Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 2676-7260
CJMS. **10**(2)(2021), 195-209

One Dimensional Dirac Operators on Time Scales

Bilender P. Allahverdiev [1] and Hüseyin Tuna [2]
[1] Department of Mathematics, Süleyman Demirel University, 32260 Isparta, Turkey
[2] Department of Mathematics, Mehmet Akif Ersoy University, 15030 Burdur, Turkey

Abstract. In this work, we study certain spectral properties of the one dimensional Dirac systems on time scales, such as formally self-adjointness, orthogonality of eigenfunctions, Green's function, the existence of a countable sequence of eigenvalues. Later, we give an expansion formula in eigenfunctions for Dirac operators on time scales. These results could provide an important contribution to the spectral theory of such operators on time scales.

Keywords: Dirac operator, time scales, self-adjoint operator, eigenvalue, Green function, eigenfunction expansion.

2000 Mathematics subject classification: 34N05, 34L10; Secondary 34L05.

1. Introduction

The study of dynamic equations on time scales is one of the new areas in mathematics. The first results in this area were obtained by Hilger [12]. Time scale calculus unites the studies of differential and difference equations. The study of time scales has several important applications, e.g. in the study of neural networks, heat transfer, insect population models, and epidemic models [1]. We refer the reader to consult the references [5], [8], [7], [11], [9], [15].

[1]Corresponding author:Hüseyin Tuna hustuna@gmail.com
Received: 09 May 2020
Revised: 10 July 2020
Accepted: 11 July 2020

Now let us consider the system

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{dy\left(x\right)}{dx} + B\left(x\right) y\left(x\right) = \lambda y\left(x\right), \ x \in [a, b], \tag{1.1}$$

where λ is a complex spectral parameter and

$$y\left(x\right) = \begin{pmatrix} y_1\left(x\right) \\ y_2\left(x\right) \end{pmatrix}, \ B\left(x\right) = \begin{pmatrix} p\left(x\right) & 0 \\ 0 & r\left(x\right) \end{pmatrix}$$

(for almost all $x \in [a, b]$); the entries of the matrix $B\left(x\right)$ are real-valued, Lebesgue measurable and integrable functions on $[a, b]$. This system is called the *Dirac system* in the literature. It is known that the system (1.1) describes a relativistic electron in the electrostatic field (see [20]).

Dirac operators are in the class of the most important operators in physics since these operators formulate the fundamental physics of realistic quantum mechanics. They predict the existence of antimatter and describe the electron spin (see [19]).

In Equation (1.1), if the differential operator is replaced with the Δ-difference operator (see Definition 2.2), then we obtain the Dirac system on the time scale. There is not much research about the Dirac system on time scales ([10, 13, 2, 3, 4]). Hence our study may fill an important gap in this subject. In [10], the authors studied an eigenvalue problem for the Dirac system with separated boundary conditions on an arbitrary time scale. In [13], the author studied the non-autonomous linear Dirac equation on a time scale containing the important discrete, continuous, and quantum time scales. A representation of the solutions is established via approximate solutions in terms of unknown phase functions with the error estimates. In [2, 3], Allahverdiev et al. proved the existence of a spectral function for one dimensional singular Dirac system on time scales. In [6], Anderson studied non-self-adjoint Hamiltonian systems on Sturmian time scales by defining Weyl–Sims sets, which replace the classical Weyl circles with a matrix-valued M-function on suitable cone-shaped domains in the complex plane. Furthermore, the author characterized the realizations of the corresponding dynamic operator and its adjoint, and then constructed their resolvents.

Now the organization of our work is as follows. In the second section, some preliminary concepts related to time scales are presented for the convenience of the reader. In the third section, we formulate a self-adjoint Dirac operator. In the last section, we construct the associated Green function of the Dirac system on the time scale.

2. Preliminaries

First, we note that, for more details in this section, we refer to [8, 9]. Now let $\mathbb{T}$ be a Sturmian time scale. The *forward jump operator* σ :

$\mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \text{ where } t \in \mathbb{T}$$

and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \text{ where } t \in \mathbb{T}.$$

Moreover, the *graininess operators* $\mu_\sigma : \mathbb{T} \to [0, \infty)$ and $\mu_\rho : \mathbb{T} \to (-\infty, 0]$ are defined by

$$\mu_\sigma(t) = \sigma(t) - t$$

and

$$\mu_\rho(t) = \rho(t) - t,$$

respectively.

Definition 2.1. A point $t \in \mathbb{T}$ is left scattered if $\mu_\rho(t) \neq 0$, and left dense if $\mu_\rho(t) = 0$. A point $t \in \mathbb{T}$ is right scattered if $\mu_\sigma(t) \neq 0$ and right dense if $\mu_\sigma(t) = 0$.

Now let us consider the sets $\mathbb{T}^k$, $\mathbb{T}_k$ and $\mathbb{T}^*$ which are derived from the time scale $\mathbb{T}$ as follows. If $\mathbb{T}$ has a left scattered maximum t_1, then $\mathbb{T}^k = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum t_2, then $\mathbb{T}_k = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. Finally, $\mathbb{T}^* = \mathbb{T}^k \cap \mathbb{T}_k$.

Definition 2.2. A function f on $\mathbb{T}$ is said to be Δ-differentiable at some point $t \in \mathbb{T}$ if there is a number $f^\Delta(t)$ such that for every $\varepsilon > 0$ there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ where } s \in U.$$

Analogously one may define the notion of ∇-differentiability of some function using the backward jump ρ. One can show that (see [11])

$$f^\Delta(t) = f^\nabla(\sigma(t)),\ f^\nabla(t) = f^\Delta(\rho(t))$$

for continuously differentiable functions.

Let $f : \mathbb{T} \to \mathbb{R}$ be a function, and $a, b \in \mathbb{T}$. If there exists a function $F : \mathbb{T} \to \mathbb{R}$ such that $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$, then F is a Δ-antiderivative of f. In this case the integral is given by the formula

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \text{ for } a, b \in \mathbb{T}.$$

Analogously one may define the notion of ∇-antiderivative of some function.

Let $L^2_\Delta(\mathbb{T}^*)$ be the space of all functions defined on $\mathbb{T}^*$ such that

$$\|f\| := \left(\int_a^b |f(t)|^2\,\Delta t\right)^{1/2} < \infty.$$

The space $L_{\Delta}^{2}(\mathbb{T}^{*})$ is a Hilbert space with the inner product (see [18])

$$(f,g) := \int_{a}^{b} f(t)\overline{g(t)}\Delta t,\ f,g \in L_{\Delta}^{2}(\mathbb{T}^{*}) .$$

Let a and b be fixed points in $\mathbb{T}$ with $a \leq b$ and $a \in \mathbb{T}_{k},\ b \in \mathbb{T}^{k}$.

Now, we introduce a convenient Hilbert space $\mathcal{H} := L_{\Delta}^{2}(\mathbb{T}^{*}; E)$ ($E := \mathbb{C}^{2}$) of vector-valued functions, by using the inner product

$$\langle f,g\rangle_{\mathcal{H}} := \int_{a}^{b} (f(x), g(x))_{E}\Delta t.$$

Let us consider the one dimensional Dirac systems on time scales:

$$\begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2^{\rho} \end{pmatrix} + \begin{pmatrix} p(t) & 0 \\ 0 & r(t) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

or

$$l(y) = \lambda y,\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ t \in \mathbb{T}^{*}, \tag{2.1}$$

$$l(y) := \begin{cases} -\Delta y_2^{\rho} + p(t)\, y_1 \\ \Delta y_1 + r(t)\, y_2, \end{cases}$$

where $\Delta f(t) = f^{\Delta}(t)$, $y_2^{\rho}(t) = y_2(\rho(t))$, λ is a complex eigenvalue parameter, $p(.)$ and $r(.)$ are real-valued functions defined on $\mathbb{T}^{*}$ and $p(.), r(.) \in L_{\Delta}^{1}(\mathbb{T}^{*})$.

Theorem 2.3. *For $c_1, c_2 \in \mathbb{C}$, Equation (2.1) has a unique solution $\Psi(t,\lambda) = \begin{pmatrix} \Psi_1(t,\lambda) \\ \Psi_2(t,\lambda) \end{pmatrix}$ in $L_{\Delta}^{2}(\mathbb{T}^{*}; E)$ which satisfies*

$$\Psi_1(t_0,\lambda) = c_1, \Psi_2(t_0,\lambda) = c_1, \lambda \in \mathbb{C}.$$

Proof. See the proof of Theorem 2.1 in [6]. □

3. Self-adjoint Dirac operator on time scales

In this section, we formulate a self-adjoint one dimensional Dirac operator in $\mathcal{H}$. We also give some spectral properties of these operators.

Now we consider the Dirac system on $\mathbb{T}^{*}$ given by

$$l(y) := \begin{cases} -\Delta y_2^{\rho} + p(t)\, y_1 \\ \Delta y_1 + r(t)\, y_2, \end{cases}$$

$$l(y) = \lambda y,\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ t \in \mathbb{T}^{*}, \tag{3.1}$$

with the boundary conditions

$$L_1(y) := k_{11} y_1(a) + k_{12} y_2^{\rho}(a) = 0, \tag{3.2}$$

$$L_2(y) := k_{21} y_1(b) + k_{22} y_2^{\rho}(b) = 0, \tag{3.3}$$

where λ is a complex eigenvalue parameter, $\{k_{ij}\}_{i,j=1,2}$ are arbitrary real numbers such that $k_{i1}^2 + k_{i2}^2 \neq 0$ $(i = 1, 2)$, $p(.)$ and $r(.)$ are real-valued functions defined on $\mathbb{T}^*$ and $p, r \in L^1_\Delta(\mathbb{T}^*)$.

Theorem 3.1. *The boundary-value problem defined by (3.1)-(3.3) is formally self-adjoint on $\mathcal{H}$.*

Proof. First, we will prove the Green's formula. Let $y(.), z(.) \in \mathcal{H}$. Then, we have

$$\begin{aligned}
&\langle l(y), z\rangle_{\mathcal{H}} - \langle y, l(z)\rangle_{\mathcal{H}} \\
&= \int_a^b \left(-\Delta y_2^\rho + p(t) y_1\right) \overline{z_1} \Delta t + \int_a^b \left(\Delta y_1 + r(t) y_2\right) \overline{z_2} \Delta t \\
&- \int_a^b y_1 \overline{\left(-\Delta z_2^\rho + p(t) z_1\right)} \Delta t - \int_a^b y_2 \overline{\left(\Delta z_1 + r(t) z_2\right)} \Delta t \\
&= -\int_a^b \left[\left(\Delta y_2^\rho\right) \overline{z_1} + y_2 \overline{\left(\Delta z_1\right)}\right] \Delta t + \int_a^b \left[\left(\Delta y_1\right) \overline{z_2} + y_1 \overline{\left(\Delta z_2^\rho\right)}\right] \Delta t.
\end{aligned}$$

Since

$$\begin{aligned}
\Delta\left(\overline{z_1(t)} y_2^\rho(t)\right) &= \overline{z_1(t)} \left(\Delta y_2^\rho(t)\right) + \left(y_2^\rho(t)\right)^\sigma \overline{\left(\Delta z_1(t)\right)} \\
&= \Delta y_2^\rho(t) \overline{z_1(t)} + y_2(t) \overline{\left(\Delta z_1(t)\right)}
\end{aligned}$$

and

$$\begin{aligned}
\Delta\left(\overline{z_2^\rho(t)} y_1(t)\right) &= \overline{\left(\Delta z_2^\rho(t)\right)} y_1(t) + \overline{\left(z_2^\rho(t)\right)^\sigma} \left(\Delta y_1(t)\right) \\
&= \overline{\left(\Delta z_2^\rho(t)\right)} y_1(t) + \overline{z_2(t)} \Delta y_1(t),
\end{aligned}$$

we get

$$\begin{aligned}
\langle l(y), z\rangle_{\mathcal{H}} - \langle y, l(z)\rangle_{\mathcal{H}} &= -\int_a^b \Delta\left(\overline{z_1(t)} y_2^\rho(t)\right) \Delta t + \int_a^b \Delta\left(y_1(t) \overline{z_2^\rho(t)}\right) \Delta t \\
&= \int_a^b \Delta\left[y_1(t) \overline{z_2^\rho(t)} - \overline{z_1(t)} y_2^\rho(t)\right] \Delta t.
\end{aligned}$$

Let us define $[y, z]_t := y_1(t) \overline{z_2^\rho(t)} - \overline{z_1(t)} y_2^\rho(t)$. Hence we obtain

$$\langle l(y), z\rangle_{\mathcal{H}} - \langle y, l(z)\rangle_{\mathcal{H}} = [y, z]_b - [y, z]_a. \tag{3.4}$$

We proceed to show that the operator L is formally self-adjoint. Let $y(.), z(.) \in \mathcal{H}$. Then, we have

$$\langle l(y), z\rangle_{\mathcal{H}} - \langle y, l(z)\rangle_{\mathcal{H}} = [y, z]_b - [y, z]_a.$$

From the boundary conditions (3.2) and (3.3), we get $[y, z]_b = 0$ and $[y, z]_a = 0$. Consequently,

$$\langle l(y), z\rangle_{\mathcal{H}} = \langle y, l(z)\rangle_{\mathcal{H}}. \tag{3.5}$$

This completes the proof. □

Corollary 3.2. *All eigenvalues of the problem defined by (3.1)-(3.3) are real. Further, if μ_1 and μ_2 are two different eigenvalues of the problem defined by (3.1)-(3.3), then the corresponding eigenfunctions v_1 and v_2 are orthogonal.*

Now let $y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, $z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \in \mathcal{H}$. Then, we define the Wronskian of $y(t)$ and $z(t)$ by

$$W(y,z)(t) = y_1(t)\, z_2^{\rho}(t) - z_1(t)\, y_2^{\rho}(t).$$

Theorem 3.3. *The Wronskian of any solution of Equation (3.1) is independent of t.*

Proof. Let $y(t)$ and $z(t)$ be two solutions of Equation (3.1). By Green's formula (3.4), we have

$$\langle l(y), z\rangle_{\mathcal{H}_t} - \langle y, l(z)\rangle_{\mathcal{H}_t} = [y,z]_t - [y,z]_a,$$

where $\mathcal{H}_t := L^2_{\Delta}(\mathbb{T}^*_t; E), \mathbb{T}^*_t := \mathbb{T}^* \cap [a,t]$. Since $l(y) = \lambda y$ and $l(z) = \lambda z$, we have

$$\langle \lambda y, z\rangle_{\mathcal{H}_t} - \langle y, \lambda z\rangle_{\mathcal{H}_t} = [y,z]_t - [y,z]_a,$$

$$\left(\lambda - \overline{\lambda}\right) \langle y, z\rangle_{\mathcal{H}_t} = [y,z]_t - [y,z]_a.$$

Since $\lambda \in \mathbb{R}$, we have $[y,z]_t = [y,z]_a = W(y,\overline{z})(a)$, i.e., the Wronskian is independent of t. □

Corollary 3.4. *If $y(t)$ and $z(t)$ are both the solutions of Equation (3.1), then either $W(y,z)(t) = 0$ or $W(y,z)(t) \neq 0$ for all $t \in \mathbb{T}^*$.*

Theorem 3.5. *Any two solutions of Equation (3.1) are linearly dependent if and only if their Wronskian is zero.*

Proof. Let $y(t)$ and $z(t)$ be two linearly dependent solutions of Equation (3.1). Then, there exists a constant $c > 0$ such that $y(t) = c\, z(t)$. Hence

$$W(y,z)(t) = \begin{vmatrix} y_1(t) & y_2^{\rho}(t) \\ z_1(t) & z_2^{\rho}(t) \end{vmatrix} = \begin{vmatrix} cz_1(t) & cz_2^{\rho}(t) \\ z_1(t) & z_2^{\rho}(t) \end{vmatrix} = 0.$$

Conversely, if $W(y,z)(t) = 0$, then $y(t) = c\ z(t)$, i.e., $y(t)$ and $z(t)$ are linearly dependent. □

Lemma 3.6. *All eigenvalues of the problem defined by (3.1)-(3.3) are simple from the geometric point of view.*

Proof. Let μ be an eigenvalue with eigenfunctions $z_1(t)$ and $z_2(t)$. From the boundary condition (3.2), we have

$$W(z_1, z_2)(a) = z_{11}(a)\, z_{22}^{\rho}(a) - z_{12}(a)\, z_{21}^{\rho}(a) = 0.$$

Then, the set $\{z_1(t),\ z_2(t)\}$ is linearly dependent. □

Now, our next goal is to determine the eigenvalues and the corresponding eigenfunctions. Let

$$\phi_1(t,\lambda) = \begin{pmatrix} \phi_{11}(t,\lambda) \\ \phi_{12}(t,\lambda) \end{pmatrix} \text{ and } \phi_2(t,\lambda) = \begin{pmatrix} \phi_{21}(t,\lambda) \\ \phi_{22}(t,\lambda) \end{pmatrix}$$

be linearly independent solutions of (3.1) which satisfy the initial conditions

$$\phi_{ij}(a,\lambda) = \delta_{ij}, \ i,j = 1,2, \ \lambda \in \mathbb{C}.$$

Then, every solution of Equation (3.1) has the form

$$y(t,\lambda) = K_1\phi_1(t,\lambda) + K_2\phi_2(t,\lambda),$$

where K_1 and K_2 do not depend on t. If we can find a nontrivial solution of the linear system

$$\begin{aligned} K_1L_1(\phi_1) + K_2L_1(\phi_2) = 0, \\ K_1L_2(\phi_1) + K_2L_2(\phi_2) = 0, \end{aligned}$$

then the solution $y(t,\lambda)$ is called an *eigenfunction* of (3.1). Hence $\lambda \in \mathbb{R}$ is an eigenvalue if and only if

$$\omega(\lambda) = \begin{vmatrix} L_1(\phi_1) & L_1(\phi_2) \\ L_2(\phi_1) & L_2(\phi_2) \end{vmatrix} = 0.$$

The function $\omega(\lambda)$ is called the *characteristic determinant* associated with the Dirac system defined by (3.1)-(3.3). The eigenvalues of the problem defined by (3.1)-(3.3) are the zeros of the function $\omega(\lambda)$. On the other hand, $\omega(\lambda)$ is an entire function in λ because $\phi_1(t,\lambda)$ and $\phi_2(t,\lambda)$ are entire in λ for each fixed $t \in \mathbb{T}^*$. Hence the eigenvalues of the Dirac system defined by (3.1)-(3.3) are at most countable with no finite limit points.

Theorem 3.7. *All eigenvalues μ_n of the problem defined by (3.1)-(3.3) are simple zeros of the function $\omega(\lambda)$.*

Proof. Let us define

$$\theta_1(t,\lambda) = \begin{pmatrix} \theta_{11}(t,\lambda) \\ \theta_{12}(t,\lambda) \end{pmatrix} \text{ and } \theta_2(t,\lambda) = \begin{pmatrix} \theta_{21}(t,\lambda) \\ \theta_{22}(t,\lambda) \end{pmatrix}$$

by

$$\begin{aligned} \theta_1(t,\lambda) &= L_1(\phi_2)\phi_1(x,\lambda) - L_1(\phi_1)\phi_2(t,\lambda), \\ \theta_2(t,\lambda) &= L_2(\phi_2)\phi_1(x,\lambda) - L_2(\phi_1)\phi_2(t,\lambda). \end{aligned} \tag{3.6}$$

Then, $\theta_1(t,\lambda)$ and $\theta_2(t,\lambda)$ are solutions of (3.1) such that

$$\theta_1(a,\lambda) = \begin{pmatrix} k_{12} \\ -k_{11} \end{pmatrix} \text{ and } \theta_2(b,\lambda) = \begin{pmatrix} k_{22} \\ -k_{21} \end{pmatrix}. \tag{3.7}$$

On the other hand, we have

$$
\begin{aligned}
& W\left(\theta_1(t,\lambda),\theta_2(t,\lambda)\right) \\
& =\theta_{11}(t,\lambda)\,\theta_{22}^{\rho}(t,\lambda)-\theta_{12}^{\rho}(t,\lambda)\,\theta_{21}(t,\lambda) \\
& =\left(L_1(\phi_2)\,\phi_{11}(t,\lambda)-L_1(\phi_1)\,\phi_{21}(t,\lambda)\right) \\
& \times\left(L_2(\phi_2)\,\phi_{12}^{\rho}(t,\lambda)-L_2(\phi_1)\,\phi_{22}^{\rho}(t,\lambda)\right) \\
& -\left(L_1(\phi_2)\,\phi_{12}^{\rho}(t,\lambda)-L_1(\phi_1)\,\phi_{22}^{\rho}(t,\lambda)\right) \\
& \times\left(L_2(\phi_2)\,\phi_{11}(t,\lambda)-L_2(\phi_1)\,\phi_{21}(t,\lambda)\right) \\
& =L_1(\phi_2)\,L_2(\phi_1)\left(-\phi_{11}(t,\lambda)\,\phi_{22}^{\rho}(t,\lambda)+\phi_{12}^{\rho}(t,\lambda)\,\phi_{21}(t,\lambda)\right) \\
& +L_1(\phi_1)\,L_2(\phi_2)\left(\phi_{11}(t,\lambda)\,\phi_{22}^{\rho}(t,\lambda)-\phi_{12}^{\rho}(t,\lambda)\,\phi_{21}(t,\lambda)\right) \\
& =\left(\phi_{11}(t,\lambda)\,\phi_{22}^{\rho}(t,\lambda)-\phi_{12}^{\rho}(t,\lambda)\,\phi_{21}(t,\lambda)\right) \\
& \times\left(L_1(\phi_1)\,L_2(\phi_2)-L_1(\phi_2)\,L_2(\phi_1)\right) \\
& =W\left(\phi_1(t,\lambda),\phi_2(t,\lambda)\right)\omega(\lambda)=\omega(\lambda)\,. \qquad (3.8)
\end{aligned}
$$

Let λ_0 be an eigenvalue of the problem defined by (3.1)-(3.3). Since λ_0 is a real number, $\theta_i(t,\lambda_0)$ $(i=1,2)$ can be taken to be real-valued. Then, by (3.8), $\theta_1(t,\lambda_0)$ and $\theta_2(t,\lambda_0)$ are linearly dependent eigenfunctions. Hence there exists a nonzero constant η_0 such that

$$\theta_2(t,\lambda_0)=\eta_0\theta_1(t,\lambda_0)\,.$$

From (3.6) and (3.7), we have

$$\theta_{21}(a,\lambda_0)=\eta_0 k_{12}=\eta_0\theta_{11}(a,\lambda)\,,\ \ \theta_{22}(a,\lambda_0)=-\eta_0 k_{11}=-\eta_0\theta_{12}(a,\lambda)\,. \qquad (3.9)$$

If we take $y(t)=\theta_2(t,\lambda)$ and $z(t)=\theta_2(t,\lambda_0)$ in (3.4), then we get

$$
\begin{aligned}
& (\lambda-\lambda_0)\int_a^b \theta_2(t,\lambda)\,\theta_2(t,\lambda_0)\,\Delta t \\
& =-\left(\theta_{21}(a,\lambda)\,\theta_{22}(a,\lambda_0)-\theta_{21}(a,\lambda_0)\,\theta_{22}(a,\lambda)\right) \\
& =-\left(\theta_{21}(a,\lambda)\left(-\eta_0\theta_{12}(a,\lambda)\right)-\eta_0\theta_{11}(a,\lambda)\,\theta_{22}(a,\lambda)\right) \\
& =\eta_0\left(\theta_{11}(a,\lambda)\,\theta_{22}(a,\lambda)-\theta_{12}(a,\lambda)\,\theta_{21}(a,\lambda)\right) \\
& =\eta_0 W\left(\theta_1(t,\lambda),\theta_2(t,\lambda)\right)=\eta_0\omega(\lambda)\,.
\end{aligned}
$$

Since $\omega(\lambda)$ is an entire function in λ, we have

$$\frac{d}{d\lambda}\omega(\lambda)\,|_{\lambda_0}:=\lim_{\lambda\to\lambda_0}\frac{\omega(\lambda)}{\lambda-\lambda_0}=\frac{1}{\eta_0}\int_a^b\theta_2^2(t,\lambda_0)\,\Delta t\neq 0.$$

Consequently, λ_0 is a simple zero of $\omega(\lambda)$. □

4. Green's function and eigenfunction expansion formula

In this section, we will investigate the solution of the nonhomogeneous system

$$-\Delta y_2^{\rho}+\{-\lambda+p(t)\}\,y_1=f_1(t), \tag{4.1}$$

$$\Delta y_1+\{-\lambda+r(t)\}\,y_2=f_2(t), \tag{4.2}$$

where $t\in\mathbb{T}^*$, which fulfills the boundary conditions

$$k_{11}y_1(a)+k_{12}y_2^{\rho}(a)=0,\ k_{11}^2+k_{12}^2\neq 0, \tag{4.3}$$

$$k_{21}y_1(b)+k_{22}y_2^{\rho}(b)=0,\ k_{21}^2+k_{22}^2\neq 0, \tag{4.4}$$

and

$$f(.)=\begin{pmatrix} f_1(.)\\ f_2(.)\end{pmatrix}\in\mathcal{H}.$$

For this, we construct the Green's function of the nonhomogeneous system (4.1)-(4.4). We also obtain an eigenfunction expansion for the above system.

Theorem 4.1. *If λ is not an eigenvalue of the problem defined by (3.1)-(3.3), then the nonhomogeneous system (4.1)-(4.4) is solvable for any vector-valued function $f(.)$. Conversely if λ is an eigenvalue of the problem defined by (3.1)-(3.3), then the nonhomogeneous system (4.1)-(4.4) is, generally unsolvable.*

Proof. Let us define

$$G(t,s,\lambda)=\begin{cases}\frac{\theta_2(t,\lambda)\theta_1^T(s,\lambda)}{\omega(\lambda)}, & a\le t\le s\\ \frac{\theta_1(t,\lambda)\theta_2^T(s,\lambda)}{\omega(\lambda)}, & s<t\le b\end{cases} \tag{4.5}$$

which is called the *Green's matrix.* We will show that the function

$$y(t,\lambda)=\int_a^b\left(G(t,s,\lambda),f(s)\right)_E\Delta s \tag{4.6}$$

is the solution of the nonhomogeneous system (4.1)-(4.4).

By definition of the Green's matrix, we have

$$G(t,s,\lambda)=\begin{cases}\frac{1}{\omega(\lambda)}\begin{pmatrix}\theta_{21}(t,\lambda)\theta_{11}(s,\lambda) & \theta_{21}(t,\lambda)\theta_{12}(s,\lambda)\\ \theta_{22}(t,\lambda)\theta_{11}(s,\lambda) & \theta_{22}(t,\lambda)\theta_{12}(s,\lambda)\end{pmatrix}, & \\ \qquad a\le t\le s, & \\ \frac{1}{\omega(\lambda)}\begin{pmatrix}\theta_{11}(t,\lambda)\theta_{21}(s,\lambda) & \theta_{11}(t,\lambda)\theta_{22}(s,\lambda)\\ \theta_{12}(t,\lambda)\theta_{21}(s,\lambda) & \theta_{12}(t,\lambda)\theta_{22}(s,\lambda)\end{pmatrix}, & \\ \qquad s<t\le b. & \end{cases}$$

From (4.6), we have

$$y_1(t,\lambda) = \frac{1}{\omega(\lambda)}\theta_{21}(t,\lambda)\int_a^t (\theta_{11}(s,\lambda) f_1(s) + \theta_{12}(s,\lambda) f_2(s))\Delta s$$
$$\frac{1}{\omega(\lambda)}\theta_{11}(t,\lambda)\int_t^b (\theta_{21}(s,\lambda) f_1(s) + \theta_{22}(s,\lambda) f_2(s))\Delta s, \tag{4.7}$$

$$y_2(t,\lambda) = \frac{1}{\omega(\lambda)}\theta_{22}(t,\lambda)\int_a^t (\theta_{11}(s,\lambda) f_1(s) + \theta_{12}(s,\lambda) f_2(s))\Delta s$$
$$\frac{1}{\omega(\lambda)}\theta_{12}(t,\lambda)\int_t^b (\theta_{21}(s,\lambda) f_1(s) + \theta_{22}(s,\lambda) f_2(s))\Delta s. \tag{4.8}$$

From (4.7), it follows that

$$\begin{aligned}
&y_1^{\Delta}(t,\lambda)\\
&= \frac{1}{\omega(\lambda)}\theta_{21}^{\Delta}(t,\lambda)\int_a^t (\theta_{11}(s,\lambda) f_1(s) + \theta_{12}(s,\lambda) f_2(s))\Delta s\\
&+ \frac{1}{\omega(\lambda)}\theta_{11}^{\Delta}(t,\lambda)\int_t^b (\theta_{21}(s,\lambda) f_1(s) + \theta_{22}(s,\lambda) f_2(s))\Delta s\\
&+ \frac{1}{\omega(\lambda)}W(\theta_1,\theta_2) f_2(t)\\
&= -\frac{1}{\omega(\lambda)}\{-\lambda + r(t)\}\theta_{22}(t,\lambda)\int_a^t (\theta_{11}(s,\lambda) f_1(s) + \theta_{12}(s,\lambda) f_2(s))\Delta s\\
&- \frac{1}{\omega(\lambda)}\{-\lambda + r(t)\}\theta_{12}(t,\lambda)\int_t^b (\theta_{21}(s,\lambda) f_1(s) + \theta_{22}(s,\lambda) f_2(s))\Delta s\\
&+ f_2(t)\\
&= -\{-\lambda + r(t)\}\frac{1}{\omega(\lambda)}\theta_{22}(t,\lambda)\int_a^t (\theta_{11}(s,\lambda) f_1(s) + \theta_{12}(s,\lambda) f_2(s))\Delta s\\
&- \{-\lambda + r(t)\}\frac{1}{\omega(\lambda)}\theta_{12}(t,\lambda)\int_x^b (\theta_{21}(s,\lambda) f_1(s) + \theta_{22}(s,\lambda) f_2(s))\Delta s\\
&+ f_2(t) = -\{-\lambda + r(t)\} y_2(t) + f_2(t).
\end{aligned}$$

The validity of (4.1) is proved similarly. Hence the function $y(t,\lambda)$ in (4.6) is the solution of the system (4.1)-(4.2). We check at once that (4.6) satisfies the boundary conditions (4.3)-(4.4). □

Theorem 4.2. *The Green's matrix defined by the formula (4.5) has the following properties:*

i) *The Green's matrix* $G(t,s,\lambda)$ *is unique, i.e., if there exists another Green's matrix* $\widetilde{G}(t,s,\lambda)$ *for the nonhomogeneous system (26)-(29), then* $G(t,s,\lambda)=\widetilde{G}(t,s,\lambda)$ *in* $L^2_\Delta(\mathbb{T}^*\times\mathbb{T}^*;E)$.

ii) $G(t,s,\lambda)$ *is continuous at the point* (a,a).

iii) $G(t,s,\lambda)=G(s,t,\lambda)$.

iv) *Let* λ_0 *be a zero of* $\omega(\lambda)$. *Then* λ_0 *can be a simple pole of the matrix* $G(t,s,\lambda)$. *Therefore, we have*

$$G(t,s,\lambda)=\frac{-\upsilon(t)\,\upsilon(s)}{\lambda-\lambda_0}+\widetilde{G}(t,s,\lambda),$$

where $\widetilde{G}(t,s,\lambda)$ *is an analytic function of* λ *in the neighborhood of* λ_0 *and* $\upsilon(t)$ *is a normalized eigenfunction corresponding to* λ_0.

Proof. Since the proof is similar to that of Sturm-Liouville equations on time scales (see [7]), we omit it. □

We next prove the existence of a countable sequence of eigenvalues of l with no finite limit points. Later, we will prove that the corresponding eigenfunctions form an orthonormal basis of $\mathcal{H}$. Hence we need the following definition and theorems.

Definition 4.3. A complex-valued function $M(t,s)$ of two variables with $a\le t,s\le b$ is called the $\Delta-$Hilbert-Schmidt kernel if

$$\int_a^b\int_a^b |M(t,s)|^2\,\Delta t\Delta s<+\infty.$$

Theorem 4.4. *(*[17]*) If*

$$\sum_{i,k=1}^{\infty}|a_{ik}|^2<+\infty, \tag{4.9}$$

then the operator A *defined by the formula*

$$A\{x_i\}=\{y_i\},$$

where

$$y_i=\sum_{k=1}^{\infty}a_{ik}x_k,\ \ i=1,2,... \tag{4.10}$$

is compact in the sequence space l^2.

Theorem 4.5 (Hilbert-Schmidt)**.** *Let* A *be a compact self-adjoint operator mapping a Hilbert space* H *into itself. Then there is an orthonormal system* $\varphi_1,\varphi_2,...$ *of eigenvectors of* A, *with corresponding nonzero* *eigenvalues* $\lambda_1, \lambda_2, ...,$ *such that every element* $x \in H$ *has a unique representation of the form*

$$x = \sum_n c_n \varphi_n + x',$$

where x' *satisfies the condition* $Ax' = 0$. *Moreover,*

$$Ax = \sum_n \lambda_n c_n \varphi_n$$

and

$$\lim_{n \to \infty} \lambda_n = 0$$

in the case where there are infinitely many nonzero eigenvalues ([14]*).*

Let us denote by D_L the linear set of all vectors $y(.) \in \mathcal{H}$ such that $\Delta y(t)$ is a continuous function on $\mathbb{T}^*$, $l(y) \in \mathcal{H}$ and $L_1(y) = 0$, $L_2(y) = 0$. We define the operator L on D_L by the equality $Ly = l(y)$. It is clear that the operator L has the same eigenvalues of the Dirac problem defined by (3.1)-(3.3). Without loss of generality, we can assume that $\lambda = 0$ is not an eigenvalue. Then, $\ker L = \{0\}$. Thus the solution of the problem $(Ly)(t) = f(t)$, $f(.) \in \mathcal{H}$ is given by

$$y(x) = \int_a^b (G(t,s), f(s))_E \Delta s,$$

where

$$G(t,s) = G(t,s,0) = \begin{cases} -\frac{\theta_2(t)\theta_1^T(s)}{W(\theta_1,\theta_2)}, & a \le t \le s \\ -\frac{\theta_1(t)\theta_2^T(s)}{W(\theta_1,\theta_2)}, & s < t \le b. \end{cases} \tag{4.11}$$

Theorem 4.6. *$G(t,s)$ defined by (4.11) is a $\Delta-$Hilbert-Schmidt kernel.*

Proof. By the upper half of the formula (4.11), we have

$$\int_a^b \Delta t \int_a^t \|G(t,s)\|^2 \Delta s < +\infty;$$

and by the lower half of (4.11), we have

$$\int_a^b \Delta t \int_t^b \|G(t,s)\|^2 \Delta s < +\infty$$

since the inner integral exists and is a linear combination of the products $\theta_{1i}(x)\overline{\theta_{2k}(t)}$ $(i,k = 1,2)$, and these products belong to $\mathcal{H}$ because each of the factors belongs to $\mathcal{H}$. Then, we obtain

$$\int_a^b \int_a^b \|G(t,s)\|^2 \Delta t \Delta s < +\infty. \tag{4.12}$$

□

Theorem 4.7. *The operator K defined by the formula*

$$(Kf)(x) = \int_a^b (G(t,s), f(s))_E \Delta s$$

is compact and self-adjoint.

Proof. Let $\phi_i = \phi_i(s)$, $i = 1, 2, ...$ be a complete, orthonormal basis of $\mathcal{H}$. Since $G(t,s)$ is a Hilbert-Schmidt kernel, we can define

$$x_i = \langle f, \phi_i \rangle_{\mathcal{H}} = \int_a^b (f(s), \phi_i(s))_E \Delta s,$$

$$y_i = \langle g, \phi_i \rangle_{\mathcal{H}} = \int_a^b (g(s), \phi_i(s))_E \Delta s,$$

$$a_{ik} = \int_a^b \int_a^b (G(t,s)\phi_i(t), \phi_k(s))_E \Delta t \Delta s.$$

Then, $\mathcal{H}$ is mapped isometrically into l^2. Consequently, our integral operator transforms into the operator defined by the formula (4.10) in the space l^2 by this mapping, and the condition (4.12) is translated into the condition (4.9). By Theorem 4.4, this operator is compact. Therefore, the original operator is compact.

Let $f, g \in \mathcal{H}$. As $G(t,s) = G(s,t)$ and $G(t,s)$ is a real matrix-valued function defined on $\mathbb{T}^* \times \mathbb{T}^*$, we have

$$\begin{aligned}\langle Kf, g \rangle_{\mathcal{H}} &= \int_a^b ((Kf)(t), g(t))_E \Delta t \\ &= \int_a^b \int_a^b (G(t,s) f(s), g(t))_E \Delta s \Delta t \\ &= \int_a^b \left(f(s), \int_a^b G(s,t) g(t) \Delta t \right)_E \Delta s \\ &= \langle f, Kg \rangle_{\mathcal{H}}.\end{aligned}$$

Thus we have proved that K is self-adjoint. □

Theorem 4.8. *The eigenvalues of the operator L form an infinite sequence $\{\lambda_n\}_{n=1}^{\infty}$ of real numbers which can be ordered such that*

$$|\lambda_1| < |\lambda_2| < ... < |\lambda_n| < ... \to \infty \text{ as } n \to \infty.$$

The set of all normalized eigenfunctions of L forms an orthonormal basis for $\mathcal{H}$.

Proof. By Theorems 4.5 and 4.7, K has an infinite sequence of non-zero real eigenvalues $\{\xi_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \xi_n = 0$. Then,

$$|\lambda_n| = \frac{1}{|\xi_n|} \to \infty \text{ as } n \to \infty.$$

Furthermore, let $\{\chi_n\}_{n=1}^{\infty}$ denote an orthonormal set of eigenfunctions corresponding to $\{\xi_n\}_{n=1}^{\infty}$. Thus we have ($y \in \mathcal{H}$, $Ky = f$, $y = Lf$, $L\chi_n = \lambda_n \chi_n$, $n = 1, 2, ...$)

$$y = Lf = \sum_{n=1}^{\infty} \langle y, \chi_n \rangle_{\mathcal{H}} \chi_n = \sum_{n=1}^{\infty} \langle Lf, \chi_n \rangle_{\mathcal{H}} \chi_n$$
$$= \sum_{n=1}^{\infty} \langle f, L\chi_n \rangle_{\mathcal{H}} \chi_n = \sum_{n=1}^{\infty} \lambda_n \langle f, \chi_n \rangle_{\mathcal{H}} \chi_n.$$

□

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