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Existence of Periodic Solutions in Totally Nonlinear Neutral Differance Equations with Variable Delay

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Abstract. We study the existence of periodic solutions of the totally nonlinear neutral difference equation with variable delay

$$\begin{aligned}\triangle x(t) &= -a(t)\, x^3(t+1) + c(t)\, \triangle x(t-g(t)) \\ &\quad + G\left(t, x^3(t), x^3(t-g(t))\right), \forall t \in \mathbb{Z}.\end{aligned}$$

We invert the given equation to construct a fixed point mapping expressed as a sum of a large contraction and a compact map. We show that such a sum of mappings fits very nicely into the framework of Krasnoselskii-Burton's fixed point theorem so that the existence of periodic solutions is readily concluded. The obtained results extend the work of Ardjouni and Djoudi [1].

Keywords: Fixed point, Large contraction, Periodic solutions, Totally nonlinear neutral differential equations.

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1. INTRODUCTION

The purpose of this paper is to present an analysis of qualitative theory of periodic solutions for a nonlinear difference equation. Motivated

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by the papers [1],[4]-[7],[9]-[11],[13] and the references therein, we focus on the following totally nonlinear neutral difference equation with variable delay, $\forall t \in \mathbb{Z}$

$$\triangle x(t) = -a(t)\, x^3(t+1) + c(t)\, \triangle x(t - g(t)) + G\left(t, x^3(t), x^3(t - g(t))\right) \tag{1.1}$$

where

$$G : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x(t) = x(t+1) - x(t)$ for any sequence $\{x(t),\ t \in \mathbb{Z}\}$.

Special cases of the above equation have been recently considered. Particularly, when $c(t) = 0$, the authors in [1] have obtained an existence of periodic solutions result of (1.1) by means of Krasnoselskii-Burton's fixed point theorem. Our objective here is to generalize and extend the analysis made in [1] to the equation (1.1).

Clearly, the equation (1.1) is totally nonlinear and we have to transform it into a tractable form by adding a linear term to both sides of the equation. Although the added term destroys a contraction already present, but it will replace it with the so called large contraction which is suitable for the fixed point theory. The inversion of the transformed equation gives rise to an equivalent difference equation that does not change the basic properties of the first one. We express the resulting difference equation as a sum of large contraction an a compact map and then use a modification of Krasnoselskii's fixed point theorem due to Burton (see [2] Theorem 3) to show that (1.1) possesses periodic solutions.

For details on Krasnoselskii's theorem we refer the reader to [12]. Also, the reader can find in Kelly and Peterson book [8] most of the materials and basic properties on the calculus of difference equations used in this paper.

In Section 2, we present the inversion of difference equations (1.1) and the modification of Krasnoselskii's fixed point theorem. We devote Section 3 to the new results on existence and periodicity of (1.1).

2. INVERSION OF THE EQUATION

Let T be an integer such that $T \geq 1$. Define $C_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(t+T) = \varphi(t)\}$ where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(C_T, \|.\|)$ is a Banach space with the maximum norm

$$\|\varphi\| = \max_{t \in [0, T-1]} |\varphi(t)|.$$

In our consideration we assume the periodicity conditions

$$a(t+T) = a(t),\ c(t+T) = c(t),\ g(t+T) = g(t),\ g(t) \geq g^* > 0, \tag{2.1}$$

for some constant g^*. Also, we suppose that

$$a(t) > 0. \tag{2.2}$$

We also require that $G(t,x,y)$ is periodic in t and Lipschitz continuous in x and y. That is

$$G(t+T,x,y) = G(t,x,y), \tag{2.3}$$

and there are positive constants k_1, k_2 such that

$$|G(t,x,y) - G(t,z,w)| \leq k_1 \|x-z\| + k_2 \|y-w\|,\ \text{for } x,y,z,w \in \mathbb{R}. \tag{2.4}$$

Having these assumptions in mind, we can now integrate equation (1.1). To do that, we use the variation of parameter formula to obtain an equivalent equation suitable for the Krasnoselskii-Burton's theorem and from which we define our fixed point mapping. Besides, the summation by parts will be applied.

Lemma 2.1. *Suppose (2.1) and (2.3) hold. If $x \in C_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) = {} & \frac{c(t-1)}{1+a(t-1)} x(t-g(t)) + \left(1 - \prod_{s=t-T}^{t-1} (1+a(s))^{-1}\right)^{-1} \\ & \times \left[\sum_{r=t-T}^{t-1} a(r)\left(x(r+1) - x^3(r+1)\right) \prod_{s=r}^{t-1} (1+a(s))^{-1} \right. \\ & \left. + \sum_{r=t-T}^{t-1} \left\{x(r-g(r))h(r) + G\left(r, x^3(r), x^3(r-g(r))\right)\right\} \prod_{s=r}^{t-1} (1+a(s))^{-1} \right], \end{aligned} \tag{2.5}$$

where

$$h(r) = \frac{c(r-1)}{1+a(r-1)} - c(r). \tag{2.6}$$

Proof. Let $x \in C_T$ be a solution of (1.1). First we write this equation as

$$\begin{aligned} \triangle x(t) + a(t)x(t+1) = {} & a(t)x(t+1) - a(t)x^3(t+1) \\ & + c(t)\triangle x(t-g(t)) + G\left(t, x^3(t), x^3(t-g(t))\right). \end{aligned}$$

We consider two cases; $t \geq 1$ and $t \leq 0$. For $t \geq 1$, by multiplying both sides of the above equation by $\prod_{s=0}^{t-1}(1+a(s))$ and by summing from

$(t-T)$ to $(t-1)$ we obtain

$$\sum_{r=t-T}^{t-1} \triangle \left[\prod_{s=0}^{r-1} (1+a(s))\, x(r)\right]$$
$$= \sum_{r=t-T}^{t-1} a(r) \left\{x(r+1) - x^3(r+1)\right\} \prod_{s=0}^{r-1} (1+a(s))$$
$$+ \sum_{r=t-T}^{t-1} \left\{c(r) \triangle x(r-g(r)) + G\left(r, x^3(r), x^3(r-g(r))\right)\right\} \prod_{s=0}^{r-1} (1+a(s)).$$

As a consequence, we arrive at

$$\prod_{s=0}^{t-1} (1+a(s))\, x(t) - \prod_{s=0}^{t-T-1} (1+a(s))\, x(t-T)$$
$$= \sum_{r=t-T}^{t-1} a(r) \left\{x(r+1) - x^3(r+1)\right\} \prod_{s=0}^{r-1} (1+a(s))$$
$$+ \sum_{r=t-T}^{t-1} \left\{c(r) \triangle x(r-g(r)) + G\left(r, x^3(r), x^3(r-g(r))\right)\right\} \prod_{s=0}^{r-1} (1+a(s)).$$

By dividing both sides of the above expression by $\prod_{s=0}^{t-1} (1+a(s))$ and the fact that $x(t) = x(t-T)$, we get

$$x(t) = \left(1 - \prod_{s=t-T}^{t-1} (1+a(s))^{-1}\right)^{-1} \tag{2.7}$$
$$\times \left[\sum_{r=t-T}^{t-1} a(r) \left(x(r+1) - x^3(r+1)\right) \prod_{s=r}^{t-1} (1+a(s))^{-1}\right.$$
$$\left. + \sum_{r=t-T}^{t-1} \left\{c(r) \triangle x(r-g(r)) + G\left(r, x^3(r), x^3(r-g(r))\right)\right\} \prod_{s=r}^{t-1} (1+a(s))^{-1}\right].$$

Rewrite

$$\sum_{r=t-T}^{t-1} c(r) \triangle x(r-g(r)) \prod_{s=r}^{t-1} (1+a(s))^{-1}$$
$$= \sum_{r=t-T}^{t-1} c(r) \prod_{s=r}^{t-1} (1+a(s))^{-1} \triangle x(r-g(r)).$$

By considering $z = x(r - g(r))$ and $Ey = c(r)\prod_{s=r}^{t-1}(1 + a(s))^{-1}$ we get $y = c(r-1) \times \prod_{s=r-1}^{t-1}(1 + a(s))^{-1}$. Thus, by performing a summation by parts on the above equation using the summation by parts formula

$$\sum Ey\triangle z = yz - \sum z\triangle y,$$

we obtain

$$\begin{aligned}
&\sum_{r=t-T}^{t-1} c(r)\triangle x(r - g(r))\prod_{s=r}^{t-1}(1 + a(s))^{-1} \\
&= \left[c(r-1)\prod_{s=r-1}^{t-1}(1 + a(s))^{-1}x(r - g(r))\right]_{t-T}^{t} \\
&- \sum_{r=t-T}^{t-1} x(r - g(r))\triangle\left(c(r-1)\prod_{s=r-1}^{t-1}(1 + a(s))^{-1}\right) \\
&= c(t-1)\prod_{s=t-1}^{t-1}(1 + a(s))^{-1}x(t - g(t)) \\
&- c(t-T-1)\prod_{s=t-T-1}^{t-1}(1 + a(s))^{-1}x(t - T - g(t-T)) \\
&- \sum_{r=t-T}^{t-1} x(r - g(r))\triangle\left(c(r-1)\prod_{s=r-1}^{t-1}(1 + a(s))^{-1}\right) \\
&= \frac{c(t-1)}{1 + a(t-1)}x(t - g(t)) - c(t-1)\prod_{s=t-T-1}^{t-1}(1 + a(s))^{-1}x(t - g(t)) \\
&- \sum_{r=t-T}^{t-1} x(r - g(r))\left\{c(r)\prod_{s=r}^{t-1}(1 + a(s))^{-1} - c(r-1)\prod_{s=r-1}^{t-1}(1 + a(s))^{-1}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{r=t-T}^{t-1} c(r)\triangle x(r - g(r))\prod_{s=r}^{t-1}(1 + a(s))^{-1} \qquad (2.8)\\
&= \frac{c(t-1)}{1 + a(t-1)}x(t - g(t))\left(1 - \prod_{s=t-T}^{t-1}(1 + a(s))^{-1}\right) \\
&+ \sum_{r=t-T}^{t-1} x(r - g(r))\,h(r)\prod_{s=r}^{t-1}(1 + a(s))^{-1},
\end{aligned}$$

where h is given by (2.6). Finally, substituting (2.8) into (2.7) completes the proof.

Now, for $t \leq 0$, equation (1.1) is equivalent to

$$\begin{aligned}
&\triangle\left[\prod_{s=t-1}^{0}(1+a(s))\,x(t)\right]\\
&= a(t)\left\{x(t+1)-x^{3}(t+1)\right\}\prod_{s=t-1}^{0}(1+a(s))\\
&+\left\{c(t)\triangle x(t-g(t))+G\left(t,x^{3}(t),x^{3}(t-g(t))\right)\right\}\prod_{s=t-1}^{0}(1+a(s)).
\end{aligned}$$

Summing the above expression from $(t-T)$ to $(t-1)$ we obtain (2.5) by a similar argument. □

As mentioned above, in our analysis we use a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. First, we give the following definition which can be found in [2] or [3].

Definition 2.2. (**Large Contraction**) Let (M,d) be a metric space and $B:M\to M$. B is said to be a large contraction if $\phi,\varphi\in M$, with $\phi\neq\varphi$ then $d(B\phi,B\varphi)\leq d(\phi,\varphi)$ and if for all $\epsilon>0$, there exists a $\delta<1$ such that

$$[\phi,\varphi\in M, d(\phi,\varphi)\geq\epsilon]\Rightarrow d(B\phi,B\varphi)\leq\delta d(\phi,\varphi).$$

Burton studied the work of Krasnoselskii on fixed point. He pointed out that Krasnoselskii theorem can be more useful with certain changes. Accordingly, he formulated the following theorem (see [2]-[3]).

Theorem 2.3. ***(Krasnoselskii-Burton)*** *Let M be a closed bounded convex nonempty subset of a Banach space $(\mathbb{B},\|.\|)$. Suppose that A and B map M into $\mathbb{B}$ such that*

i. $x,y\in M$, implies $Ax+By\in M$;

ii. A is continuous and AM is contained in a compact subset of M;

iii. B is a large contraction mapping.

Then there exists $z\in M$ with $z=Az+Bz$.

We will use this theorem to prove the existence of periodic solutions for equation (1.1). But, let us first begin with the following proposition.

Proposition 2.4. *If $\|.\|$ is the maximum norm,*

$$M=\left\{\varphi\in C(\mathbb{Z},\mathbb{R}):\|\varphi\|\leq\sqrt{3}/3\right\},$$

and $(\mathfrak{B}\varphi)(t) = \varphi(t+1) - \varphi^3(t+1)$, *then* $\mathfrak{B}$ *is a large contraction of the set* M.

Proof. For each $t \in \mathbb{Z}$ we have for real functions φ, ψ

$$\begin{aligned} &|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \\ &= |\varphi(t+1) - \psi(t+1)| \\ &\times \left|1 - \left(\varphi^2(t+1) + \varphi(t+1)\psi(t+1) + \psi^2(t+1)\right)\right|. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\varphi(t+1) - \psi(t+1)|^2 &= \varphi^2(t+1) - 2\varphi(t+1)\psi(t+1) + \psi^2(t+1) \\ &\leq 2\left(\varphi^2(t+1) + \psi^2(t+1)\right). \end{aligned}$$

Using $\varphi^2(t+1) + \psi^2(t+1) < 1$ we have

$$\begin{aligned} &|(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| \\ &\leq |\varphi(t+1) - \psi(t+1)| \\ &\times \left[1 - \left(\varphi^2(t+1) + \psi^2(t+1)\right) + |\varphi(t+1)\psi(t+1)|\right] \\ &\leq |\varphi(t+1) - \psi(t+1)| \\ &\times \left[1 - \left(\varphi^2(t+1) + \psi^2(t+1)\right) + \frac{\varphi^2(t+1) + \psi^2(t+1)}{2}\right] \\ &\leq |\varphi(t+1) - \psi(t+1)| \left[1 - \frac{\varphi^2(t+1) + \psi^2(t+1)}{2}\right] \\ &\leq \|\varphi - \psi\|. \end{aligned}$$

Then

$$\|\mathfrak{B}\varphi - \mathfrak{B}\psi\| \leq \|\varphi - \psi\|.$$

Now, let $\epsilon \in (0,1)$ be given and let $\varphi, \psi \in M$ with $\|\varphi - \psi\| \geq \epsilon$.

a) Suppose that for some t we have

$$\epsilon/2 \leq |\varphi(t+1) - \psi(t+1)|.$$

Then

$$(\epsilon/2)^2 \leq |\varphi(t+1) - \psi(t+1)|^2 \leq 2\left(\varphi^2(t+1) + \psi^2(t+1)\right),$$

that is

$$\varphi^2(t+1) + \psi^2(t+1) \geq \epsilon^2/8.$$

For all such t we have

$$\begin{aligned} |(\mathfrak{B}\varphi)(t) - (\mathfrak{B}\psi)(t)| &\leq |\varphi(t+1) - \psi(t+1)| \left[1 - \frac{\epsilon^2}{16}\right] \\ &\leq \left[1 - \frac{\epsilon^2}{16}\right] \|\varphi - \psi\|. \end{aligned}$$

b) Suppose that for some t we have

$$\left|\varphi\left(t+1\right)-\psi\left(t+1\right)\right|\leq\epsilon/2,$$

then

$$\left|\left(\mathfrak{B}\varphi\right)\left(t\right)-\left(\mathfrak{B}\psi\right)\left(t\right)\right|\leq\left|\varphi\left(t+1\right)-\psi\left(t+1\right)\right|\leq\left(1/2\right)\left\|\varphi-\psi\right\|.$$

So, for all t we have

$$\left|\left(\mathfrak{B}\varphi\right)\left(t\right)-\left(\mathfrak{B}\psi\right)\left(t\right)\right|\leq\max\left\{1/2,1-\frac{\epsilon^{2}}{16}\right\}\left\|\varphi-\psi\right\|.$$

Hence, for each $\epsilon>0$, if $\sigma=\max\left\{1/2,1-\frac{\epsilon^{2}}{16}\right\}<1$, then

$$\left\|\mathfrak{B}\varphi-\mathfrak{B}\psi\right\|\leq\sigma\left\|\varphi-\psi\right\|.$$

Consequently, $\mathfrak{B}$ is a large contraction. □

3. EXISTENCE OF PERIODIC SOLUTIONS

To apply Theorem 2.3, we need to define a Banach space $\mathbb{B}$, a bounded convex subset M of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B},\|.\|)=(C_T,\|.\|)$ and $M=\{\varphi\in\mathbb{B}\mid\|\varphi\|\leq L\}$, where $L=\sqrt{3}/3$. We express equation (2.5) as

$$\varphi\left(t\right)=\left(B\varphi\right)\left(t\right)+\left(A\varphi\right)\left(t\right):=\left(H\varphi\right)\left(t\right),$$

where $A,B:M\to\mathbb{B}$ are defined as follow

$$\begin{aligned}\left(A\varphi\right)\left(t\right)=&\frac{c\left(t-1\right)}{1+a\left(t-1\right)}\varphi\left(t-g\left(t\right)\right)+\left(1-\prod_{s=t-T}^{t-1}\left(1+a\left(s\right)\right)^{-1}\right)^{-1}\\&\times\sum_{r=t-T}^{t-1}\left\{\varphi\left(r-g\left(r\right)\right)h\left(r\right)+G\left(r,\varphi^{3}\left(r\right),\varphi^{3}\left(r-g\left(r\right)\right)\right)\right\}\prod_{s=r}^{t-1}\left(1+a\left(s\right)\right)^{-1},\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}\left(B\varphi\right)\left(t\right)=&\left(1-\prod_{s=t-T}^{t-1}\left(1+a\left(s\right)\right)^{-1}\right)^{-1}\\&\times\sum_{r=t-T}^{t-1}a\left(r\right)\left(\varphi\left(r+1\right)-\varphi^{3}\left(r+1\right)\right)\prod_{s=r}^{t-1}\left(1+a\left(s\right)\right)^{-1}.\end{aligned}\tag{3.2}$$

Further, we need the following assumptions

$$\left((k_1+k_2)\,L^3+|G(t,0,0)|\right)\le \beta L a(t)\,, \tag{3.3}$$

$$|h(t)|\le \delta a(t)\,, \tag{3.4}$$

$$\max_{t\in[0,T-1]}\left|\frac{c(t-1)}{1+a(t-1)}\right|=\alpha \tag{3.5}$$

$$J(\beta+\alpha+\delta)\le 1, \tag{3.6}$$

where α, β, δ and J are positive constants with $J\ge 3$.

We shall prove that the mapping H has a fixed point which solves (1.1).

Lemma 3.1. *For A defined in (3.1), suppose that (2.1)-(2.4) and (3.3)-(3.6) hold. Then $A:M\to M$ is continuous in the maximum norm and maps M into a compact subset of M.*

Proof. First we show that $A:M\to M$.
Let $\varphi\in M$. Evaluate (3.1) at $t+T$.

$$\begin{aligned}
&(A\varphi)(t+T)\\
&=\frac{c(t+T-1)}{1+a(t+T-1)}\varphi(t+T-g(t+T))+\left(1-\prod_{s=t}^{t+T-1}(1+a(s))^{-1}\right)^{-1}\\
&\times\sum_{r=t}^{t+T-1}\left\{\varphi(r-g(r))\,h(r)+G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right\}\prod_{s=r}^{t+T-1}(1+a(s))^{-1}\\
&=\frac{c(t-1)}{1+a(t-1)}\varphi(t-g(t))+\left(1-\prod_{s=t}^{t+T-1}(1+a(s))^{-1}\right)^{-1}\\
&\times\sum_{r=t}^{t+T-1}\left\{\varphi(r-g(r))\,h(r)+G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right\}\prod_{s=r}^{t+T-1}(1+a(s))^{-1}.
\end{aligned}$$

Let $j=r-T$, then

$$\begin{aligned}
&(A\varphi)(t+T)\\
&=\frac{c(t-1)}{1+a(t-1)}\varphi(t-g(t))+\left(1-\prod_{s=t}^{t+T-1}(1+a(s))^{-1}\right)^{-1}\\
&\times\sum_{j=t-T}^{t-1}\{\varphi(j+T-g(j+T))\,h(j+T)\\
&+G\left(j+T,\varphi^3(j+T),\varphi^3(j+T-g(j+T))\right)\}\prod_{s=j+T}^{t+T-1}(1+a(s))^{-1}.
\end{aligned}$$

Now let $k = s - T$, then

$$
\begin{aligned}
&(A\varphi)(t+T) \\
&= \frac{c(t-1)}{1+a(t-1)}\varphi(t-g(t)) + \left(1 - \prod_{k=t-T}^{t-1}(1+a(k))^{-1}\right)^{-1} \\
&\times \sum_{j=t-T}^{t-1}\left\{\varphi(j-g(j))h(j) + G\left(j,\varphi^3(j),\varphi^3(j-g(j))\right)\right\}\prod_{k=j}^{t-1}(1+a(k))^{-1} \\
&= (A\varphi)(t).
\end{aligned}
$$

Consequently, $A : C_T \to C_T$.
Making use (2.4) we obtain

$$
\begin{aligned}
|G(t,x,y)| &= |G(t,x,y) - G(t,0,0) + G(t,0,0)| \\
&\le |G(t,x,y) - G(t,0,0)| + |G(t,0,0)| \\
&\le k_1\|x\| + k_2\|y\| + |G(t,0,0)|.
\end{aligned}
$$

Note that from (2.2), we have $1 - \prod_{s=t-T}^{t-1}(1+a(s))^{-1} > 0$. So, for any $\varphi \in M$, we obtain

$$
\begin{aligned}
&|(A\varphi)(t)| \\
&\le \left|\frac{c(t-1)}{1+a(t-1)}\right||\varphi(t-g(t))| + \left(1 - \prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1} \\
&\times \sum_{r=t-T}^{t-1}\left\{|\varphi(r-g(r))||h(r)| + \left|G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right|\right\}\prod_{s=r}^{t-1}(1+a(s))^{-1} \\
&\le \alpha L + \left(1 - \prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1} \\
&\times \sum_{r=t-T}^{t-1}\left(\delta a(r)L + (k_1+k_2)L^3 + |G(r,0,0)|\right)\prod_{s=r}^{t-1}(1+a(s))^{-1} \\
&\le \alpha L + \left(1 - \prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1}\sum_{r=t-T}^{t-1}(\delta+\beta)La(r)\prod_{s=r}^{t-1}(1+a(s))^{-1} \\
&= (\alpha+\delta+\beta)L \le \frac{L}{J} < L.
\end{aligned}
$$

Thus $A\varphi \in M$.
Consequently, we have $A : M \to M$.

We show that A is continuous in the maximum norm. Let $\varphi, \psi \in M$, and let

$$\lambda_1 = \max_{t\in[0,T-1]} \left|\frac{c(t-1)}{1+a(t-1)}\right|, \ \lambda_2 = \max_{t\in[0,T-1]} \left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1},$$
$$\lambda_3 = \max_{r\in[t-T,t-1]} |h(r)|.$$

Note that from (2.2), we have $\max_{r\in[t-T,t-1]} \prod_{s=r}^{t-1}(1+a(s))^{-1} \le 1$. So,

$$\begin{aligned}
&|(A\varphi)(t)-(A\psi)(t)| \\
&\le \left|\frac{c(t-1)}{1+a(t-1)}\right| \|\varphi-\psi\| \\
&+\left(1-\prod_{s=t-T}^{t-1}(1+a(s))^{-1}\right)^{-1} \sum_{r=t-T}^{t-1}\{\|\varphi-\psi\| |h(r)| \\
&+\left|G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)-G\left(r,\varphi^3(r),\varphi^3(r-g(r))\right)\right|\} \prod_{s=r}^{t-1}(1+a(s))^{-1} \\
&\le \lambda_1\|\varphi-\psi\|+\lambda_2\sum_{r=t-T}^{t-1}\left\{\lambda_3\|\varphi-\psi\|+(k_1+k_2)\left\|\varphi^3-\psi^3\right\|\right\} \\
&\le \left\{\lambda_1+\lambda_2 T\left(\lambda_3+3(k_1+k_2)L^2\right)\right\}\|\varphi-\psi\|.
\end{aligned}$$

Let $\epsilon>0$ be arbitrary. Define $\eta=\epsilon/K$ with $K=\lambda_1+\lambda_2 T\left(\lambda_3+3(k_1+k_2)L^2\right)$, where k_1 and k_2 are given by (2.4). Then, for $\|\varphi-\psi\|<\eta$ we obtain

$$\|A\varphi-A\psi\| \le K\|\varphi-\psi\|<\epsilon.$$

This proves that A is continuous.

Next, we show that A maps bounded subsets into compact sets. Since M is bounded and A is continuous, then AM is a subset of $\mathbb{R}^T$ which is bounded. So, AM is contained in a compact subset of M. Therefore, A is continuous in M and AM is contained in a compact subset of M. □

Lemma 3.2. *Let B be defined by (3.2) and assume that (2.1)-(2.2) hold. Then $B: M \to M$ is a large contraction.*

Proof. We show that $B: M\to M$.
Let $\varphi\in M$. We evaluate (3.2) at $t+T$.

$$(B\varphi)(t+T)=$$
$$\left(1-\prod_{s=t}^{t+T-1}(1+a(s))^{-1}\right)^{-1}\sum_{r=t}^{t+T-1}a(r)\left(\varphi(r+1)-\varphi^3(r+1)\right)\prod_{s=r}^{t+T-1}(1+a(s))^{-1}.$$

Let $j = r - T$, then

$$\begin{aligned}
&(B\varphi)(t+T) \\
&= \left(1 - \prod_{s=t}^{t+T-1} (1+a(s))^{-1}\right)^{-1} \\
&\times \sum_{j=t-T}^{t-1} a(j+T)\left(\varphi(j+T+1) - \varphi^3(j+T+1)\right) \prod_{s=j+T}^{t+T-1} (1+a(s))^{-1}.
\end{aligned}$$

Now let $k = s - T$, then

$$\begin{aligned}
&(B\varphi)(t+T) \\
&= \left(1 - \prod_{k=t-T}^{t-1} (1+a(k))^{-1}\right)^{-1} \sum_{j=t-T}^{t-1} a(j)\left(\varphi(j+1) - \varphi^3(j+1)\right) \prod_{k=j}^{t-1} (1+a(k))^{-1} \\
&= (B\varphi)(t).
\end{aligned}$$

That is, $B : C_T \to C_T$.

Note that from (2.2), we have $1 - \prod_{s=t-T}^{t-1} (1+a(s))^{-1} > 0$. So, for any $\varphi \in M$, we obtain

$$\begin{aligned}
&|(B\varphi)(t)| \\
&\leq \left(1 - \prod_{s=t-T}^{t-1} (1+a(s))^{-1}\right)^{-1} \sum_{r=t-T}^{t-1} a(r)\left|\varphi(r+1) - \varphi^3(r+1)\right| \prod_{s=r}^{t-1} (1+a(s))^{-1} \\
&\leq \left(1 - \prod_{s=t-T}^{t-1} (1+a(s))^{-1}\right)^{-1} \sum_{r=t-T}^{t-1} a(r)\left\|\varphi - \varphi^3\right\| \prod_{s=r}^{t-1} (1+a(s))^{-1} \\
&= \left\|\varphi - \varphi^3\right\|.
\end{aligned}$$

Since $\|\varphi\| \leq L$, we have $\left\|\varphi - \varphi^3\right\| \leq \left(2\sqrt{3}\right)/9 < L$. This implies that, for any $\varphi \in M$,

$$\|B\varphi\| < L.$$

That is $B\varphi \in M$ and consequently we have $B : M \to M$.

It remains to show that B is large contraction. From the proof of Proposition 2.4 we have for $\varphi, \psi \in M$, with $\varphi \neq \psi$

$$\begin{aligned}
|(B\varphi)(t) - (B\psi)(t)| &\leq \left(1 - \prod_{s=t-T}^{t-1} (1+a(s))^{-1}\right)^{-1} \sum_{r=t-T}^{t-1} a(r)\|\varphi - \psi\| \prod_{s=r}^{t-1} (1+a(s))^{-1} \\
&= \|\varphi - \psi\|.
\end{aligned}$$

Then $\|B\varphi - B\psi\| \leq \|\varphi - \psi\|$. Now, let $\epsilon \in (0,1)$ be given and let $\varphi, \psi \in M$ with $\|\varphi - \psi\| \geq \varepsilon$. From the proof of the Proposition 2.4 we have found $\sigma < 1$ such that

$$\begin{aligned} &|(B\varphi)(t) - (B\psi)(t)| \\ &\leq \left(1 - \prod_{s=t-T}^{t-1} (1 + a(s))^{-1}\right)^{-1} \sum_{r=t-T}^{t-1} a(r)\,\delta\,\|\varphi - \psi\| \prod_{s=r}^{t-1} (1 + a(s))^{-1} \\ &= \sigma \|\varphi - \psi\|. \end{aligned}$$

Then $\|B\varphi - B\psi\| \leq \sigma \|\varphi - \psi\|$. Hence, B is a large contraction. □

Theorem 3.3. *Let $(C_T, \|.\|)$ be the Banach space of T-periodic real valued functions and $M = \{\varphi \in C_T \mid \|\varphi\| \leq L\}$, where $L = \sqrt{3}/3$. Suppose (2.1)-(2.4) and (3.3)-(3.6) hold. Then equation (1.1) has a T-periodic solution φ in the subset M.*

Proof. By Lemma 3.1, $A : M \to M$ is continuous and AM is contained in a compact set. Also, from Lemma 3.2 the mapping $B : M \to M$ is a large contraction. Moreover, if $\varphi, \psi \in M$, we see that

$$\|A\varphi + B\psi\| \leq \|A\varphi\| + \|B\psi\| \leq L/J + \left(2\sqrt{3}\right)/9 \leq L.$$

Thus $A\varphi + B\psi \in M$.

Clearly, all the hypotheses of the Krasnoselskii-Burton's theorem are satisfied. Thus there exists a fixed point $\varphi \in M$ such that $\varphi = A\varphi + B\varphi$. This shows that equation (1.1) has a T-periodic solution which lies in M. □

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