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Inverse connective eccentricity index and its applications

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Abstract. The inverse connective eccentricity index of a connected graph G is defined as $\xi_{ce}^{-1}(G) = \sum\limits_{u \in V(G)} \frac{\epsilon_G(u)}{d_G(u)}$, where $\epsilon_G(u)$ and $d_G(u)$ are the eccentricity and degree of a vertex u in G, respectively. In this paper, we obtain an upper bounds for inverse connective eccentricity indices for various classes of graphs such as generalized hierarchical product graph and F-sum of graphs.

Keywords: Eccentricity, Topological index, Product graph.

2000 Mathematics subject classification: 05C12; Secondary 05C76.

1. Introduction

All graphs considered in this paper are simple and connected. Let $N_G(v)$ be the set of all neighbors of a vertex v in a graph G. The degree $d_G(v)$ of a vertex v in G is the cardinality of the set $N_G(v)$. A vertex with degree one is called a pendent vertex. The eccentricity of a vertex u,

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denoted by $\epsilon_G(u)$, is the largest distance from u to any other vertex v of G.

A topological index $Top(G)$ of G is a real number with the property that for every graph H isomorphic to G, $Top(H) = Top(G)$. In organic Chemistry, topological indices have been found to be useful in chemical documentation, isomer discrimination, structure-property relationships, structure-activity relationships and pharmaceutical drug design. Wiener index is the first distance-based topological index which was introduced by Wiener [?] in 1947. Wiener used his index for the calculation of boiling points of alkanes. The Wiener index of a graph G is denoted by $W(G)$ and defined as the sum of distances between all pairs of vertices in graph G, that is, $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)$.

In recent years, Some indices have been derived related to eccentricity such as eccentric connectivity index,eccentric distance sum, adjacent distance sum, total eccentricity index. The eccentric connectivity index was successfully used for mathematical models of biological activities of diverse nature [?, ?, ?]. It has been shown to give a high degree of predictability of pharmaceutical properties, and provide leads for the development of safe and potent anti-HIV compounds [?]. Eccentric connectivity index is also proposed as a measure of branching in alkanes [?].

Gupta et al. [?] introduced a novel topological descriptor which is called eccentric-distance sum index. This index offers a vast potential for structure-activity/property relationships; it also displays high discriminating power with respect to both biological activity and physical properties [?]. From [?], we also know that some structure-activity and quantitative structure-property studies using eccentric-distance sum were better than the corresponding values obtained using the Wiener index. More recently, the mathematical properties of eccentric-distance sum have been investigated. Yu et al. [?] computed the EDS of trees and unicyclic graphs. Hua et al. [?] obtained the sharp lower bound on EDS of n-vertex cacti. Hua et al. [?] studied the graphs with graph parameters having the minimum EDS. Ilić et al. [?] studied the various lower and upper bounds for the EDS in terms of the other graph invariant. Li et al. [?] determined the trees with the third and fourth minimal EDS among the n-vertex trees.

Motivated above indices Malik [?] proposed another topological descriptors related to eccentricity. The inverse connective eccentricity index of G is denoted by $\xi_{ce}^{-1}(G)$, is defined as $\xi_{ce}^{-1}(G) = \sum_{u \in V(G)} \frac{\epsilon_G(u)}{d_G(u)}$. Inverse degree index of G is defined as $ID(G) = \sum_{u \in V(G)} \frac{1}{d_G(u)}$. For notions not defined here we refer the reader to [?]. The inverse degree first attracted attention through numerous conjectures generated by the computer programme Graffiti[?].

A number of topological indices based on vertex eccentricity are already subject to various studies. The total eccentricity index of G is defined as $\zeta(G) = \sum_{u \in V(G)} \epsilon_G(u)$. Similar to this index, Dankelmann et. al. [?] and Tang et al. [?] studied average eccentricity of graphs. Fathalikhani et al. in [?], studied total eccentricity of some graph operations. In this paper, we obtain an upper bounds for inverse connective eccentricity indices for various classes of graphs such as generalized hierarchical product graph and F-sum graphs.

2. Generalized hierarchical product of graphs

A graph G with a specified vertex subset $U \subseteq V(G)$ is denoted by $G(U)$. Barriere et al. [?, ?] defined a new product of graphs, namely, the generalized hierarchical product, as follows: Let G and H be two graphs with a nonempty vertex subset $U \subseteq V(G)$. Then the *generalized hierarchical product*, denoted by $G(U) \sqcap H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') are adjacent if and only if $g = g' \in U$ and $hh' \in E(H)$ or, $gg' \in E(G)$ and $h = h'$. *The Cartesian product*, $G \Box H$ of graphs G and H has the vertex set $V(G \Box H) = V(G) \times V(H)$ and $(u, x)(v, y)$ is an edge of$G \Box H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$.

To each vertex $u \in V(G)$, there is an isomorphic copy of H in $G \Box H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G \Box H$. But in the generalized hierarchical product, to each vertex $u \in U$, there is an isomorphic copy of H and to each vertex $v \in V(H)$, there is an isomorphic copy of G. In particular, if $U = V(G)$. then

$G\square H = G(U) \sqcap H$. The generalized hierarchical product and Cartesian product of P_5 and P_4 are shown in Figure 1.

(0,-3.3789062)(13.300938,3.3589063) [linewidth=0.04cm](1.8209375,2.6589062)(1.7809376,-1.5610938) [linew

A path between the vertices $u, v \in V(G)$ through $U \subseteq V(G)$ is a uv-path in G containing some vertex $x \in U$ (vertex x could be the vertex u or v). The distance between u and v through U is the length of the shortest path between u and v through U and is denoted by $d_{G(U)}(u,v)$. If $u \in U$ then we have, $\epsilon_{G(U)}(u) = max_{u \in V(G)} d_{G(U)}(u,v)$ and $\xi_{ce}^{-1}(G(U)) = \sum\limits_{u_i \in V(G)} \frac{\epsilon_{G(U)}(u_i)}{d_{G(U)}(u_i)}$.
From the structure of a graph $G(U) \sqcap H$, we have the following Lemma.

Lemma 2.1.. *Let G and H be two connected graphs and $U \subseteq V(G)$. Then $\epsilon_{G(U)\sqcap H}((u,v)) = \epsilon_{G(U)}(u) + \epsilon_H(v)$.*

Lemma 2.2.. [?] *Let a and b be real number. Then $\frac{1}{a+b} \leq \frac{1}{4}\left(\frac{1}{a} + \frac{1}{b}\right)$ with equality if and only if $a = b$.*

Theorem 2.3.. *Let G and H be two connected graphs and $U \subseteq V(G)$. Then $\xi_{ce}^{-1}(G(U) \sqcap H) \leq |V(H)|\, \xi_{ce}^{-1}(G(U)) + \frac{ID(H)}{4}\zeta(G(U)) + ID(G(U))\zeta(H) + \frac{|U|}{4}\xi_{ce}^{-1}(H)$.*

Proof: By the definition of inverse connective eccentricity index

$$\begin{aligned}
\xi_{ce}^{-1}(G(U) \sqcap H) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{\epsilon_{G(U)\sqcap H}((u_i,v_j))}{d_{G(U)\sqcap H}((u_i,v_j))} \\
&= \Big(\sum_{u_i \in U}\sum_{j=1}^{m} + \sum_{u_i \in V(G)|U}\sum_{j=1}^{m}\Big) \frac{\epsilon_{G(U)\sqcap H}((u_i,v_j))}{d_{G(U)\sqcap H}((u_i,v_j))} \\
&= \sum_{u_i \in U}\sum_{j=1}^{m} \frac{\epsilon_{G(U)}(u_i) + \epsilon_H(v_j)}{d_G(u_i) + d_H(v_j)} + \sum_{u_i \in V(G)|U}\sum_{j=1}^{m} \frac{\epsilon_{G(U)}(u_i) + \epsilon_H(v_j)}{d_G(u_i)}, \text{ by Lemma ??}
\end{aligned}$$

By Lemma **??**, we have $\frac{1}{d_G(u_i)+d_H(v_j)} \le \frac{1}{4}\Big(\frac{1}{d_G(u_i)} + \frac{1}{d_H(v_j)}\Big)$ with equality if and only if $d_G(u_i) = d_H(v_j)$. Hence

$$\begin{aligned}
\xi_{ce}^{-1}(G(U) \sqcap H) &\le \frac{1}{4}\sum_{u_i\in U}\sum_{j=1}^{m}(\epsilon_{G(U)}(u_i)+\epsilon_H(v_j))\Big(\frac{1}{d_G(u_i)}+\frac{1}{d_H(v_i)}\Big)\\
&+ \sum_{u_i\in V(G)}\sum_{j=1}^{m}\frac{\epsilon_{G(U)}(u_i)+\epsilon_H(v_j)}{d_G(u_i)}\\
&= \frac{1}{4}\sum_{u_i\in U}\sum_{j=1}^{m}\frac{\epsilon_{G(U)}(u_i)}{d_G(u_i)} + \frac{1}{4}\sum_{u_i\in U}\sum_{j=1}^{m}\frac{\epsilon_{G(U)}(u_i)}{d_H(v_j)} + \frac{1}{4}\sum_{u_i\in U}\sum_{j=1}^{m}\frac{\epsilon_H(v_j)}{d_G(u_i)}\\
&+ \frac{1}{4}\sum_{u_i\in U}\sum_{j=1}^{m}\frac{\epsilon_H(v_j)}{d_H(v_j)} + \sum_{u_i\in V(G)|U}\sum_{j=1}^{m}\frac{\epsilon_{G(U)}(u_i)}{d_G(u_i)} + \sum_{u_i\in V(G)|U}\sum_{j=1}^{m}\frac{\epsilon_H(v_j)}{d_G(u_i)}
\end{aligned}$$

From the definition of inverse degree, we obtain

$$\begin{aligned}
\xi_{ce}^{-1}(G(U) \sqcap H) &\le \frac{1}{4}|V(H)|\sum_{u_i\in U}\frac{\epsilon_{G(U)}(u_i)}{d_G(u_i)} + \frac{1}{4}ID(H)\sum_{u_i\in U}\epsilon_{G(U)}(u_i)\\
&+ \frac{1}{4}\sum_{u_i\in U}\frac{1}{d_G(u_i)}\sum_{j=1}^{m}\epsilon_H(v_j) + \frac{1}{4}|U|\sum_{j=1}^{m}\frac{\epsilon_H(v_j)}{d_H(v_j)}\\
&+ |V(H)|\sum_{u_i\in V(G)|U}\frac{\epsilon_{G(U)}(u_i)}{d_G(u_i)} + \sum_{u_i\in V(G)|U}\frac{1}{d_G(u_i)}\sum_{j=1}^{m}\epsilon_H(v_j)\\
&= \frac{|V(H)|}{4}\sum_{u_i\in U}\frac{\epsilon_{G(U)}(u_i)}{d_G(u_i)} + \frac{ID(H)}{4}\zeta(G(U)) + \frac{1}{4}\sum_{u_i\in U}\frac{1}{d_G(u_i)}\zeta(H)\\
&+ \frac{|U|}{4}\xi_{ce}^{-1}(H) + |V(H)|\sum_{u_i\in V(G)|U}\frac{\epsilon_{G(U)}(u_i)}{d_G(u_i)} + \zeta(H)\sum_{u_i\in V(G)|U}\frac{1}{d_G(u_i)}
\end{aligned}$$

By the definition of inverse connective eccentricity index, we have

$$\xi_{ce}^{-1}(G(U) \sqcap H) \le |V(H)|\,\xi_{ce}^{-1}(G(U)) + \frac{ID(H)}{4}\zeta(G(U)) + ID(G(U))\zeta(H) + \frac{|U|}{4}\xi_{ce}^{-1}(H).$$

By setting $U = V(G)$ in Theorem **??** , we obtain the upper bound for inverse connective eccentricity index of Cartesian product of two graphs G and H.

Corollary 2.4.. *Let G and H be a connected graph. Then* $\xi_{ce}^{-1}(G\Box H) \le |V(H)|\,\xi_{ce}^{-1}(G) + \frac{ID(H)}{4}\zeta(G) + ID(G)\zeta(H) + \frac{|V(G)|}{4}\xi_{ce}^{-1}(H)$.

Example 2.5.. *The $TC_4(m,n)$ nanotorns is a graph which is isomorphic to a Cartesian product of two cycles. Hence by Corollary* **??**, *we obtain*

$$\xi_{ce}^{-1}(TC_4(m,n)) \leq \begin{cases} \frac{5mn(2m+n-3)}{16} & \textit{if } n,m \textit{ are odd,} \\ \frac{mn(5m+6n-6)}{16} & \textit{if } n \textit{ is odd } m \textit{ is even,} \\ \frac{5mn(m+n-1)}{16} & \textit{if } n \textit{ is even } m \textit{ is odd,} \\ \frac{5mn(m+n)}{16} & \textit{if } n,m \textit{ are even.} \end{cases}$$

3. F-sum of graphs

Let G be a connected graph. Then

(i)The *subdivision graph* of a graph G, denoted by $S(G)$, is obtained from G by replacing each edge of G by a path of length two.

(ii) The *triangle parallel graph* of a graph G is denoted by $R(G)$ and is obtained from G by replacing each edge of G by a triangle.

(iii) The *line superposition graph* $Q(G)$ of a graph G is obtained from G by inserting a new vertex into each edge of G and then joining with edges each pair of new vertices on adjacent edges of G.

(iv) The *total graph* $T(G)$ of a graph G has its vertices as the edges and vertices of G and adjacency in $T(G)$ is defined by the adjacency or incidence of the corresponding elements of G, see Figure 2.

(0,-4.688906)(5.9,4.668906) [linewidth=0.04,dimen=outer](2.67,3.1489062)(1.86,1.46) [linewidth=0.04,dimen

Let F be one of the subdivision operations S, R, Q or T. For two connected graphs G and H, the $F-$sum, denoted by $G+_F H$, is the graph with vertex set $(V(G) \cup E(G)) \times V(H)$, and any two vertices (u,v) and $(u^{'},v^{'})$ of $G+_F H$ are adjacent if and only $\Big[u = u^{'} \in V(G)$ and $(v,v^{'}) \in E(H)\Big]$ or $\Big[v = v^{'} \in V(H)$ and $(u,u^{'}) \in E(F(G))\Big]$. The graph $P_3 +_F P_2$ are shown in Figure 3.

(0,-4.578906)(8.26,4.558906) [linewidth=0.04cm](0.1,3.8989062)(0.1,2.5189064) [linewidth=0.04cm](1.48,3.8

The concept of $F-$sum graph was first introduced by Eliasi and Taeri [**?**] and the Weiner indices of the resulting graphs were studied therein. Li and Wang [**?**] derived explicit expression of the PI indices of four sums of two graphs. The hyper and reverse Weiner indices of F sum graphs were studied by Metsidik et al. [?]. Eskender and Vumar [?] determined the eccentric connectivity index of $F-$sum graphs in terms of some invariants of the factors. In this sequence, now find the upper bounds for inverse connective eccentricity index of $G +_F H$, where$F = S, R, Q$ and T.

Lemma 3.1.. [?] *Let G and H be two connected graphs. Then*
(i) $\zeta(S(G)(U)) = 2\zeta(G) + \zeta(L(G)) + |E(G)|$,
(ii) $\zeta(R(G)(U)) = \zeta(G) + \zeta(L(G)) + |E(G)|$,
(iii) $\zeta(Q(G)(U)) = \zeta(G) + \zeta(L(G)) + |V(G)| + |E(G)|$,
(iv) $\zeta(T(G)(U)) = \zeta(G) + \zeta(L(G)) + |E(G)|$.

Theorem 3.2.. *Let G and H be two connected graphs. Then* $\xi_{ce}^{-1}(G +_S H) \leq 2|V(H)|\xi_{ce}^{-1}(G) + \frac{\zeta(L(G))}{4}\Big(|V(H)| + ID(H)\Big) + \frac{|E(G)|}{4}(2|V(H)| + ID(H)) + \frac{ID(H)}{2}\zeta(G) + ID(S(G)(U))\zeta(H) + \frac{|V(G)|+|E(G)|}{4}\xi_{ce}^{-1}(H)$.

Proof: Let U be a subset of $V(S(G))$. From the structure of $S(G)$, the number of vertices and edge of $S(G)$ are $|V(G)| + |E(G)|$ and $2|E(G)|$, respectively. Further, for each vertex $v \in U$, $\epsilon_{S(G)(U)}(v) = 2\epsilon_G(v)$ and for each vertex $v \in V(S(G))|U$, $\epsilon_{S(G)(U)}(v) = 2\epsilon_{L(G)}(v) + 1$, where $L(G)$ is a line graph of G. Hence by the definition of inverse connective eccentricity index, we have

$$\begin{aligned}
\xi_{ce}^{-1}(S(G)(U)) &= \sum_{v\in V(S(G)(U))} \frac{\epsilon_{S(G)(U)}(v)}{d_{S(G)(U)}(v)} \\
&= \sum_{v\in U} \frac{\epsilon_{S(G)(U)}(v)}{d_{S(G)(U)}(v)} + \sum_{v\in V(S(G))|U} \frac{\epsilon_{S(G)(U)}(v)}{d_{S(G)(U)}(v)} \\
&= \sum_{v\in U} \frac{2\epsilon_G(v)}{d_G(v)} + \sum_{v\in V(S(G))|U} \frac{\epsilon_{S(G)(U)}(v)}{2} \\
&= 2\xi_{ce}^{-1}(G) + \frac{1}{2}\sum_{v\in V(L(G))}(2\epsilon_{L(G)}(v) + 1) \\
&= 2\xi_{ce}^{-1}(G) + \zeta(L(G)) + \frac{|E(G)|}{2}.
\end{aligned}$$

Combining this result with result of $\xi^{-1}(G(U)\sqcap H)$ and by use of Lemma **??**, we obtain

$\xi_{ce}^{-1}(G +_S H) \leq 2\,|V(H)|\,\xi_{ce}^{-1}(G) + \frac{\zeta(L(G))}{4}\Big(\,|V(H)| + ID(H)\Big) + \frac{|E(G)|}{4}(2\,|V(H)| + ID(H)) + \frac{ID(H)}{2}\zeta(G) + ID(S(G)(U))\zeta(H) + \frac{|V(G)|+|E(G)|}{4}\xi_{ce}^{-1}(H)$.

Example 3.3.. *The Zig-Zag polyhex nanotube (Figure 4) is the graph $S(C_n)(U) \sqcap P_2$, where $U = V(C_n) \subseteq V(S(C_n))$. Hence by Theorem **??**, we obtain the following*

$$\xi_{ce}^{-1}(S(C_n(U) \sqcap P_2)) \leq \begin{cases} \frac{8n^3+3n^2+7n}{4} & \text{if } n \text{ odd,} \\ \frac{11n^2+18n}{4} & \text{if } n \text{ is even.} \end{cases}$$

(0,-1.6389062)(8.635938,1.6189063) [linewidth=0.04,dimen=outer](3.6940625,0.7189062)(3.52,0.84) [linewidt

Theorem 3.4.. *Let G and H be two connected graphs. Then $\xi_{ce}^{-1}(G +_R H) \leq |V(H)|\Big(\xi_{ce}^{-1}(G) + \frac{1}{2}(2\zeta(L(G)) + |E(G)|\Big) + \frac{ID(H)}{4}\Big(\zeta(R(G)) + \zeta(L(G)) + |E(G)|\Big) + ID(R(G)(U))\zeta(H) + \frac{|U|}{4}\xi_{ce}^{-1}(H)$.*

Proof: Let U be a subset of $V(R(G))$. From the structure of $R(G)$, we get $|V(R(G))| = |V(G)| + |E(G)|$ and $|E(R(G))| = 3\,|E(G)|$. Further, for each vertex $v \in U$, $\epsilon_{R(G)(U)}(v) = \epsilon_G(v)$ and for each vertex $v \in V(R(G))|U$, $\epsilon_{R(G)(U)}(v) = \epsilon_{L(G)}(v) + 1$, where $L(G)$ is a line graph of G. Thus by the definition of inverse connective eccentricity index, we obtain

$$\begin{aligned} \xi_{ce}^{-1}(R(G)(U)) &= \sum_{v \in V(R(G))} \frac{\epsilon_{R(G)(U)}(v)}{d_{R(G)(U)}(v)} \\ &= \sum_{v \in U} \frac{\epsilon_{R(G)(U)}(v)}{d_{R(G)(U)}(v)} + \sum_{v \in V(R(G))|U} \frac{\epsilon_{R(G)(U)}(v)}{d_{R(G)(U)}(v)} \\ &= \sum_{v \in U} \frac{\epsilon_G(v)}{d_G(v)} + \sum_{v \in V(R(G))|U} \frac{\epsilon_{R(G)(U)}(v)}{2} \\ &= \xi_{ce}^{-1}(G) + \frac{1}{2} \sum_{v \in V(L(G))} (2\epsilon_{L(G)}(v) + 1) \\ &= \xi_{ce}^{-1}(G) + \frac{1}{2}(\zeta(L(G)) + |E(G)|). \end{aligned}$$

Combining this result with result of $\xi^{-1}(G(U) \sqcap H)$ and by use of Lemma **??**, we obtain

$\xi_{ce}^{-1}(G +_R H) \leq |V(H)|\Big(\xi_{ce}^{-1}(G) + \frac{1}{2}(2\zeta(L(G)) + |E(G)|\Big) + \frac{ID(H)}{4}\Big(\zeta(G)+\zeta(L(G))+|E(G)|\Big) + ID(R(G)(U))\zeta(H) + \frac{|U|}{4}\xi_{ce}^{-1}(H).$

Theorem 3.5.. *Let G and H be two connected graphs. Then* $\xi_{ce}^{-1}(G+_Q H) \leq |V(H)|\Big(\xi_{ce}^{-1}(G)+ID(G)+\frac{1}{4}\Big(\xi_{ce}^{-1}(L(G))+ID(L(G))+\frac{1}{2}\Big(\zeta(L(G))+|E(G)|\Big)\Big)\Big)+\frac{ID(H)}{4}\Big(\zeta(G)+|V(G)|+\zeta(L(G))+|E(G)|\Big)+ID(Q(G)(U))\zeta(H)+\frac{|U|}{4}\xi_{ce}^{-1}(H).$

Proof: Let U be a subset of $V(Q(G))$. From the structure of $Q(G)$, the number of vertices and edge of $Q(G)$ are $|V(G)| + |E(G)|$ and $2|E(G)| + |E(L(G))|$, respectively. Further, for each vertex $v \in U$, $\epsilon_{Q(G)(U)}(v) = \epsilon_G(v) + 1$ and for each vertex $v \in V(Q(G))|U$, $\epsilon_{Q(G)(U)}(v) = \epsilon_{L(G)}(v) + 1$, where $L(G)$ is a line graph of G. Hence by the definition of inverse connective eccentricity index, we have

$$\begin{aligned}
\xi_{ce}^{-1}(Q(G)(U)) &= \sum_{v\in V(Q(G))}\frac{\epsilon_{Q(G)(U)}(v)}{d_{Q(G)(U)}(v)}\\
&= \sum_{v\in U}\frac{\epsilon_{Q(G)(U)}(v)}{d_{Q(G)(U)}(v)} + \sum_{v\in V(Q(G))|U}\frac{\epsilon_{Q(G)(U)}(v)}{d_{Q(G)(U)}(v)}\\
&= \sum_{v\in U}\frac{\epsilon_G(v)+1}{d_G(v)} + \sum_{v\in V(L(G))}\frac{\epsilon_{L(G)(v)}+1}{d_{L(G)}(v)+2}.
\end{aligned}$$

By Lemma **??**, we have $\frac{1}{d_{L(G)(v)+2}} \leq \frac{1}{4}\Big(\frac{1}{d_{L(G)(v)}}+\frac{1}{2}\Big)$ with equality if and only if $d_{L(G)}(v) = 2$. Hence

$$\begin{aligned}
\xi_{ce}^{-1}(Q(G)(U)) &\leq \sum_{v\in U}\frac{\epsilon_G(v)}{d_G(v)} + \sum_{v\in U}\frac{1}{d_G}(v) + \frac{1}{4}\Big(\sum_{v\in V(L(G))}\frac{\epsilon_{L(G)(v)}+1}{d_{L(G)}(v)} + \sum_{v\in V(L(G))}\frac{\epsilon_{L(G)(v)}+1}{2}\Big)\\
&\leq \xi_{ce}^{-1}(G) + ID(G) + \frac{1}{4}\Big(\xi_{ce}^{-1}(L(G)) + ID(L(G)) + \frac{1}{2}\Big(\zeta(L(G)) + |E(G)|\Big)\Big).
\end{aligned}$$

Combining this result with result of $\xi^{-1}(G(U)\sqcap H)$ and by use of Lemma **??**, we obtain

$\xi_{ce}^{-1}(G+_Q H) \leq |V(H)|\Big(\xi_{ce}^{-1}(G)+ID(G)+\frac{1}{4}\Big(\xi_{ce}^{-1}(L(G))+ID(L(G))+\frac{1}{2}\Big(\zeta(L(G))+|E(G)|\Big)\Big)\Big)+\frac{ID(H)}{4}\Big(\zeta(G)+|V(G)|+\zeta(L(G))+|E(G)|\Big)+ID(Q(G)(U))\zeta(H)+\frac{|U|}{4}\xi_{ce}^{-1}(H).$

Theorem 3.6.. *Let G and H be two connected graphs. Then* $\xi_{ce}^{-1}(G+_T H) \leq |V(H)|\Big(2\xi_{ce}^{-1}(G)+\frac{1}{4}\Big(\xi_{ce}^{-1}(L(G))+ID(L(G))+\frac{\zeta(L(G))}{2}+\frac{|E(G)|}{2}\Big)+\frac{ID(H)}{4}\Big(\zeta(G)+\zeta(L(G))+|E(G)|\Big)+ID(T(G)(U))\zeta(H)+\frac{|U|}{4}\xi_{ce}^{-1}(H).$

Proof: Let U be a subset of $V(T(G))$. From the structure of $T(G)$, the number of vertices and edge of $T(G)$ are $|V(G)|+|E(G)|$ and $3|E(G)|+|E(L(G))|$, respectively. Further, for each vertex $v \in U$, $\epsilon_{T(G)(U)}(v) = \epsilon_G(v)$ and for each vertex $v \in V(T(G))|U$, $\epsilon_{T(G)(U)}(v) = \epsilon_{L(G)}(v)+1$, where $L(G)$ is a line graph of G. Thus by the definition of inverse connective eccentricity index, we get

$$\begin{aligned}
\xi_{ce}^{-1}(T(G)(U)) &= \sum_{v\in V(T(G))} \frac{\epsilon_{T(G)(U)}(v)}{d_{T(G)}(v)} \\
&= \sum_{v\in U} \frac{\epsilon_{T(G)(U)}(v)}{d_{T(G)}(v)} + \sum_{v\in V(T(G))|U} \frac{\epsilon_{T(G)(U)}(v)}{d_{T(G)}(v)} \\
&= \sum_{v\in U} \frac{\epsilon_G(v)}{d_G(v)} + \sum_{v\in V(L(G))} \frac{\epsilon_{L(G)(v)}+1}{d_{L(G)}+2}.
\end{aligned}$$

By Lemma **??**, we have $\frac{1}{d_{L(G)(v)+2}} \leq \frac{1}{4}\Big(\frac{1}{d_{L(G)}(v)}+\frac{1}{2}\Big)$ with equality if and only if $d_{L(G)}(v)=2$. Therefore

$$\begin{aligned}
\xi_{ce}^{-1}(T(G)(U)) &\leq 2\xi_{ce}^{-1}(G)+\frac{1}{4}\Big(\sum_{v\in L(G)}\frac{\epsilon_{L(G)}(v)+1}{d_L(G)(v)}+\frac{\epsilon_L(G)+1}{2}\Big) \\
&\leq 2\xi_{ce}^{-1}(G)+\frac{1}{4}\Big(\xi_{ce}^{-1}(L(G))+ID(L(G))+\frac{\zeta(L(G))}{2}+\frac{|E(G)|}{2}\Big).
\end{aligned}$$

Combining this result with result of $\xi^{-1}(G(U)\sqcap H)$ and by the use of Lemma **??**, we obtain
$\xi_{ce}^{-1}(G +_T H) \leq |V(H)|\Big(2\xi_{ce}^{-1}(G) + \frac{1}{4}\Big(\xi_{ce}^{-1}(L(G))+ID(L(G))+\frac{\zeta(L(G))}{2}+\frac{|E(G)|}{2}\Big)+\frac{ID(H)}{4}\Big(\zeta(G)+\zeta(L(G))+|E(G)|\Big)+ID(T(G)(U))\zeta(H)+\frac{|U|}{4}\xi_{ce}^{-1}(H).$

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