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# Golden numbers

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Abstract. The golden ratio $\phi = \frac{1+\sqrt{5}}{2} = 1/61803398874...$ is the root of the polynomial $x^2 - x - 1 = 0$, and is the one of the important numbers in mathematics. The golden ratio is also used in many fields of science. The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves and other plant parts. In this paper, we present a sequence of golden numbers $\{\phi_n\}_n$ and study their properties.

Keywords: Golden ratio, Polynomial, Golden numbers, Golden polynomials.

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## 1. Introduction

The golden ratio is a special number equal to $\phi = \frac{1+\sqrt{5}}{2}$. It appears many times in geometry, art, architecture and other areas. Mathematicians since Euclid have studied the properties of the golden ratio, including its appearance in the dimensions of a regular pentagon and in a golden rectangle, which may be cut into a square and a smaller rectangle with the same aspect ratio. The golden ratio has also been used to analyze the proportions of natural objects as well as man-made systems such as financial markets.

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**Definition 1.1.** Let $n \geq 2$ be a natural number. We define the golden polynomials $g_n(x)$ by

$$g_n(x) = x^n - \sum_{k=0}^{n-1} x^k.$$

Let $G_n(x) = (x-1)g_n(x) = x^{n+1} - 2x^n + 1$. It is easy to see that the roots of the polynomial $g_n(x)$ are the roots of the polynomial $G_n(x)$ else 1. We have $G_n'(x) = x^{n-1}((n+1)x - 2n) = 0$, so $x = 0, x = \frac{2n}{n+1}$ are the roots of $G_n'$.

| $x$ | | $0$ | | $\frac{2n}{n+1}$ | |
|---|---|---|---|---|---|
| ($n$ even) $G_n'(x)$ | $+$ | $0$ | $-$ | $0$ | $+$ |

,

| $x$ | | $0$ | | $\frac{2n}{n+1}$ | |
|---|---|---|---|---|---|
| ($n$ odd) $G_n'(x)$ | $-$ | $0$ | $-$ | $0$ | $+$ |

Since $G_n(0) = 1,\ G_n(1) = 0,\ G_n(2) = 1, G_{2n}(-1) = -2$ and $G_{2n+1}(-1) =$

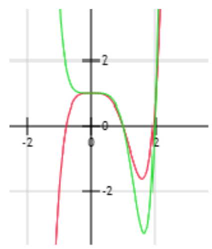

FIGURE 1. $y = G_n(x)$

4, if $n = 2k$ then $G_n(x)$ has only three real roots 1, $\phi_n$ and $\psi_n$ such that $1 < \phi_n < 2$ and $-1 < \psi_n < 0$, and if $n = 2k+1$ then $G_n(x)$ has only two real roots 1 and $\phi_n$ such that $1 < \phi_n < 2$, for every $n \in \mathbb{N}$ [see figure **??**]. Hence if $n = 2k$, then $g_n(x)$ has only two real roots $1 < \phi_n < 2$ and $-1 < \psi_n < 0$ and if $n = 2k+1$, then $g_n(x)$ has only one root $1 < \phi_n < 2$. We call $\phi_n$ the $n$'th golden number for every $n (n \geq 2)$.

## 2. Golden polynomials

In this section we studied the properties of the golden polynomials and golden numbers.

*Remark* 2.1. Let $n \geq 2$ be a natural number and $g_n(x)$ be the golden polynomial, then

(1)

$$g_n(-1) = \begin{cases} 1 & n = 2k \\ -2 & n = 2k+1 \end{cases}$$

(2) $g_n(0) = -1 < 0,\ g_n(1) = 1 - n < 0,\ g_n(2) = 1 > 0.$

(3) If $n = 2k$ then $g_n(x)$ has only two real roots $1 < \phi_n < 2$ and $-1 < \psi_n < 0$.
(4) If $n = 2k+1$ then $g_n(x)$ has only one root $1 < \phi_n < 2$.
(5) For every $n \in \mathbb{N}$, $n \geq 2$, $\frac{2n}{n+1} < \phi_n < 2$, thus $\lim_{n\to\infty} \phi_n = 2$.
(6) The sequence $\{\psi_{2n}\}_n$ is a decreasing sequence and $\lim_{n\to} \psi_{2n} = -1$.

**Lemma 2.2.** *Let $n \geq 2$ be a natural number. If $\phi_n$ is the n'th golden number, then*

(1) *For every $n \in \mathbb{N}$, $n \geq 2$*

$$\phi_n^n = \sum_{i=0}^{n-1} \phi_n^i.$$

(2) *Let $n \geq 2$ be a natural number*

$$\phi_n^{n+k} = \sum_{i=0}^{n-1} \phi_n^{i+k},$$

*for every $k \in \mathbb{N}$.*
(3) *For every $n \in \mathbb{N}$, $n \geq 2$*

$$\phi_n = \sum_{i=0}^{n-1} \frac{1}{\phi_n^i}, \text{ and } \sum_{i=1}^{n} \frac{1}{\phi_n^i} = 1.$$

*Proof.* (1) Since $\phi_n$ is the root of the polynomial $g_n(x)$, $\phi_n^n = \sum_{i=0}^{n-1} \phi_n^i$ $\forall n \geq 2$.
(2) Multiply $\phi_n^n = \sum_{i=0}^{n-1} \phi_n^i$ by $\phi_n^k$ to obtain the equality $\phi_n^{n+k} = \sum_{i=0}^{n-1} \phi_n^{i+k}$.
(3) By the use of part (1), it is easy to see.
(4) By the use of part (1), it is easy to see.

□

**Lemma 2.3.** *Let $n \geq 2$ be a natural number and $g_n$ be the golden polynomial.*

(1) *If $x \neq 2, 0$, then $\frac{g_{n+1}(x) - g_n(x)}{g_n(x) - g_{n-1}(x)} \equiv x$, for every $n \geq 2$.*
(2) *If $x \neq 2, 0$, then $\lim_{n\to\infty} \frac{g_{n+1}(x)}{g_n(x)} = x$.*
(3) *If $n \geq 3$, then $g_{n+1}(x) = (x+1)g_n(x) - xg_{n-1}(x)$*
(4) *If $n \geq 2$, then*

$$g_n(x) \equiv \begin{cases} x^n - \frac{x^n - 1}{x-1} & x \neq 1 \\ 1 - n & x = 1 \end{cases}.$$

*Proof.* (1) Let $n \geq 2$. We have

$$\frac{g_{n+1}(x) - g_n(x)}{g_n(x) - g_{n-1}(x)} = \frac{x^{n+1} - \sum_{k=0}^{n} x^k - x^n + \sum_{k=0}^{n-1} x^k}{x^n - \sum_{k=0}^{n-1} x^k - x^{n-1} + \sum_{k=0}^{n-2} x^k}$$
$$= \frac{x^{n+1}(x-2)}{x^n(x-2)} \equiv x,$$

for all $x \neq 0, 2$.

(2) Let $n \geq 2$. We have

$$\lim_{n\to\infty} \frac{g_{n+1}(x)}{g_n(x)} = \lim_{n\to\infty} \frac{x^{n+1} - \sum_{k=0}^{n} x^k}{x^n - \sum_{k=0}^{n-1} x^k} = x,$$

for all $x \neq 0, 2$.

(3) Let $n \geq 3$. We have

$$g_{n+1}(x) + x g_{n-1}(x) = x^{n+1} - \sum_{k=0}^{n} x^k + x(x^{n-1} - \sum_{k=0}^{n-2} x^k)$$
$$= x^{n+1} - \sum_{k=0}^{n} x^k + x^n - \sum_{k=0}^{n-2} x^{k+1}$$
$$= (x+1) g_n(x).$$

(4) It is easy to see.

□

**Lemma 2.4.** *Let $n \geq 2$ be a natural number and $g_n$ be the golden polynomial.*

(1) *If n is an odd number, then (see Figure* **??***, part A),*

$$\begin{cases} g_{n+1}(x) > g_n(x) & x > 2 \text{ or } x < 0 \\ g_{n+1}(x) < g_n(x) & 0 < x < 2 \end{cases},$$

*and if n is an even number, then (see Figure* **??***, part B)*

$$\begin{cases} g_{n+1}(x) > g_n(x) & x > 2 \\ g_{n+1}(x) < g_n(x) & x < 2 \end{cases}.$$

(2) *If $n \geq 2$, then $g_{n+2}(x) - g_n(x) = x^n g_1(x)$.*

(3) *If n is an odd number, then (see Figure* **??***, part C)*

$$\begin{cases} g_{n+2}(x) < g_n(x) & x < -1 \text{ or } 0 < x < 2 \\ g_{n+2}(x) > g_n(x) & -1 < x < 0 \text{ or } x > 2 \end{cases},$$

*and if n is an even number, then (see Figure* **??***, part D)*

$$\begin{cases} g_{n+2}(x) < g_n(x) & -1 < x < 2 \\ g_{n+2}(x) > g_n(x) & x < -1 \text{ or } x > 2 \end{cases}.$$

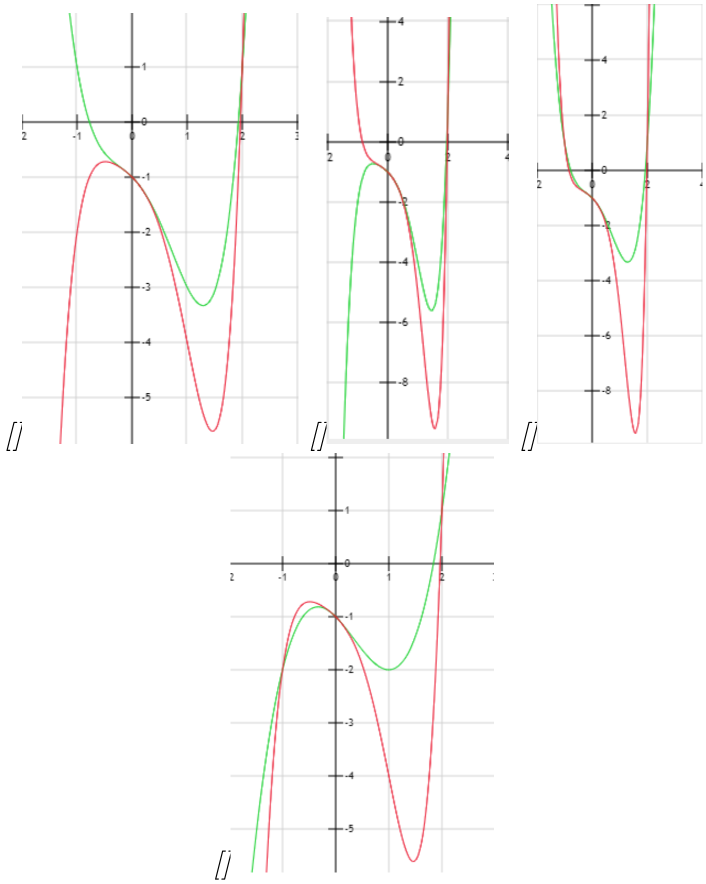

FIGURE 2. compare $g_n,\ g_{n+1}$ and $g_n,\ g_{n+2}$

*Proof.* (1) Statement (1) is trivial because $g_{n+1}(x) - g_n(x) = x^{n+1}(x - 2)$.

(2) Let $n \geq 2$ be a natural numer.

$$g_{n+2}(x) - g_n(x) = x^{n+2} - \sum_{k=0}^{n+1} x^k - x^n + \sum_{k=0}^{n-1} x^k$$
$$= x^{n+2} - x^{n+1} - 2x^n = x^n g_1(x).$$

(3) Statement (3) is trivial because $g_{n+2}(x) - g_n(x) = x^n(x^2 - x - 1)$. □

The sequence $\{\phi_n\}_n$ is a bounded and increasing sequence.

*Proof.* Since $\frac{2n}{n+1} < \phi_n < 2$ and $G_n(x)$ is an increasing function on the interval $(\frac{2n}{n+1}, 2)$ and $G_{n+1}(x) < G_n(x)$, so $\phi_n < \phi_{n+1} < 2$. □

**Lemma 2.5.** *The number $\phi_n$ is an irrational number, $\forall n \geq 2$*

*Proof.* Let $\phi_n = \frac{p_n}{q_n}$, $(p_n, q_n) = 1$. Therefore

$$(\frac{p_n}{q_n})^n - \sum_{k=0}^{n-1} (\frac{p_n}{q_n})^k = 0.$$

So

$$(p_n)^n - \sum_{k=0}^{n-1} (p_n)^k (q_n)^{n-k} = 0.$$

Hence,

$$q_n | (p_n)^n \Longrightarrow q_n | p_n \Longrightarrow q_n = 1.$$

But if $q_n = 1$, then $\phi_n = p_n$, thus $(p_n)^n - \sum_{k=0}^{n-1} (p_n)^k = 0$, this means that $p_n | 1$ or $p_n = 1$ which is a contradiction. □

**Theorem 2.6.** *If $\{F_k\}_{k=0}^{\infty}$ is a sequence of positive real numbers that for every $k \geq 1$ have*

$$F_{k+n} = F_{k+n-1} + F_{k+n-2} + ... + F_k, \tag{2.1}$$

*then*

$$\lim_{k \to \infty} \frac{F_{k+1}}{F_k} = \phi_n.$$

*Proof.* Let $\lim_{k \to \infty} \frac{F_{k+1}}{F_k} = a$. Since $F_{k+n} = F_{k+n-1} + F_{k+n-2} + ... + F_k$, so

$$\frac{F_{k+n}}{F_k} = \frac{F_{k+n-1}}{F_k} + \frac{F_{k+n-2}}{F_k} + ... + \frac{F_k}{F_k}.$$

Thus,

$$\frac{F_{k+n}}{F_{k+n-1}} \times \frac{F_{k+n-1}}{F_{k+n-2}} \times \ldots \times \frac{F_{k+1}}{F_k} = \frac{F_{k+n-1}}{F_{k+n-2}} \times \frac{F_{k+n-2}}{F_{k+n-3}} \times \ldots \times \frac{F_{k+1}}{F_k} + \ldots + \frac{F_k}{F_k},$$

therefore if $k \to \infty$, then $a^n = a^{n-1} + \ldots + a + 1$. Since $\phi_n$ is uniqe, so $a = \phi_n$. □

Theorems **??** and Remark **??** (5) yield the following theorem.

**Theorem 2.7.** *Let $F : \mathbb{N} \times \mathbb{N}_0 \longrightarrow \mathbb{R}_{\geq 0}$ be a function such that*

$$F(k, n) = F(k, n-1) + F(k, n-2) + \ldots + F(k, 0),$$

*for every $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$, then*

$$\lim_{n \to \infty} \lim_{k \to \infty} \frac{F(k+1, n)}{F(k, n)} = 2.$$

*Proof.* It follows from Theorem **??** and Remark **??**. □

**Theorem 2.8.** *Suppose that $b_0$ and $b_N$ are two fixed complex numbers.Then there is not polynomial with the complex coefficients $p(z)$ such that $g_3(z)p(z) \equiv b_N z^N + b_0$.*

*Proof.* Let $p(z) = \sum_{k=0}^{n} a_k z^k,\ a_n \neq 0,\ N = n+3$ and

$$q(z) = p(z)(z^3 - z^2 - z - 1) = b_N z^N + b_0.$$

Therefore, $q'(z) = N b_N z^{N-1}$ and

$$z | q^{(m)}(z) \tag{2.2}$$

for every $1 \leq m \leq n+2$. On the other hand we have

$$q'(z) = (3z^2 - 2z - 1)p(z) + (z^3 - z^2 - z - 1)p'(z) \tag{2.3}$$

$$q''(z) = (6z-2)p(z) + 2(3z^2 - 2z - 1)p'(z) + (z^3 - z^2 - z - 1)p''(z). \tag{2.4}$$

Also we have

$$\begin{aligned} q^{(m)}(z) = m(m-1)(m-2)p^{(m-3)}(z) + \frac{m(m-1)}{2}(6z-2)p^{(m-2)}(z) \\ + m(3z^2 - 2z - 1)p^{(m-1)}(z) + (z^3 - z^2 - z - 1)p^{(m)}(z) \end{aligned} \tag{2.5}$$

for every $3 \leq m \leq n$, and

$$\begin{aligned} q^{(n+1)}(z) = (n+1)n(n-1)p^{(n-2)}(z) + \frac{(n+1)n}{2}(6z-2)p^{(n-1)}(z) \\ + (n+1)(3z^2 - 2z - 1)p^{(n)}(z), \end{aligned} \tag{2.6}$$

$$q^{(n+2)}(z) = (n+2)(n+1)np^{(n-1)}(z) + \frac{(n+2)(n+1)}{2}(6z-2)p^{(n)}(z) \tag{2.7}$$

and

$$q^{(n+3)}(z) = (n+3)(n+2)(n+1)p^{(n)}(z). \tag{2.8}$$

By the using of (**??**), have $z|q^{(m)}(z)$ for all $1 \leq m \leq n+2$, and by the use of (**??**) we have

$$-a_0 - a_1 = 0 \Longrightarrow a_1 = -a_0. \tag{2.9}$$

By the using of (**??**), we have

$$-2a_0 - 2a_1 - 2a_2 = 0 \overset{a_1 = -a_0}{\Longrightarrow} a_2 = 0, a_1 = -a_0. \tag{2.10}$$

So $n \neq 2$. Now, we consider two cases:
Case 1: Let $n = 1$. In this case $p(z) = a_1 z - a_1$ and

$$(z^3 - z^2 - z - 1)(a_1 z - a_1) = b_4 z^4 + b_0.$$

So,

$$(z^3 - z^2 - z - 1)(a_1 z - a_1) = a_1(z^4 - 2z^3 + 1),$$

and

$$a_1(z^4 - 2z^3 + 1) = b_4 z^4 + b_0,$$

which is a contradiction.

Case 2: Let $n \geq 3$. By the using of (**??**), we get

$$m(m-1)(m-2)(m-3)!a_{m-3} + \frac{m(m-1)}{2}(-2)(m-2)!a_{m-2} + m(-1)(m-1)!a_{m-1} - m!a_m = 0,$$

for every $3 \leq m \leq n$. So,

$$m!a_{m-3} - m!a_{m-2} - m!a_{m-1} - m!a_m = 0.$$

Hence,

$$a_m = a_{m-3} - a_{m-2} - a_{m-1}, \tag{2.11}$$

for every $3 \leq m \leq n$, and by the use of (**??**) we have

$$(n+1)n(n-1)(n-2)!a_{n-2} + \frac{n(n+1)}{2}(-2)(n-1)!a_{n-1} - (n+1)n!a_n = 0.$$

Thus,

$$a_n = a_{n-2} - a_{n-1}. \tag{2.12}$$

Also by the use of (**??**) have

$$(n+2)(n+1)n(n-1)!a_{n-1} + \frac{(n+2)(n+1)}{2}(-2)n!a_n = 0$$

or

$$a_n = a_{n-1}. \tag{2.13}$$

Finally from Relations (**??**), (**??**), (**??**) we conclude that the coefficients $a_2, a_3, ..., a_n$ are the multiple of each others. Since $a_2 = 0$, $a_2 = a_3 = ... = a_n = 0$, which is a contradiction. □

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