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k-distance enclaveless number of a graph

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Abstract. For an integer $k \geq 1$, a k-distance enclaveless number (or k-distance B-differential) of a connected graph $G = (V, E)$ is $\Psi^k(G) = max\{|(V - X) \cap N_{k,G}(X)| : X \subseteq V\}$. In this paper, we establish upper bounds on the k-distance enclaveless number of a graph in terms of its diameter, radius and girth. Also, we prove that for connected graphs G and H with orders n and m respectively, $\Psi^k(G \times H) \leq mn - n - m + \Psi^k(G) + \Psi^k(H) + 1$, where $G \times H$ denotes the direct product of G and H. In the end of this paper, we show that the k-distance enclaveless number $\Psi^k(T)$ of a tree T on $n \geq k + 1$ vertices and with n_1 leaves satisfies inequality $\Psi^k(T) \leq \frac{k(2n-2+n_1)}{2k+1}$ and we characterize the extremal trees.

Keywords: k-distance enclaveless number, diameter, radius, girth, direct product.

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1. Introduction

Distance in graphs is a fundamental concept in graph theory. Let G be a connected graph. *The distance between two vertices* u and v in G, denoted $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The *eccentricity* $ecc_G(v)$ *of* v in G is the distance between v and a vertex farthest from v in G. The minimum eccentricity among all vertices of

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G is *the radius* of G, denoted by $rad(G)$, while the maximum eccentricity among all vertices of G is *the diameter* of G, denoted by $diam(G)$. Thus, the diameter of G is the maximum distance among all pairs of vertices of G. A vertex v with $ecc_G(v) = diam(G)$ is called a *peripheral vertex* of G. A *diametral path* in G is a shortest path in G whose length is equal to the diameter of the graph. Thus, a diametral path is a path of length $diam(G)$ joining two peripheral vertices of G. If S is a set of vertices in G, then the *distance,* $d_G(v, S)$, from a vertex v to the set S is the minimum distance from v to a vertex of S; that is, $d_G(v, S) = min\{d_G(u, v) : u \in S\}$. In particular, if $v \in S$, then $d(v, S) = 0$. Enclaveless number (B-differential) of graphs is also very well studied in graph theory. An *enclaveless number* (B-differential) of a set X in a graph G is $\Psi(X) = |bd(X)| = |B(X)| = |(V - X) \cap N_G(X)|$ so that $B(X)$ is called *boundary* of X. The *enclaveless number* (B-differential) of G, denoted by $\Psi(G)$, is $\Psi(G) = max\{|B(X)| : X \subseteq V\}$. In this paper, we start the study of k-distance enclaveless number in graphs which combines the concepts of both distance and enclaveless number in graphs [4]. Let $k \geq 1$ be an integer and let G be a graph. A k-distance enclaveless number (k-distance B-differential) of a set X in a graph G is $\Psi^k(X) = |bd^k(X)| = |B^k(X)| = |(V - X) \cap N_{k,G}(X)|$ so that $B^k(X)$ is called k-boundary of X. The *k-distance enclaveless number (k-distance B-differential)* of G, denoted by $\Psi^k(G)$, is $\Psi^k(G) = max\{|B^k(X)| : X \subseteq V\}$. A set D is called *k-distance enclaveless set of G if $B^k(D) = V - D$. A set D is called maximum k-distance enclaveless set of G if $\Psi^k(G) = |B^k(D)|$. When $k = 1$, the 1-distance enclaveless number* of G is precisely the enclaveless number of G; that is, $\Psi^1(G) = \Psi(G)$. In 1977, Slater [6] introduced the concept of a enclaveless set (or B-differential set) in a graph.

Let $G = (V, E)$ be a simple undirected graph with the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. We refer the reader to [1],[7] for any terminology and notation not here in. We denote minimum degree of a graph G with $\delta(G)$ and maximum degree with $\Delta(G)$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u : uv \in E(G)\}$, while the *closed neighborhood* of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$. The *closed neighborhood* of a set $S \subseteq V$ is the set $N[S] = N(S) \cup S$. Let E_v be the set of edges incident with v in G that is, $E_v = \{uv \in E(G) : u \in N(v)\}$. We denote the degree of v by $\deg_G(v) = |E_v|$.

Let k be a positive integer. For a vertex $v \in V(G)$, the *open k-neighborhood* $N_{k,G}(v)$ is the set $\{u \in V(G) : u \neq v \text{ and } d(u, v) \leq k\}$

and the *closed k-neighborhood* $N_{k,G}[v]$ is the set $N_{k,G}(v) \cup \{v\}$. The *open k-neighborhood* $N_{k,G}(S)$ of a set $S \subseteq V$ is the set $\cup_{v\in S} N_{k,G}(v)$, and the *closed k-neighborhood* $N_{k,G}[S]$ of a set $S \subseteq V$ is the set $N_{k,G}(S) \cup S$. The *k-degree of a vertex* v is defined as $\delta_{k,G}(v) = \deg_{k,G}(v) = |N_{k,G}(v)|$. The *minimum and maximum* k-degree of a graph G are denoted by $\delta_k(G)$ and $\Delta_k(G)$, respectively. For a non-empty subset $S \subseteq V$, and any vertex $v \in V$ we denote by $N_{k,S}(v)$ the set of k-neighbors v has in S: $N_{k,S}(v) := \{u \in S : d(u, v) \leq k\}$ and $\delta_{k,S}(v) = |N_{k,S}(v)|$. The graph G is called *distance k-regular* if $\delta_k(G) = \Delta_k(G)$. The *$k$-th power* G^k of a graph G is the graph with vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{xy : d(x, y) \leq k\}$. Now clearly, we have $N_{k,G}(v) = N_{1,G^k}(v) = N_{G^k}(v)$, $N_{k,G}[v] = N_{1,G^k}[v] = N_{G^k}[v]$, $deg_{k,G}(v) = deg_{1,G^k}(v) = deg_{G^k}(v)$, $\delta_k(G) = \delta_1(G^k) = \delta(G^k)$ and $\Delta_k(G) = \Delta_1(G^k) = \Delta(G^k)$. A vertex v is called k-adjacent to (or k-neighbor with) a vertex w if $d(v, w) = k$. A vertex of degree one is called a leaf and the set of leaves of a graph G is denoted by $\Omega(G)$. The number of leaves $\Omega(G)$ will be denoted by $n_1(G)$. For a tree T and an edge $xy \in E(T)$, let T_x and T_y denote the components of $T - xy$ in which the vertices x and y belong to T_x and T_y, respectively. A complete bipartite graph $K_{m,n}$ with partite sets X, Y such that $|X| = m$ and $|Y| = n$. If $m = 1$, then $K_{1,n}$ is called an star with $n + 1$ vertices. The edge subdivision in a graph G is the following operation; remove one edge $e = xy$ of G and add a new vertex z and the edges xz and zy. A k-times subdivided star SS^k_t is obtained from a star $K_{1,t}$ by subdividing each edge by exactly k vertices.

This paper is organized as follows: In Section 2 we study some elementary results on k-distance enclaveless number of G. We establish upper bounds on the k-distance enclaveless number of a graph in terms of its diameter, radius and girth in Section 3. Also, we prove that for the connected graphs G and H with orders n and m respectively, $\Psi^k(G \times H) \leq mn - n - m + \Psi^k(G) + \Psi^k(H) + 1$ in Section 4. Finally, in Section 5, we show that the k-distance enclaveless number $\Psi^k(T)$ of a tree T on $n \geq k + 1$ vertices and with n_1 leaves satisfies inequality $\Psi^k(T) \leq \frac{k(2n-2+n_1)}{2k+1}$ and we characterize the extremal trees.

2. Preliminary results

In order to prove recent inequality, the techniques of article [3] have been used. It is well known that, if H is a subgraph of G and u, v be two vertices in G, then $d_G(u, v) \leq d_H(u, v)$ and $N_{k,H}(u) \subseteq N_{k,G}(u)$. Therefore we have following observation.

Observation 2.1. *For $k \geq 1$, if H is a spanning subgraph of a graph G, then $\Psi^k(G) \leq \Psi^k(H)$.*

The following theorem shows that the study of k-distance enclaveless set of a graph G is lead to the study of k-distance enclaveless set of a spanning tree T of G.

Theorem 2.2. *For $k \geq 1$, every connected graph G has a spanning tree T such that $\Psi^k(T) = \Psi^k(G)$.*

Proof. Let $D = \{v_1, \ldots, v_t\}$ be a maximum k-distance enclaveless set of G. Thus, $|D| = t = \Psi^k(G)$. We now partition the vertex set $V(G)$ into t sets $V_1, \ldots, V_t$ as follows. Initially, we let $V_i = \{v_i\}$ for all $i \in [t]$. We then consider sequentially the vertices not in D. For each vertex $v \in V(G) - D$, we select a vertex $v_i \in D$ at minimum distance from v in G and add the vertex v to the set V_i. We note that if $v \in V(G) - D$ and $v \in V_i$ for some $i \in [t]$, then $d_G(v, v_i) = d_G(v, D)$, although the vertex v_i is not necessarily the unique vertex of D at minimum distance from v in G. Further, since D is a k-distance enclaveless set of G , we note that $d_G(v, v_i) \leq k$. For each $i \in [t]$, let T_i be a spanning tree of $G[V_i]$ that is distance preserving from the vertex v_i; that is, $V(T_i) = V_i$ and for every vertex $v \in V(T_i)$, we have $d_{T_i}(v, v_i) = d_G(v, v_i)$. We now let T be the spanning tree of G obtained from the disjoint union of the t trees $T_1, \ldots, T_t$ by adding $t-1$ edges of G. We remark that these added $t-1$ edges exist as G is connected. We now consider an arbitrary vertex, v say, of G. The vertex $v \in V_i$ for some $i \in [t]$. Thus, $d_T(v, v_i) \leq d_{T_i}(v, v_i) = d_G(v, v_i) = d_G(v, D) \leq k$. Therefore, the set D is a k-distance enclaveless set of T, and so $\Psi^k(T) \leq |D| = \Psi^k(G)$. However, by Observation 2.1, $\Psi^k(G) \leq \Psi^k(T)$. Consequently, $\Psi^k(T) = \Psi^k(G)$. □

We shall also need the following lemma.

Lemma 2.3. *Let G be a connected graph that is not a tree, and let C be a shortest cycle in G. If v is a vertex of G outside of C that $|B^k(\{v\}) \cap V(C)| \geq 2k$, then there exist two vertices $u, w \in V(C) \cup B^k(\{v\})$ such that a shortest (u, v)-path does not contain w and a shortest (v, w)-path does not contain u.*

Proof. Since v is not on C, it has a distance of at least 1 to every vertex of C. Let u be a vertex of C at minimum distance from v in G. We put $Q = V(C) \cap B^k(\{v\})$. Thus, $Q \subseteq V(C)$ and, by assumption, $|Q| \geq 2k$. Among all vertices in Q, let $w \in Q$ be chosen to have maximum distance from u on the cycle C. Since there are $2k-1$ vertices within distance $k-1$ from u on C, the vertex w has distance at least k from u on the cycle C. Let P_u be a shortest (u, v)-path and let P_w be a shortest (v, w)-path in G.

If $w \in V(P_u)$, then $d_G(u,w) < d_G(u,v) \leq k$, contradicting our choice of the vertex u. Therefore, $w \notin V(P_u)$. Suppose that $u \in V(P_w)$. Since C is a shortest cycle in G, the distance between u and w on C is the same as the distance between u and w in G. Thus, $d_G(u,w) = d_C(u,w)$, implying that $d_G(v,w) = d_G(v,u)+d_G(u,w) \geq 1+d_G(u,w) = 1+d_C(u,w) \geq 1+k$, a contradiction. Therefore, $u \notin V(P_w)$. □

3. Upper bound of the k-distance enclaveless in a graph

In this section we provide various upper bounds on the k-distance enclaveless number for general graphs.

Theorem 3.1. *For $k \geq 1$, if G is a connected graph with diameter d, then*

$$\Psi^k(G) \leq \frac{2kn+n-d-1}{2k+1}.$$

This bound is sharp.

Proof. Let $P : u_0u_1 \ldots u_d$ be a diametral path in G, joining two peripheral vertices $u = u_0$ and $v = u_d$ of G. Thus, the length of P is $diam(G) = d$. We claim that for every vertex $v \in G$, $|V(P) \cap B^k(\{v\})| \leq 2k+1$. Suppose, to the contrary, that there exists a vertex $q \in V(G)$ so that we have, $|V(P) \cap B^k(\{q\})| \geq 2k+2$. (Possibly, $q \in V(P)$.) Now we put $Q = V(P) \cap B^k(\{q\})$. Then $|Q| \geq 2k+2$. Let i and j be the smallest and greatest integers respectively, such that $u_i,\ u_j \in Q$. We note that $Q \subseteq \{u_i, u_{i+1}, \ldots, u_j\}$. Thus, $2k+2 \leq |Q| \leq j-i+1$. Since P is a shortest (u,v)-path in G, we therefore note that $d_G(u_i,u_j) = d_P(u_i,u_j) = j-i \geq 2k+1$. Let P_i and P_j be shortest (u,q)-path and (q,v)-path in G. Since $u_i, u_j \in B^k(\{q\})$, both paths P_u and P_v have length at most k. Therefore, the (u_i,u_j)-path obtained by the following path P_i from u_i to q, and then proceeding along the path P_j from q to u_j, has length at most $2k$, implying that $d_G(u_i,u_j) \leq 2k$, a contradiction. Therefore, for every vertex $v \in V(G)$, $|V(P) \cap B^k(v)| \leq 2k+1$.
Let S be a maximum k-distance enclaveless set of G. Thus, $|S| = \Psi^k(G)$. For every vertex $x \in S$, we have $|V(P) \cap B^k(\{x\})| \leq 2k+1$, and so $|V(P) \cap B^k(S)| \leq (n-|S|)(2k+1)$. However, since S is a k-distance enclaveless set of G and for any vertex $y \in P$, $y \in B^k(S)$, thus we have, $|B^k(S) \cap V(P)| = d+1$. Therefore, $(n-|S|)(2k+1) \geq d+1$, or, equivalently, $\Psi^k(G) = |S| \leq (2kn+n-d-1)/(2k+1)$.

For seeing the sharpness of bound, let G be a path, $v_1v_2 \ldots v_n$, of order $n = \ell(2k+1)$ for some $\ell \geq 1$. Let $d = diam(G)$, and so $d = n-1 = \ell(2k+1)-1$. It is clear $\Psi^k(G) \leq \frac{(2kn+n-d-1)}{2k+1} = 2k\ell = n-\ell$.

Figure 1. For $n = 4$, $V_2 = V_3 = V_4 = K_2$

The set

$$S = \cup_{i=0}^{\ell-1}\{v_{v_{k+1+i(2k+1)}}\}$$

is a k-distance enclaveless set of G, and so $\Psi^k(G) \geq |S| = n - \ell$. Consequently, $\Psi^k(G) = n - \ell = \frac{(2kn+n-d-1)}{2k+1}$. We state this formally as follows. $\square$

For the family of the graphs we obtain the bound in Theorem 3.1. For this, let $P = v_1v_2 \dots v_n$ be a path. By replacing each vertex v_i, for $2 \leq i \leq n-1$, on the path with a clique (clique V_i corresponds to vertex v_i) of size at least $\delta \geq 1$, and adding all edges between v_1 and vertices in V_2, adding all edges between v_n and vertices in V_{n-1}, and adding all edges between vertices in V_i and V_{i+1} for $2 \leq i \leq n-2$, we obtain a graph with minimum degree δ achieving the upper bound of Theorem 3.1, see Figure 1.

In general, by applying Theorem 3.1, the k-distance enclaveless number of a cycle C_n or path P_n of order $n \geq 3$, are easily obtained.

Proposition 3.2. *For $k \geq 1$ and $n \geq 3$, $\Psi^k(P_n) = \Psi^k(C_n) = n - \lceil \frac{n}{2k+1} \rceil$.*

As a consequence of Theorem 3.1, we have the following upper bound on the k-distance enclaveless number of a graph in terms of its radius.

Corollary 3.3. *For $k \geq 1$, if G is a connected graph with radius r, then*

$$\Psi^k(G) \leq \frac{2kn + n - 2r}{2k+1}.$$

This bound is sharp.

Proof. By Theorem 2.2, the graph G has a spanning tree T such that $\Psi^k(T) = \Psi^k(G)$. Since adding edges to a graph cannot increase its radius, $rad(G) \leq rad(T)$. Since T is a tree, we note that $diam(T) \geq 2rad(T) - 1$. Applying Theorem 3.1 to the tree T, we have that

$$\Psi^k(G) = \Psi^k(T) \leq \frac{2kn+n-d-1}{2k+1} \leq \frac{2kn+n-2r+1-1}{2k+1} = \frac{2kn+n-r}{2k+1}.$$

For seeing the upper bound, let G be a path P_n of order $n = 2\ell(2k+1)$ for some integer $\ell \geq 1$. Let $d = diam(G)$ and let $r = rad(G)$, and so $d = 2\ell(2k+1) - 1$ and $r = \ell(2k+1)$. In particular, we note that $d = 2r - 1$. By Theorem 3.1, $\Psi^k(G) = \frac{2kn+n-d-1}{2k+1} = \frac{2kn+n-2r}{2k+1}$. Then by replacing each internal vertices on the path with a clique of size at least $\delta \geq 1$, we can obtain a graph with minimum degree δ achieving the upper bound. □

Theorem 3.4. *For $k \geq 1$, if G is a connected graph with girth g, then*

$$\Psi^k(G) \leq \frac{2kn + n - g}{2k + 1}.$$

Proof. If $g \leq 2k + 1$, then upper bound holds by using Proposition 3.2 and Corollary 3.3. Let $g \geq 2k + 2$, and C be a shortest cycle in G, of length g. We note that the distance between two vertices in C is exactly equal to the distance between them in G. Now we consider the following two cases, depending on the value of the girth of graphs.

Case 1. $2k + 2 \leq g \leq 4k + 2$. In this case, we show that $\Psi^k(G) \leq n - \lceil \frac{g}{2k+1} \rceil = n - 2$. Suppose to the contrary, that $\Psi^k(G) = n - 1$. Then, G contains a vertex v that is within distance k from every vertex of G. In particular, $d(u, v) \leq k$ for every vertex $u \in V(C)$. If $v \in V(C)$, then, since C is a shortest cycle in G, we note that $d_C(u, v) = d_G(u, v) \leq k$ for every vertex $u \in V(C)$. However, the lower bound condition on the girth, namely $g \geq 2k+2$, implies that no vertex on the cycle C is within distance k in C from every vertex of C, a contradiction. Therefore, $v \notin V(C)$. By Lemma 2.3, there exist two vertices $u, w \in V(C)$ such that a shortest (v, u)-path does not contain w and a shortest (v, w)-path does not contain u. We show that, we can choose u and w to be adjacent vertices on C. Let w be a vertex of C at maximum distance, say d_w, from v in G. Let w_1 and w_2 be the two neighbors of w on the cycle C. If $d_G(v, w_1) = d_w$, then we can take $u = w_1$, and the desired property (that a shortest (v, u)-path does not contain w and a shortest (v, w)-path does not contain u) holds. Hence we may assume that $d_G(v, w_1) \neq d_w$. By our choice of the vertex w, we note that $d_G(v, w_1) \leq d_w$, implying that $d_G(v, w_1) = d_w - 1$. Similarly, we may assume that $d_G(v, w_2) = d_w - 1$. Let P_w be a shortest (v, w)-path. At most one of w_1 and w_2 belong to the path P_w. Renaming w_1 and w_2, if necessary, we may assume that

w_1 does not belong to the path P_w. In this case, letting $u = w_1$ and letting P_u be a shortest (v, u)-path, we note that $w \notin V(P_u)$. As observed earlier, $u \notin V(P_w)$. This shows that u and w can indeed be chosen to be neighbors on C. Let x be the last vertex in common with the (v, u)-path P_u, and the (v, w)-path, P_w. Possibly, $x = v$. Then, the cycle obtained from the (x, u)-section of P_u by proceeding along the edge uw to w, and then the following (w, x)-section of P_w back to x, has length at most $d_G(v, u) + 1 + d_G(v, w) \leq 2k + 1$, contradicting the fact that the girth $g \geq 2k + 2$. Therefore, $\Psi^k(G) \leq n - 2$, as desired.

Case 2. $g \geq 4k + 3$. Let S be a maximum k-distance enclaveless set of G, and so $|S| = \Psi^k(G)$. Let $K = S \cap V(C)$ and let $L = S - V(C)$. Thus, $S = K \cup L$. If $L = \emptyset$, then $S = K$ and the set K is a k-distance enclaveless set of C, implying by Proposition 3.2, that $\Psi^k(G) = |S| = |K| \leq \Psi^k(C_g) = n - \lceil \frac{g}{2k+1} \rceil$, and the theorem holds. Hence we may assume that $|L| \geq 1$. We wish to show that $|K| + |L| = |S| \leq n - \lceil \frac{g}{2k+1} \rceil$. Suppose to the contrary that,

$$|K| \geq n - \lceil \frac{g}{1+2k} \rceil + 1 - |L|.$$

As observed earlier, the distance between two vertices in $V(C)$ is exactly the same in C as in G. This implies that each vertex of K (recall that $K \subseteq V(C)$) is within distance k from exactly $2k+1$ vertices of C. Thus, the set $B^k(K) \cap V(C)$ has at least $|K|(2k + 1)$ vertices where

$$|K|(2k + 1) \geq (n - \lceil \tfrac{g}{1+2k} \rceil + 1 - |L|)(2k + 1) \geq$$
$$(n - \tfrac{g+2k}{2k+1} + 1 - |L|)(2k + 1) = 2kn + n - g + 1 - |L|(2k + 1).$$

Thus, clearly we have $|K|(2k + 1) \geq n - g + 1 - |L|(2k + 1)$. Consequently, since $|V(C^c)| = n-g$, there are at most $-1+|L|(2k+1)$ vertices of $V(C)^c$ that do not belong to set $B^k(K)$, and so they must belong to set $B^k(L)$. Thus, by the Pigeonhole Principle, there is at least one vertex, say v, in L that $|B^k(\{v\}) \cap V(C)| \geq 2k$. By Lemma 2.3, there exist two vertices $u, w \in V(C)$ that are both $u, w \in B^k(\{v\})$ and such that a shortest (u, v)-path, P_u say, (from u to v) does not contain w and a shortest (w, v)-path, P_w say, (from w to v) does not contain u. Analogously as in the proof of Lemma 2.3, we can choose the vertex u to be a vertex of C at minimum distance from v in G. Thus, the vertex u is the only vertex on the cycle C that belongs to the path P_u. Combining the paths P_u and P_w produces a (u, w)-walk of length at most $d_G(u, v) + d_G(v, w) \leq 2k$, implying that $d_G(u, w) \leq 2k$. Since C is a shortest cycle in G, we therefore have that $d_C(u, w) = d_G(u, w) \leq 2k$. The cycle C yields two (w, u)-paths. Let P_{wu} be the (w, u)-path on the cycle C of shorter length (starting at w and ending at u). Thus, P_{wu}has length $d_C(u,w) \leq 2k$. Note that the path P_{wu} belongs entirely on the cycle C. Let $x \in V(C)$ be the last vertex in common with the (w,v)-path, P_w, and the (w,u)-path, P_{wu}. Possibly, $x = w$. However, note that $x \neq u$ since $u \notin V(P_w)$. Let y be the first vertex in common with the (x,v)-subsection of the path P_w and with the (u,v)-path P_u. Possibly, $y = v$. However, note that $y \neq x$ since $x \notin V(P_u)$ and $V(P_u) \cap V(C) = \{u\}$. Using the (x,u)-subsection of the path P_{wu}, the (x,y)-subsection of the path P_w, and the (u,y)-subsection of the path P_u produces a cycle in G of length at most $d_G(u,v) + d_G(w,v) + d_G(u,w) \leq k + k + 2k = 4k$, contradicting the fact that the girth $g \geq 4k+3$. Therefore, $\Psi^k(G) = |S| = |K| + |L|$, as desired. □

4. Direct Product Graphs

The direct product graph, $G \times H$, of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and with edges $(g_1,h_1)(g_2,h_2)$, where $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Let $A \subseteq V(G \times H)$. The projection of A onto G is defined as $P_G(A) = \{g \in V(G) : (g,h) \in A$ for some $h \in V(H)\}$. Similarly, the projection of A onto H is defined as $P_H(A) = \{h \in V(H) : (g,h) \in A$ for some $g \in V(G)\}$. For a detailed discussion on direct product graphs, we refer the reader to the handbook on graph products [2]. Recall that for every graph G, $\Psi^1(G) = \Psi(G)$.

Lemma 4.1. *Let G and H be connected graphs. If D is a k-distance enclaveless set of $G \times H$, then $P_G(D)$ is a k-distance enclaveless set of G and $P_H(D)$ is a k-distance enclaveless set of H.*

Proof. Let $D \subseteq V(G \times H)$ be a k-distance enclaveless set of $G \times H$. We firstly show that $P_G(D)$ is a k-distance enclaveless set of G. Or, equivalently, we have to show that $B^k(P_G(D)) = V(G) - P_G(D)$. If $g \in B^k(P_G(D))$, then we have clearly, $0 < d_G(g, P_G(D)) \leq k$. Thus, $g \notin P_G(D)$ and then $B^k(P_G(D)) \subseteq V(G) - P_G(D)$. Hence, we assume that $g \in V(G) - P_G(D)$. Let h be an arbitrary vertex in $V(H)$. Since $g \notin P_G(D)$, then $(g,h) \notin D$. However, the set D is a k-distance enclaveless set of $G \times H$, and so $(g,h) \in B^k(D)$; that is, $d_{G \times H}((g,h), D) \leq k$. Let $(g_0,h_0), (g_1,h_1), \ldots, (g_r,h_r)$ be a shortest path from (g,h) to D in $G \times H$, where $(g,h) = (g_0,h_0)$ and $(g_r,h_r) \in D$. By assumption, $1 \leq r \leq k$. For $i \in \{0, \ldots, r-1\}$, the vertices (g_i,h_i) and (g_{i+1},h_{i+1}) are adjacent in $G \times H$. Hence, by the definition of the direct product graph, the vertices g_i and g_{i+1} are adjacent in G, implying that $g_0g_1...g_r$ is a (g_0,g_r)-walk in G of length r. This in turn implies that there is a (g_0,g_r)-path in G of length r. Recall that $g = g_0$ and $1 \leq r \leq k$. Since $(g_r,h_r) \in D$, the vertex $g_r \in P_G(D)$. Hence, there is a path from g to a vertex of $P_G(D)$

in G of length at most k. Therefore, $g \in B^k(P_G(D))$. Analogously, the set $P_H(D)$ is a k-distance enclaveless set of H. □

Theorem 4.2. *If G and H are connected graphs of the orders n and m, respectively. Then*

$$\Psi^k(G \times H) \leq mn - n - m + \Psi^k(G) + \Psi^k(H) + 1.$$

Proof. Let $D \subseteq V(G \times H)$ be a maximum k-distance enclaveless set of $G \times H$. Suppose, to the contrary, that $|D| \geq mn - n - m + \Psi^k(G) + \Psi^k(H) + 2$. We will refer to this inequality as $(*)$. By Lemma 4.1, $P_G(D)$ is a k-distance enclaveless set of G and $P_H(D)$ is a k-distance enclaveless set of H. Therefore, we have that $|D| \leq n - |P_G(D)| \leq \Psi^k(G)$ and $|D| \leq m - |P_H(D)| \leq \Psi^k(H)$. If $\Psi^k(G) = n - 1$, then by $(*)$, $\Psi^k(H) \geq |D| \geq mn - m + 1 + \Psi^k(H)$, a contradiction. Therefore, $\Psi^k(G) \leq n - 2$. Analogously, $\Psi^k(H) \leq n - 2$. Recall that $n - |P_G(D)| \leq \Psi^k(G)$. We now remove vertices from the set $P_G(D)$ until we obtain a set, D_G say, of cardinality exactly $n - 1 - \Psi^k(G)$. Thus, D_G is a proper subset of $P_G(D)$ of cardinality $n - 1 - \Psi^k(G)$. Since D_G is not a k-distance enclaveless set of G, there exists a vertex $g \in V(G)$ such that $g \notin B^k(D_G)$; that is, $d_G(g, D_G) > k$. Let $D_G = \{g_1, \ldots, g_t\}$, where $t = n - 1 - \Psi^k(G) \geq 1$. For each $i \in [t]$, there exists a (not necessarily unique) vertex $h_i \in V(H)$ such that $(g_i, h_i) \in D$ (since $D_G \subseteq P_G(D)$). We now consider the set $D_0 = \{(g_1, h_1), \ldots, (g_t, h_t)\}$, and note that $D_0 \subset D$ and $|D_0| = n - 1 - \Psi^k(G)$. By (*), we note that

$$m - |P_H(D - D_0)| \geq |D - D_0| = |D| - |D_0| \geq$$
$$(mn - n - m + \Psi^k(G) + \Psi^k(H) + 2) - (n - 1 - \Psi^k(G))$$
$$= mn - 2n - m + 3 + 2\Psi^k(G) + \Psi^k(H) > \Psi^k(H).$$

Hence, there exists a vertex $h \in V(H)$ such that $h \notin B^k(P_H(D - D_0))$; that is, $d_H(h, P_H(D - D_0)) > k$. We now consider the vertex $(g, h) \in V(G \times H)$. Since D is a k-distance enclaveless set of $G \times H$, then there exists the vertex $(g^*, h^*) \in D$ such that $(g, h) \in B^k\{(g^*, h^*)\}$. A similar proof as the proof of Lemma 4.1 shows that $d_G(g, g^*) \leq k$ and $d_H(h, h^*) \leq k$. If $(g^*, h^*) \in D - D_0$, then $h^* \in P_H(D - D_0)$, implying that $d_H(h, P_H(D - D_0)) \leq d_H(h, h^*) \leq k$, a contradiction. Hence, $(g^*, h^*) \in D_0$. This implies that $g^* \in P_G(D_0) = D_G$. Thus, $d_G(g, D_G) \leq d_G(g, g^*) \leq k$, contradicting the fact that $d_G(g, D_G) > k$. Therefore, (*) inequality that $|D| \geq mn - n - m + \Psi^k(G) + \Psi^k(H) + 2$ must be false, and the result follows. □

5. Upper bound for k-distance enclaveless number of a tree

In this section we study the upper bound of k-distance enclaveless number of trees.

Theorem 5.1. *Let T be a tree of order $n(T) \geq k+1$ and with $n_1(T)$ leaves. Then*

$$kn_1(T) \geq 2k - 2kn(T) + (2k+1)\Psi^k(T).$$

Proof. We use induction on n, the order of a tree. The result is trivial for a tree of order $k+1$ due to $diam(T) \leq k$ or equivalently $\Psi^k(T) = n-1$. Let T be a tree of order $n > k+1$, $diam(T) \geq 2k+1$ and assume that $kn_1(T') \geq 2k - 2kn(T') + (2k+1)\Psi^k(T')$ for each tree T' with $k+1 < n(T') \leq n-1$. Let D be a maximum k-distance enclaveless set of T having property that, let $P = v_0v_1\cdots v_l$ be a longest path in T and let $T' = T - \{v_0\}$ be the subtree of T. Clearly, we have $l \geq 2k+1$. Without loss of generality we may assume that P is chosen in such a way that $d_{k,T}(v_k)$ is as large as possible. We consider two cases: $d_{k,T}(v_k) > k+1$ or $d_{k,T}(v_k) = k+1$.
Case 1. $d_{k,T}(v_k) > k+1$.

In T' we have $kn_1(T') \geq 2k - 2kn(T') + (2k+1)\Psi^k(T')$ (by induction), and as $n_1(T') = n_1(T) - 1$, $n(T') = n(T) - 1$ and $\Psi^k(T) = \Psi^k(T') + 1$, therefore, $k(n_1(T) - 1) \geq 2k - 2k(n(T) - 1) + (2k+1)(\Psi^k(T) - 1) = 2k - 2kn(T) + 2k + (2k+1)\Psi^k(T) - 2k - 1$ or equivalently, $kn_1(T) \geq 2k - 2kn(T) + (2k+1)\Psi^k(T) + k - 1 \geq 2k - 2kn(T) + (2k+1)\Psi^k(T)$ due to $k \geq 1$.

Case 2. If $d_{k,T}(v_k) = k+1$, we consider two subcases: $\Psi^k(T) < \Psi^k(T') + 1$ or $\Psi^k(T) = \Psi^k(T') + 1$. there are two subcases:

Subcase 2.1. If $\Psi^k(T) < \Psi^k(T') + 1$, then since clearly, $\Psi^k(T') \leq \Psi^k(T)$, we conclude $\Psi^k(T') = \Psi^k(T)$. By induction, $kn_1(T') \geq 2k - 2kn(T') + (2k+1)\Psi^k(T')$ and consequently $kn_1(T) \geq 2k - 2kn(T) + (2k+1)\Psi^k(T)$ as $n_1(T) = n_1(T')$, $n(T') = n(T) - 1$ and $\Psi^k(T') = \Psi^k(T)$.

Subcase 2.2. If $\Psi^k(T) = \Psi^k(T') + 1$, then $v_{k+1} \notin N_{k,T}(\Omega(T))$ (otherwise $D - \{v_k\}$ would be a k-distance enclaveless set of T and $1 + \Psi^k(T') > \Psi^k(T)$)and therefore $l \geq 2k+2$. By T_1 and T_2 we denote the subtrees of $T - v_{k+1}v_{k+2}$ to which belong vertices v_{k+2} and v_{k+1}, respectively. If $n(T_1) = k+1$, then certainly $kn_1(T_1) \geq 2k - 2kn(T_1) + (2k+1)\Psi^k(T_1)$. Thus assume that $n(T_1) \geq k+2$. Let Ω_2 denotes the set $\Omega(T_2) \cap \Omega(T)$ and let D_2 be a maximum k-distance enclaveless set of T_2 which does not contain v_{k+1}. Since $d_{k,T}(v_k) = k+1$, from the choice of P, it follows that all k-neighbours of v_{k+1} in T_2 are of degree $k+1$ and this implies $|\Omega_2| = |D_2|$. It is easy to observe that, $\Psi^k(T) = \Psi^k(T_1) + \Psi^k(T_2) = \Psi^k(T_1) + |D_2|$ and $n(T) = n(T_1) + |\Omega_2| + |D_2| + 1$. If v_{k+2} is an leaf of T_1, then we have $n_1(T) = n_1(T_1) + |\Omega_2| - 1$, otherwise $n_1(T) = n_1(T_1) + |\Omega_2| \geq n_1(T_1) + |\Omega_2| - 1$ as well. Now, since $n(T_1) \geq k+2$; we have by induction $kn_1(T_1) \geq 2k - 2kn(T_1) + (2k+1)\Psi^k(T_1)$. In both cases, for $n(T_1) = k+1$ and for $n(T_1) \geq k+2$ we get $2k - 2kn(T_1) + (2k+1)\Psi^k(T_1) \leq kn_1(T_1) \leq kn_1(T) - k|\Omega_2| + k$. Thus $2k - 2k(n(T) - |\Omega_2| - |D_2| - 1) + (2k+1)(\Psi^k(T) - |D_2|) \leq kn_1(T_1) \leq kn_1(T) - k|\Omega_2| + k$ and $2k - 2kn(T) + (2k+1)\Psi^k(T) \leq 2k - 2kn(T) + (2k+1)\Psi^k(T) + k(|\Omega_2| + 1) - |D_2| \leq kn_1(T)$. □

By $\Re$ we denote the family of all trees in which the distance between any two distinct leaves is equevalent to $2k$ modulo $2k+1$; i.e., a tree $T \in \Re$ if $d(x, y) \equiv 2k \pmod{2k+1}$ for two distinct vertices $x, y \in \Omega(T)$. The next lemma describes main properties of trees belonging to $\Re$.

Lemma 5.2. *If T is a tree belonging to the family $\Re$ and $\Psi^k(T) < n-1$, then there exists an edge xy in T such that both T_x and T_y belong to $\Re$, $\Psi^k(T) = \Psi^k(T_x) + \Psi^k(T_y) =$ and $n_1(T) = n_1(T_x) + n_1(T_y) - 2$.*

Proof. Let $T \in \Re$ with $\Psi^k(T) \leq n-2$ and let $P = v_0v_1 \dots v_l$ be a longest path in T. In addition, let D be a maximum k-distance enclaveless set of T containing the vertex v_k. Then $l \equiv 2k\ (mod\, 2k{+}1)$, $l \geq 4k+1$ and $v_k \in D$. We will show that $d(v_{k+1}) = d(v_{k+2}) = \dots = d(v_{3k}) = 2$. Suppose to the contrary that $N(v_i) - V(P) \neq \emptyset$ for some $i \in \{k+1, k+2, \dots, 3k\}$. Then there exists a leaf $u \in \Omega(T)$ such that $d(u, v_i) = d(u, P) > 0$. In order to derive a contradiction, we will compute the possible values for i. We have $d(u, v_i) = d(u, v_0) - d(v, v_0) = d(u, v_0) - i$ and $d(v_i, v_l) = d(v_0, v_l) - d(v_0, v_i) = d(v_0, v_l) - i$. It follows that $d(u, v_l) = d(u, v_i) + d(v_i, v_l) = d(u, v_0) + d(v_0, v_l) - 2i$. Since v_0, v_l and u are leaves and $T \in \Re$, it follows that $2i \equiv 2k\ (mod\ (2k+1))$. The latter together with $k+1 \leq i \leq 3k$ leads immediately to a contradiction. It follows that $d(v_{k+1}) = d(v_{k+2}) = \dots = d(v_{3k}) = 2$ which means we can choose D such that $v_{3k+l} \in D$. Let us remove the edge $xy = v_{2k}v_{2k+l}$ from T. Then $n_1(T) = n_1(T_x) + n(T_y) - 2, \Psi^k(T_x) = n-1$ and $D - v_k$ is a maximum k-distance enclaveless set of T_y. Thus, $\Psi^k(T_x) + \Psi^k(T_y) = \Psi^k(T)$. Since $T_x = SS_t^{k-1}$ is a star with all edges $(k-1)$-times subdivided, $T_x \in \Re$. As $T \in \Re$, we have $d(v_0, v) = 2k\ (mod\ (2k+1))$ for every vertex $v_0 \neq v \in \Omega(T)$. Since $d(v_0, v_{2k+1}) = 2k+1$, we obtain $d(v_{2k+1}, v) = 2k\ (mod\ (2k+1))$ for every vertex $v_{2k+1} \neq v \in \Omega(T_y)$ and consequently, $T_y \in \Re$. This completes the proof. □

Using Lemmma 5.2, we will now characterize the class of trees T which fulfill the equality $kn_1(T) = 2k - 2kn(T) + (2k+1)\Psi^k(T)$.

Theorem 5.3. *If T is a tree, then $kn_1(T) = 2k - 2kn(T) + (2k+1)\Psi^k(T)$ if and only if T belongs to $\Re$.*

Proof. Suppose first that $T \in \Re$. If $\Psi^k(T) = n - 1$, then $T = SS_t^{k-1}$ is a star with each edge $(k - l)$-times subdivided and $kn_1(T) = 2k - 2kn(T) + (2k + 1)\Psi^k(T)$ is obvious. Assume now that $\Psi^k(T) \leq n - 2$ and that $kn_1(T') = 2k - 2kn(T') + (2k+1)\Psi^k(T')$ for every tree $T' \in \Re$ with $\Psi^k(T) < \Psi^k(T') + 1$. According to Lemmma 5.2, there exists an edge xy in T such that $T_x, T_y \in \Re$, $\Psi^k(T) = \Psi^k(T_x) + \Psi^k(T_y)$ and $n_1(T) = n_1(T_x) + n_1(T_y) - 2$. By the induction hypothesis, $kn_1(T_x) = 2k - 2kn(T_x) + (2k + 1)\Psi^k(T_x)$ and $kn_1(T_y) = 2k - 2kn(T_y) + (2k + 1)\Psi^k(T_y)$. By adding these equalities we finally conclude that $kn_1(T) = 2k - 2kn(T) + (2k+1)\Psi^k(T)$. Suppose second that T fulfills the equality $kn_1(T) = 2k - 2kn(T) + (2k + 1)\Psi^k(T)$. If $\Psi^k(T) = n(T) - 1$, then the equality yields $kn_1(T) = n(T) - 1$. This together with $diam(T) \leq 2k$ implies that $T = SS_t^{k-1}$ is a star with each edge $(k-1)$-times subdivided and $T \in \Re$ is obvious. Now let T be a tree with $\Psi^k(T) < n - 1$ that fulfills the equality $kn_1(T) = 2k - 2kn(T) + (2k + 1)\Psi^k(T)$ and assume that $T' \in \Re$ for all trees T' with $\Psi^k(T) < \Psi^k(T') + 1$ and $kn_1(T') = 2k - 2kn(T') + (2k + 1)\Psi^k(T')$. According to Lemmma 5.2 there exists an edge xy in T such that $\Psi^k(T) = \Psi^k(T_x) + \Psi^k(T_y)$. Since $kn_1(T) = 2k - 2kn(T) + (2k+1)\Psi^k(T)$, it follows that $n_1(T) = n_1(T_x) + n_1(T_y) - 2$, $kn_1(T_x) = 2k - 2kn(T_x) + (2k+1)\Psi^k(T_x)$ and $kn_1(T_y) = 2k - 2kn(T_y) + (2k + 1)\Psi^k(T_y)$. Note that this means that T arises from T_x and T_y by adding the edge xy which joins the leaves x and y of T_x and T_y, respectively. In addition, we conclude that $T_x, T_y \in \Re$ by the induction hypothesis. The latter together with the observation before implies that $T \in \Re$ which completes the proof of this theorem. $\square$

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