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Some geometrical properties of the Oscillator group

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Abstract. We consider the oscillator group equipped with a bi-invariant Lorentzian metric. Some geometrical properties of this space and the harmonicity properties of left-invariant vector fields on this space are determined. In some cases, all these vector fields are critical points for the energy functional restricted to vector fields. Left-invariant vector fields defining harmonic maps are also classified, and the energy of these vector fields is explicitly calculated.

Keywords: Oscillator group, Ricci solitons, Harmonic vector fields, Harmonic maps.

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1. Introduction

Suppose $\{P, X_1...X_m, Y_1...Y_m, Q\}$ is a basis for Lie algebra $\mathfrak{g}_m(\lambda) = \{\lambda_1, ..., \lambda_m\}$ with brackets

$$[X_i, Y_j] = \delta_{ij}P, \quad [Q, X_j] = \lambda_j Y_j, \quad [Q, Y_j] = -\lambda_j X_j. \tag{1.1}$$

The corresponding simply connected Lie group $G_m(\lambda) = \{\lambda_1, ..., \lambda_m\}$ is called the oscillator group. The family of left-invariant Lorentzian metrics g_ϵ, $-1 < \epsilon < 1$ on the oscillator group $G_m(\lambda)$ given in the basis $\{P, X_1...X_m, Y_1...Y_m, Q\}$, by

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$$g_\epsilon = \begin{pmatrix} \epsilon & 0 & 1 \\ 0 & I_{2m} & 0 \\ 1 & 0 & \epsilon \end{pmatrix}. \tag{1.2}$$

The oscillator group is defined as the semidirect product of the line (time) with the Heisenberg group, with the action given by the dynamics. Thus, it is not realized as a matrix group. It is a solvable group, but not an exponential group. The oscillator groups, are not only of great importance in Lorentzian geometry but also have a variety of applications in other fields, such as Conformal Field Theory, WZWmodels (see [11]) and Supergravity. This group has many useful properties in both geometry and physics. Levichev showed that the oscillator group equipped with the Lorentzian bi-invariant metric is geometrically a Lorentzian symmetric space and is physically related to an isotropic electromagnetic field. The oscillator group has interesting properties both in terms of differential geometry and physics (see, for example, [6] and the references therein).

Up to our knowledge, no geometrical properties such as harmonicity properties of invariant vector fields have been obtained yet for the oscillator group. Studying critical points of the energy associated with vector fields is a practical goal in various fields. As an example by the Reeb vector field ξ of a contact metric manifold, one can see how the criticality of such a vector field is linked to the geometry of the manifold ([12],[13]). Recently, it has been [8] proved that critical points of the energy functional restricted to vector fields, i.e., function $E : \mathfrak{X}(M) \to R$, are parallel vector fields. Moreover, in the same paper, it also has been determined the tension field associated with a unit vector field V, and investigated the problem of determining when V defines a harmonic map. A Riemannian manifold admitting a parallel vector field is locally reducible, and a pseudo-Riemannian manifold likewise admitting either space-like or time-like parallel vector field is locally reducible too. This leads us to consider different situations, where some interesting types of non-parallel vector fields can be characterized in terms of harmonicity properties. We may refer to the recent monograph [7] and reference [1] for an overview of harmonic vector fields.

As for the contents, in Section 2, we give some preliminaries. In Section 3, we investigate some geometric properties of the oscillator group. Harmonicity properties of vector fields on the oscillator group will be determined in Sections 4. Finally, the energy of all these vector fields is explicitly calculated in Section 5.

2. PRELIMINARIES

Let (M, g) be a compact Riemannian manifold and g_s be the Sasaki metric on the tangent bundle TM, then the energy of a smooth vector field $V : (M, g) \longrightarrow (TM, g^s)$ on M is;

$$E(V) = \frac{n}{2} vol(M, g) + \frac{1}{2} \int_M ||\nabla V||^2 dv \tag{2.1}$$

(M is compact; in the non-compact case, one works over relatively compact domains see [4]). $V : (M, g) \longrightarrow (TM, g^s)$ is said to define a harmonic map if V is a critical point for the energy functional. The Euler-Lagrange equations characterize vector fields V defining harmonic maps as the ones whose tension field $\theta(V) = tr(\nabla^2 V)$ vanishes. Consequently, V defines a harmonic map from (M, g) to (TM, g^s) if and only if

$$tr[R(\nabla_. V, V).] = 0, \quad \nabla^* \nabla V = 0, \tag{2.2}$$

where with respect to an orthonormal local frame $\{e_1, ..., e_n\}$ on (M, g), with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i, one has

$$\nabla^* \nabla V = \textstyle\sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V).$$

A smooth vector field V is said to be a harmonic section if and only if it is a critical point of $E^v(V) = (1/2) \int_M ||\nabla V||^2 dv$ where E^v is the vertical energy. The corresponding Euler-Lagrange equations are given by

$$\nabla^* \nabla V = 0, \tag{2.3}$$

Let $\mathfrak{X}^\rho(M) = \{V \in \mathfrak{X}(M) : ||V||^2 = \rho^2\}$ and $\rho \neq 0$. Then, one can consider vector fields $V \in \mathfrak{X}(M)$ which are critical points for the energy functional $E|_{\mathfrak{X}^\rho(M)}$, restricted to vector fields of the same constant length. The Euler-Lagrange equations of this variational condition are given by

$$\nabla^* \nabla V \quad is \quad collinear \quad to \quad V. \tag{2.4}$$

In the non-compact case, the condition (2.4) is taken as a definition of critical points for the energy functional under the assumption $\rho \neq 0$, that is, if V is not light-like. If $\rho = 0$, then (2.4) is still a sufficient condition so that V is a critical point for the energy functional $E|_{\mathfrak{X}^0(M)}$, restricted to light-like vector fields ([4], Theorem 26).

3. SOME GEOMETRIC PROPERTIES OF OSCILLATOR GROUP

We consider the special case $\{\epsilon = 0, m = 1\}$. So, the oscillator algebra $\mathfrak{g}$ has 4-generator P, X_1, Y_1, Q and Lie brackets

$$[X_1, Y_1] = P, \quad [Q, X_1] = Y_1, \quad [Q, Y_1] = -X_1. \tag{3.1}$$

Consider the biinvariant Lorentzian metric g on the oscillator group G given in the basis $\{P, X_1, Y_1, Q\}$, by

$$g = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

The components of the Levi-Civita connection are calculated using the well known *Koszul* formula and are

$$\begin{aligned} \Lambda_1 &= 0, & \Lambda_2 &= \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \Lambda_3 &= \begin{pmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Lambda_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.3}$$

We can determine the non-zero curvature components;

$$\begin{aligned} R(X_1, Q)X_1 &= \tfrac{1}{4}P, & R(Y_1, Q)Y_1 &= \tfrac{1}{4}P, \\ R(Q, X_1)Q &= \tfrac{1}{4}X_1, & R(Q, Y_1)Q &= \tfrac{1}{4}Y_1. \end{aligned}$$

Since $R(X, Y, Z, W) = g(R(X, Y)Z, W)$ we have;

$$R(X_1, Q, X_1, Q) = R(Q, Y_1, Q, Y_1) = \tfrac{1}{4}. \tag{3.4}$$

Applying the Ricci tensor formula, we get;

$$(\rho_{ij}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{3.5}$$

which is diagonal with eigenvalue $r_1 = \frac{1}{2}$. By (3.5) and (3.2), $\rho_{ij} \neq \lambda g_{ij}$ for some indices i, j, so G can not be an Einstein manifold.
We denote the scalar curvature by τ. Let M_q^n be a pseudo-Riemannian manifold of index q. The Weyl conformal curvature tensor field C of type $(1,3)$ of M is defined by

$$\begin{aligned} C(X, Y)Z = R(X, Y)Z - (&\tfrac{1}{n-2}(QX \wedge Y + X \wedge QY)Z \\ &+ \tfrac{\tau}{(n-1)(n-2)}(X \wedge Y)Z, \end{aligned} \tag{3.6}$$

where $(X \wedge Y)Z = <Y, Z> X - <X, Z> Y$. It is well-known [2] that for a conformally flat space, the curvature tensor can be completely determined using the Ricci tensor. Moreover, if $n \geqslant 4$, then M_q^n is conformally flat if and only if $C = 0$.

Proposition 3.1. *The oscillator group equipped with bi-invariant Lorentzian metric g described in* (3.2) *is conformally flat.*

Proof. Since the scalar curvature is $\tau = \sum_i(\rho_i, \rho_i)$ (see [3]. p. 43), by (3.5), $\tau = \frac{1}{2}$. Using (3.6) and (3.5) a straightforward calculation then yields that $C = 0$, as desired. □

The oscillator group equipped with bi-invariant Lorentzian metric g is conformally flat, so we conclude that:

Corollary 3.2. *The oscillator group equipped with bi-invariant Lorentzian metric g described in* (3.2) *is conformally Einstein.*

A D' Atri space is defined as a Riemannian manifold (M, g) whose local geodesic symmetries are volume-preserving. Let us recall that the property of being a D' Atri space is equivalent to the infinite number of curvature identities called the odd Ledger conditions L_{2k+1}, $k \geq 1$ (see [5]). In particular, the two first non-trivial Ledger conditions are:

$$\begin{array}{c} L_3 : (\nabla_X \rho)(X, X) = 0, \\ L_5 : \sum_{a,b=1}^{n} R(X, E_a, X, E_b)(\nabla_X R)(X, E_a, X, E_b) = 0, \end{array} \tag{3.7}$$

where X is any tangent vector at any point $m \in M$ and $\{E_1, ..., E_n\}$ is any orthonormal basis of $T_m M$. Here R denotes the curvature tensor and ρ the Ricci tensor of (M, g), respectively, and $n = dim M$.
Thus, it is natural to start with the investigation of the oscillator group satisfying the simplest Ledger condition L_3, which is the first approximation of the D' Atri property. This condition is called "the class $\mathcal{A}$ condition". Equivalently Ledger condition L_3 holds if and only if the Ricci tensor is cyclic-parallel, i.e.

$$(\nabla_X \rho)(Y, Z) + (\nabla_Y \rho)(Z, X) + (\nabla_Z \rho)(X, Y) = 0.$$

Proposition 3.3. *The oscillator group equipped with bi-invariant Lorentzian metric described in* (3.2) *is a D' Atri space which its first approximation holds.*

Proof. In Ledger condition L_3,

$$\nabla_i \rho_{jk} = -\sum_t (\varepsilon_j B_{ijt} \rho_{tk} + \varepsilon_k B_{ikt} \rho_{tj}),$$

where B_{ijk} components can be obtained by using the relation $\nabla_{e_i} e_j = \sum_k \varepsilon_j B_{ijk} e_k$ with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i. But $\nabla_1 \rho_{11} = \nabla_2 \rho_{22} = \nabla_3 \rho_{33} = 0$, hence the Ricci tensor is cyclic-parallel, and the first approximation of the D' Atri property holds. □

A pseudo-Riemannian manifold which admits a parallel degenerate distribution is called a *Walker* manifold. Walker spaces were introduced by Arthur Geoffrey Walker, in 1949. The existence of such structures causes many interesting properties for the manifold with no Riemannian counterpart. Walker also determined standard local coordinates for these kinds of manifolds [14, 15].

Proposition 3.4. *Let G be the oscillator group equipped with a bi-invariant Lorentzian metric g described in* (3.2)*, then (G, g) admits invariant parallel degenerate line field $\mathcal{D}$ with the generator $\{P\}$.*

Proof. Set $X = aP + bX_1 + cY_1 + dQ \in \mathfrak{g}$ and suppose that $\mathcal{D} = \mathrm{span}(X)$ is an invariant null parallel line field. Then, the following equations must satisfy for some parameters $\omega_1, \dots, \omega_4$

$$\nabla_P X = \omega_1 X, \quad \nabla_{X_1} X = \omega_2 X, \quad \nabla_{Y_1} X = \omega_3 X, \quad \nabla_Q X = \omega_4 X.$$

By straight forward calculations, we conclude that the following equations must satisfy

$$\begin{array}{llll} \omega_1 a = 0, & \omega_1 b = 0, & \omega_1 c = 0, & \omega_1 d = 0, \\ \omega_2 b = 0, & \omega_2 d = 0, & -\omega_2 a + \frac{1}{2}c = 0, & -\omega_2 c + \frac{1}{2}d = 0, \\ \omega_3 c = 0, & \omega_3 d = 0, & -\omega_3 a - \frac{1}{2}b = 0, & -\omega_3 b + \frac{1}{2}d = 0, \\ \omega_4 a = 0, & \omega_4 d = 0, & -\omega_4 b - \frac{1}{2}c = 0, & -\omega_4 c + \frac{1}{2}b = 0. \end{array}$$

X is null, hence X must satisfy $g(X, X) = 2ad + b^2 + c^2 = 0$ described in (3.2). By solving the above system of equations, we obtain that $X = aP$. It means that $b = c = d = 0$. □

4. Harmonicity of vector fields on oscillator group

In this section, we investigate the harmonicity of invariant vector fields on the oscillator group equipped with bi-invariant Lorentzian metric g described in (3.2).

Theorem 4.1. *Let G be the oscillator group equipped with bi-invariant Lorentzian metric g described in* (3.2) *and $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ be a left-invariant vector field on G for some real constants a, b, c, d, then the following conditions are equivalent:*

(1) *V defines a harmonic map;*
(2) *V is harmonic;*
(3) *V is a critical point for the energy functional restricted to vector fields of the same length;*
(4) *$V = ae_1 + bu + ce_3$, that is, $b = -d$.*

Proof. We can construct an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ with respect to g;

$$e_1 = -P + X_1, \quad e_2 = X_1 + Q, \quad e_3 = Y_1, \quad e_4 = -P + X_1 + Q,$$

with e_1, e_2, e_3 space-like and $e4$ time-like. We get;

$$\begin{array}{lll} [e_1, e_2] = -e_3, & [e_1, e_3] = e_2 - e_4, & [e_1, e_4] = -e_3, \\ [e_2, e_3] = -e_1, & [e_3, e_4] = e_1. & \end{array} \tag{4.1}$$

The connection components are;

$$\begin{array}{lll} \nabla_{e_1} e_2 = -\frac{1}{2} e_3, & \nabla_{e_1} e_3 = \frac{1}{2} e_2 - \frac{1}{2} e_4, & \nabla_{e_1} e_4 = -\frac{1}{2} e_3, \\ \nabla_{e_2} e_1 = -\frac{1}{2} e_3, & \nabla_{e_2} e_3 = -\frac{1}{2} e_1, & \\ \nabla_{e_3} e_1 = -\frac{1}{2} e_2 + \frac{1}{2} e_4, & \nabla_{e_3} e_2 = \frac{1}{2} e_1, & \nabla_{e_3} e_4 = \frac{1}{2} e_1, \\ \nabla_{e_4} e_1 = \frac{1}{2} e_3, & \nabla_{e_4} e_3 = -\frac{1}{2} e_1. & \end{array} \tag{4.2}$$

while $\nabla_{e_i} e_j = 0$ in the remaining cases.

Set $u = e_2 - e_4$. Then, from (4.2) we get $\nabla_{e_i} u = 0$ for all indices i. Therefore, u is a parallel light-like vector field. Finding a light-like parallel vector field is an important issue that lacks Riemannian interpretation, and represents a class of pseudo-Riemannian manifolds that illustrate the great differences between Riemannian and pseudo-Riemannian spaces.

For an arbitrary left-invariant vector field $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ we can now use (4.2) to calculate $\nabla_{e_i} V$ for all indices i. We get

$$\begin{array}{ll} \nabla_{e_1} V = -\frac{1}{2}(b+d)e_3 + \frac{1}{2} cu, & \nabla_{e_2} V = -\frac{1}{2} ce_1 + \frac{1}{2} ae_3, \\ \nabla_{e_3} V = \frac{1}{2}(b+d)e_1 - \frac{1}{2} au, & \nabla_{e_4} V = -\frac{1}{2} ce_1 + \frac{1}{2} ae_3. \end{array} \tag{4.3}$$

where the special role of $u = e_2 - e_4$ is clear. We can now calculate $\nabla_{e_i} \nabla_{e_i} V$ for all indices i. We obtain

$$\begin{array}{ll} \nabla_{e_1} \nabla_{e_1} V = -\frac{1}{4}(b+d)u, & \nabla_{e_2} \nabla_{e_2} V = -\frac{1}{4}(ae_1 + ce_3), \\ \nabla_{e_3} \nabla_{e_3} V = -\frac{1}{4}(b+d)u, & \nabla_{e_4} \nabla_{e_4} V = -\frac{1}{4}(ae_1 + ce_3). \end{array} \tag{4.4}$$

And for $\nabla_{\nabla_{e_i} e_i} V$ for all indices i

$$\nabla_{\nabla_{e_1} e_1} V = \nabla_{\nabla_{e_2} e_2} V = \nabla_{\nabla_{e_3} e_3} V = \nabla_{\nabla_{e_4} e_4} V = 0. \tag{4.5}$$

Thus, we find

$$\nabla^* \nabla V = \textstyle\sum_i \varepsilon_i (\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) = -\frac{1}{2}(b+d)u. \tag{4.6}$$

Thus, condition (2.3) is satisfied if and only if $b = -d$. It means that, V is harmonic if and only if $b = -d$. The condition (2.4) is again equivalent to $b = -d$. Because, if $\nabla^* \nabla V$ is collinear to V, then, by (4.6), either $b = -d$, or V is collinear to $u = e_2 - e_4$, which again implies $b = -d$. Now, using (4.3), we find

$$\begin{array}{ll} R(\nabla_{e_1} V, V)e_1 = 0, & R(\nabla_{e_2} V, V)e_2 = \dfrac{1}{8}(b+d)(ce_1 - ae_3), \\ R(\nabla_{e_3} V, V)e_3 = 0, & R(\nabla_{e_4} V, V)e_4 = \dfrac{1}{8}(b+d)(ce_1 - ae_3). \end{array}$$

Next, suppose now that $b = -d$, that is, $V = ae_1 + bu + ce_3$. Then, $R(\nabla_{e_i} V, V)e_i = 0$ for all indices i. Therefore,

$$tr[R(\nabla_{.}V, V)_{.}] = \sum_i \varepsilon_i R(\nabla_{e_i} V, V)e_i = 0,$$

with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i. Thus, condition (2.2) is satisfied if and only if $b = -d$. □

Therefore, left-invariant vector fields defining a harmonic map form a three-parameter family. As $||ae_1 + bu + ce_3||^2 = a^2 + c^2$ such vector fields are either space-like or light-like.
A vector field V is geodesic if $\nabla_V V = 0$, and is Killing if $\mathcal{L}_V g = 0$, where $\mathcal{L}$ denotes the Lie derivative. Parallel vector fields are both geodesic and Killing, and vector fields with these special geometric features often have particular harmonicity properties [9, 10].

Proposition 4.2. *Let G be the oscillator group equipped with bi-invariant Lorentzian metric g described in* (3.2) *and $V \in \mathfrak{g}$ be a left-invariant vector field on G, then V is geodesic. Moreover, V is Killing too.*

Proof. Suppose that $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$, using (4.3), it is clear $\nabla_V V = 0$. So, V is geodesic.
Because g is fixed and bi-invariant, for an arbitrary left-invariant vector field $V \in \mathfrak{g}$, we have

$$\mathcal{L}_V g = 0. \tag{4.7}$$

However, for $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ using (4.3) and the relation $(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X)$ simply results $\mathcal{L}_V g = 0$. □

Also, about harmonicity properties of invariant vector fields, the oscillator groups display some particular features. The main geometrical reasons for the special behavior of these groups are the existence of a parallel light-like vector field.
We have the following classification result which emphasizes once again the special role played by the parallel vector field u.

Theorem 4.3. *Let $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{g}$ be a left-invariant vector field on the oscillator group, then the following conditions are equivalent:*

(1) *V is geodesic;*
(2) *V is Killing;*
(3) *V is parallel if and only if $a = c = b - d = 0$, that is, V is collinear to u.*

Proof. Conditions (1) and (2) are straight results of Proposition 4.2. Using (4.3), $\nabla_{e_i} V$ for all indices i, is collinear to V if and only if $a = c = b - d = 0$. So, V is parallel if and only if $a = c = b - d = 0$, that is, V is collinear to u. □

5. The energy of vector fields on oscillator group

We calculate explicitly the energy of a vector field $V \in \mathfrak{g}$ on the oscillator group. This allows us to determine some critical values of the energy functional on the oscillator group. We shall first discuss geometric properties of the map V defined by a vector field $V \in \mathfrak{g}$.

Proposition 5.1. *Let G be the oscillator group, $V = ae_1 + be_2 + ce_3 + de_4 \in \mathfrak{su}(2)$ be a left-invariant vector field on the oscillator group for some real constants a, b, c, d. Denote by $\mathcal{D}$ a relatively compact domain of G and by $E_{\mathcal{D}}(V)$ the energy of $V|_{\mathcal{D}}$. The energy of V is;*

$$E_{\mathcal{D}}(V) = (2 + \tfrac{1}{4}(b+d)^2)vol(\mathcal{D}).$$

Proof. Let G be the oscillator group. Consider a local orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of vector fields. Then, locally,

$$||\nabla V||^2 = \textstyle\sum_{i=1}^{n} \varepsilon_i g(\nabla_{e_i} V, \nabla_{e_i} V),$$

with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i. Let $V \in \mathfrak{g}$ be a left-invariant vector field on the oscillator group, then (4.3) easily yields

$$||\nabla V||^2 = \tfrac{1}{2}(b+d)^2.$$

The conclusion follows from the expression of $E(V)$ and the fact that $||\nabla V||^2$ is constant. □

We already know from Theorem 4.3 which vector fields in $\mathfrak{g}$ on the oscillator group are critical points for the energy functional. Taking into account the Proposition (5.1), we then have the following.

Theorem 5.2. *Let G be the oscillator group, then $2vol(\mathcal{D})$ is the absolute minimum value of the energy functional $E_{\mathcal{D}}$. Such a minimum is attained by all vector fields $V = ae_2 + bu + ce_3 \in \mathfrak{g}$.*

Proof. By Proposition 5.1, $E_{\mathcal{D}}(V) = (2 + \frac{1}{4}(b+d)^2)vol(\mathcal{D})$. Therefore, $E_{\mathcal{D}}(V) = 2vol(\mathcal{D})$ if and only if $b = -d$. Thus, among vector fields of the same length, the ones with $b = -d$ minimize the energy. □

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