^p_{0,0,0},\ldots,u^p_{0,0,m},\ldots,u^p_{0,m,m},\ldots,u^p_{m,m,m}]^T, \tag{4.4}$$

where $p \geq 1$, is a positive integer.
Proof. The proof is like a similar lemma in [15]

To approximate the integral part in Eq. (1.1) with Eq. (4.1), from Eq. (4.2) we get
$\int_0^x \int_0^1 \int_0^1 k(x,y,z,s,t,r)[u(s,t,r)]^p drdtds \simeq$

$\int_0^x \int_0^1 \int_0^1 \Phi^T_{m,\varepsilon}(x,y,z)K_{m,\varepsilon}\Phi_{m,\varepsilon}(s,t,r)\Phi^T_{m,\varepsilon}(s,t,r)U_{m,\varepsilon,p}drdtds =$

$$\Phi^T_{m,\varepsilon}(x,y,z)K_{m,\varepsilon}\left(\int_0^x \int_0^1 \int_0^1 \Phi_{m,\varepsilon}(s,t,r)\Phi^T_{m,\varepsilon}(s,t,r)drdtds\right)U_{m,\varepsilon,p}. \tag{4.5}$$

Now by using Eq.s (2.3) and (2.7), denoting R_j for the $(j+1)$th row of the conventional integration operational matrix $P_{m,\varepsilon}$ $\Big((P_{m,\varepsilon})_{(m+1)\times(m+1)}$ is operational matrix of 1D-BFs defined over [0,1), see Maleknejad et al., 2011$\Big)$ and considering $\int_0^1 \phi_i(\tau)d\tau = \triangle(I_{i,\varepsilon})$ follows:

$$\int_0^x\int_0^1\int_0^1 \Phi_{m,\varepsilon}(s,t,r)\Phi^T_{m,\varepsilon}(s,t,r)drdtds = \begin{pmatrix} D_0 & 0 & \dots & 0 \\ 0 & D_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D_m \end{pmatrix}_{(m+1)^3\times(m+1)^3},$$

where $D_i =$

$$\begin{pmatrix} \int_0^x\int_0^1\int_0^1 \phi_i(s)\phi_0(t)\phi_0(r)drdtds & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \int_0^x\int_0^1\int_0^1 \phi_i(s)\phi_0(t)\phi_m(r)drdtds & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \int_0^x\int_0^1\int_0^1 \phi_i(s)\phi_m(t)\phi_m(r)drdtds \end{pmatrix}$$

$$= \begin{pmatrix} (h-\varepsilon)(h-\varepsilon)R_i\Phi_{m,\varepsilon}(x) & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & (h-\varepsilon)hR_i\Phi_{m,\varepsilon}(x) & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (h-\varepsilon)\varepsilon R_i\Phi_{m,\varepsilon}(x) & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & \varepsilon(h-\varepsilon)R_i\Phi_{m,\varepsilon}(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & \varepsilon\varepsilon R_i\Phi_{m,\varepsilon}(x) \end{pmatrix}, \quad (4.6)$$

where

$$\Phi_{m,\varepsilon}(x) = [\phi_0(x), \phi_1(x), \dots, \phi_m(x)]. \quad (4.7)$$

Also by using Eq. (2.3), Eq. (2.6) can be reformulated as:

$$\Phi_{m,\varepsilon}(x,y,z) = \begin{pmatrix} \phi_0(x) & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \phi_0(x) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \phi_m(x) & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & \phi_m(x) \end{pmatrix}_{(m+1)^3\times(m+1)^3} \times$$

$$[\phi_{0,0}(y,z), \dots, \phi_{0,m}(y,z), \dots, \phi_{m,m}(y,z), \dots, \phi_{0,0}(y,z), \dots, \phi_{0,m}(y,z), \dots, \phi_{m,m}(y,z)]^T_{(m+1)^3\times 1}. \quad (4.8)$$

So, we have

$$\Phi^T_{m,\varepsilon}(x,y,z)K_{m,\varepsilon} = [\phi_{0,0}(y,z), \dots, \phi_{0,m}(y,z), \dots, \phi_{m,m}(y,z), \dots, \phi_{0,0}(y,z), \dots, \phi_{0,m}(y,z), \dots, \phi_{m,m}(y,z)]\times$$

$$\begin{pmatrix} k_{1,1}\phi_0(x) & \dots & k_{1,(m+1)^2}\phi_0(x) & \dots & k_{1,m(m+1)^2}\phi_0(x) & \dots & k_{1,(m+1)^3}\phi_0(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{(m+1)^2,1}\phi_0(x) & \dots & k_{(m+1)^2,(m+1)^2}\phi_0(x) & \dots & k_{(m+1)^2,m(m+1)^2}\phi_0(x) & \dots & k_{(m+1)^2,(m+1)^3}\phi_0(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{m(m+1)^2,1}\phi_m(x) & \dots & k_{m(m+1)^2,(m+1)^2}\phi_m(x) & \dots & k_{m(m+1)^2,m(m+1)^2}\phi_m(x) & \dots & k_{m(m+1)^2,(m+1)^3}\phi_m(x) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ k_{(m+1)^3,1}\phi_m(x) & \dots & k_{(m+1)^3,(m+1)^2}\phi_m(x) & \dots & k_{(m+1)^3,m(m+1)^2}\phi_m(x) & \dots & k_{(m+1)^3,(m+1)^3}\phi_m(x) \end{pmatrix}. \quad (4.9)$$

Also, we have:

$$R_i\Phi(x) = \begin{cases} \frac{(h-\varepsilon)}{2}\phi_0(x) + (h-\varepsilon)\phi_1(x) + \dots + (h-\varepsilon)\phi_m(x), & i = 0 \\ \\ \frac{h}{2}\phi_i(x) + h\phi_{i+1}(x) + \dots + h\phi_m(x), & i = 1(1)(m-1) \\ \\ \frac{\varepsilon}{2}\phi_m(x), & i = m, \end{cases} \quad (4.10)$$

and

$$\phi_i(x)\phi_j(x) = \begin{cases} \phi_i(x), & i = j \\ 0, & \text{otherwise} \end{cases}.$$

By using Eq.s (4.6), (4.9) and (4.10), Eq. (4.5) can be reformulated as:

$$[\phi_{0,0}(y,z), \dots, \phi_{0,m}(y,z), \dots, \phi_{m,m}(y,z), \dots, \phi_{0,0}(y,z), \dots, \phi_{0,m}(y,z), \dots, \phi_{m,m}(y,z)]_{(m+1)^3\times 1} \times$$

$$\begin{pmatrix} A_{00} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ A_{10} & A_{11} & \mathbf{0} & \dots & \mathbf{0} \\ A_{20} & A_{21} & A_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m0} & A_{m1} & A_{2m} & \dots & A_{mm} \end{pmatrix}_{(m+1)^3\times(m+1)^3} .U_{m,\varepsilon,p}, \quad (4.11)$$

where

$$A_{i,j} = \begin{cases} \frac{\triangle(I_{j,\varepsilon})}{2}\triangle(I_{\left[\frac{l}{(m+1)}\right],\varepsilon})\triangle(I_{g,\varepsilon})(k_{lq})\phi_i(x), & i = j \\ \\ \triangle(I_{j,\varepsilon})\triangle(I_{\left[\frac{l}{(m+1)}\right],\varepsilon})\triangle(I_{g,\varepsilon})(k_{lq})\phi_i(x), & \text{otherwise} \end{cases}, \quad (4.12)$$

where

$$l = ((m+1)^2 i + 1), ((m+1)^2 + 1), \dots, ((m+1)^2(i+1)),$$

$$q = ((m+1)^2 j + 1), ((m+1)^2 j + 2), \dots, ((m+1)^2(j+1)),$$

$$g = q - (m+1)\left[\frac{q}{(m+1)}\right],$$

and $\mathbf{0}$ is a zero matrix. Also

$$\begin{pmatrix} A_{00} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ A_{10} & A_{11} & \mathbf{0} & \dots & \mathbf{0} \\ A_{20} & A_{21} & A_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m0} & A_{m1} & A_{2m} & \dots & A_{mm} \end{pmatrix}_{(m+1)^3\times(m+1)^3} = \begin{pmatrix} \phi_0(x) & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \phi_0(x) & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & \phi_m(x) & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & \phi_m(x) \end{pmatrix} Q, \quad (4.13)$$

where

$$Q = \begin{pmatrix} Q_{00} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ Q_{10} & Q_{11} & \mathbf{0} & \dots & \mathbf{0} \\ Q_{20} & Q_{21} & Q_{22} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{m0} & Q_{m1} & Q_{2m} & \dots & Q_{mm} \end{pmatrix}_{(m+1)^3\times(m+1)^3}, \quad (4.14)$$

$$Q_{i,j} = \begin{cases} \frac{\triangle(I_{j,\varepsilon})}{2}\triangle(I_{\left[\frac{l}{(m+1)}\right],\varepsilon})\triangle(I_{g,\varepsilon})(k_{lq}), & i=j \\ \\ \triangle(I_{j,\varepsilon})\triangle(I_{\left[\frac{l}{(m+1)}\right],\varepsilon})\triangle(I_{g,\varepsilon})(k_{lq}), & \text{otherwise} \end{cases}. \quad (4.15)$$

So, we have :

$$\int_0^x\int_0^1\int_0^1 k(x,y,z,s,t,r)[u(s,t,r)]^p drdtds \simeq \Phi^T_{m,\varepsilon}(x,y,z)QU_{m,\varepsilon,p}. \quad (4.16)$$

Substituting Eq.s (4.2) and (4.16) into Eq. (1.1) with Eq. (4.1) gives:

$$\Phi^T_{m,\varepsilon}(x,y,z)U_{m,\varepsilon} = \Phi^T_{m,\varepsilon}(x,y,z)F_{m,\varepsilon} + \Phi^T_{m,\varepsilon}(x,y,z)QU_{m,\varepsilon,p} \Rightarrow U_{m,\varepsilon} - QU_{m,\varepsilon,p} = F_{m,\varepsilon}. \quad (4.17)$$

After solving the above nonlinear system by using Newton-Raphson method, we can find $U_{m,\varepsilon}$ and then

$$u_{m,\varepsilon}(x,y,z) \simeq U^T_{m,\varepsilon}\Phi_{m,\varepsilon}(x,y,z) \ . \quad (4.18)$$

Then

$$u(x,y,z) \simeq \bar{u}_{m,k}(x,y,z) = \frac{1}{k}\sum_{i=0}^{k-1} u_{m,\varepsilon_i}(x,y,z) \ , \quad (4.19)$$

where $\varepsilon_i = \frac{ih}{k},\ i = 0(1)(k-1)$ is the estimation of the solution of nonlinear mixed Volterra-Fredholm integral equation of the second kind.

5. **Numerical examples**

In this section, two examples are given to certify the convergence and error bound of the presented method. All results are computed by using a program written in the Matlab. The numerical experiments are carried out for the selected grid point which are proposed as $(2^{-l}\ ;\ l = 1,2,3,4,5,6)$ and m terms and k times of modifications of the M3D-BFs series. The following problems have been tested.

Example 5.1. Consider the following linear mixed Volterra-Fredholm integral equation [3]:

$$u(x,y,z)=f(x,y,z)+\int_0^x\int_0^1\int_0^1 \log(y(t+1)+1)\log(z(r+1)+1)\cos(x-s)u(s,t,r)drdtds; \quad (5.1)$$

where $(x,y,z)\in D$ and

$$\begin{aligned} f(x,y,z) = yz\cos(x) - \tfrac{1}{32}\Big(&4yz - 8zy\log(2z+1) + 4zy^2\log(z+1) - 4z^2y\log(2y+1) - 8y\log(z+1) \\ &\log(2y+1) - 8y\log(2z+1)\log(y+1) - 4z^2\log(z+1)\log(2y+1) + 8z\log(2z+1)\log(2y+1) - \\ &8z\log(z+1)\log(2y+1) + 8y\log(y+1)\log(z+1) + 8zy\log(y+1) + 4zy^2\log(y+1) + 8zy\log(z+1) - \\ &8zy\log(2y+1) + 8y\log(2z+1)\log(2y+1) + 4z^2y\log(z+1) - 4zy^2\log(2z+1) + 2z^2y^2\log(y+1) + \\ &2z^2y^2\log(z+1) + 4z^2\log(z+1)\log(y+1) - 8z\log(2z+1)\log(y+1) + 4y^2\log(z+1)\log(y+1) + \\ &4z^2y\log(y+1) - 4y^2\log(2z+1)\log(y+1) + 8z\log(z+1)\log(y+1) + 2zy^2 + z^2y^2 + 2z^2y + 4\log(2z+1) \\ &\log(2y+1) - 4y\log(2z+1) + 4y\log(z+1) - 4z\log(2y+1) - 2z^2\log(2y+1) - 2y^2\log(2z+1) + 2z^2 \\ &\log(y+1) + 4z\log(y+1) - 8z^2y\log(z+1)\log(2y+1) - 2\log(2z+1)\log(y+1) + 2\log(z+1)\log(y+1) - \\ &4\log(z+1)\log(2y+1) - 8zy^2\log(2z+1)\log(y+1) - 16zy\log(2z+1)\log(y+1) + 4z^2y^2\log(z+1) \\ &\log(y+1) + 8zy^2\log(z+1)\log(y+1) + 16zy\log(z+1)\log(y+1) + 2y^2\log(z+1) + 8z^2y\log(z+1) \\ &\log(y+1) - 16yz\log(z+1)\log(2y+1) + 16zy\log(2z+1)\log(2y+1)\Big)\frac{\sin(x)+x\cos(x)}{y^2z^2}. \end{aligned} \quad (5.2)$$

The exact solution is $u(x,y,z)=yz\cos(x)$. Table 1 illustrates the numerical results for this example.

Table 1: Numerical results of Example 1 with M3D-BFs

Nodes (x,y,z)	Error for m=2		Error for m=3	
(x,y,z)=2^{-l}	k=1	k=2	k=1	k=2
$l=1$	0.0383523	0.0261456	0.0173785	0.0083283
$l=2$	0.0281769	0.0197749	0.0187439	0.0076244
$l=3$	0.0382961	0.0148927	0.0135762	0.0094286
$l=4$	0.0412896	0.0187870	0.0096827	0.0037828
$l=5$	0.0431665	0.0197624	0.0087069	0.0028917
$l=6$	0.0434095	0.0200065	0.0084628	0.0021735

Example 5.2. Consider the following nonlinear mixed Volterra-Fredholm integral equation:

$$u(x,y,z)=f(x,y,z)+2\int_0^x\int_0^1\int_0^1 (x+s)(y^2+r)ztu^2(s,t)drdtds\ ; \quad (5.3)$$

where $(x, y, z) \in D$ and

$$f(x, y, z) = x^2yz - \frac{11}{240}x^6z - \frac{11}{180}x^6y^2z\ . \tag{5.4}$$

The exact solution is $u(x, y, z) = x^2yz$. Table 2 illustrates the numerical results for this example.

Table 2: Numerical results of Example 2 with M3D-BFs

Nodes (x,y,z)	Error for m=2		Error for m=3	
(x,y,z)=2^{-l}	k=1	k=2	k=1	k=2
$l = 1$	0.0021167	0.0017309	0.0005209	0.0002829
$l = 2$	0.0047691	0.0023303	0.0005947	0.0001125
$l = 3$	0.0051760	0.0027372	0.0010060	0.0005194
$l = 4$	0.0052014	0.0027626	0.0010270	0.0005448
$l = 5$	0.0052030	0.0027642	0.0010286	0.0005464
$l = 6$	0.0052031	0.0027643	0.0010287	0.0005465

6. CONCLUSIONS

In this paper, we have worked out a computational method to approximate solution of nonlinear mixed Volterra-Fredholm integral equations of the second kind, based on the expansion of the solution as series of M3D-BFs. This method converts a nonlinear mixed Volterra-Fredholm integral equation whose answers are the coefficients of M3D-BFs expansion of the solution of nonlinear mixed Volterra-Fredholm integral equation. Moreover, the numerical results show that typical convergence rate is $O(\frac{1}{km})$ which make this method rather competitive, compared with methods presented in other works. It is to be noted that this method can easily be extended and applied to the nonlinear system of mixed Volterra-Fredholm integral equations.

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