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Composition operators between growth spaces on circular and strictly convex domains in complex Banach spaces

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Abstract. Let Ω_X be a bounded, circular and strictly convex domain in a complex Banach space X, and $\mathcal{H}(\Omega_X)$ be the space of all holomorphic functions from Ω_X to $\mathbb{C}$. The growth space $\mathcal{A}^\nu(\Omega_X)$ consists of all $f \in \mathcal{H}(\Omega_X)$ such that

$$|f(x)| \leqslant C\nu(r_{\Omega_X}(x)), \quad x \in \Omega_X,$$

for some constant $C > 0$, whenever r_{Ω_X} is the Minkowski functional on Ω_X and $\nu : [0,1) \to (0,\infty)$ is a nondecreasing, continuous and unbounded function. For complex Banach spaces X and Y and a holomorphic map $\varphi : \Omega_X \to \Omega_Y$, put $C_\varphi(f) = f \circ \varphi, f \in \mathcal{H}(\Omega_Y)$. We characterize those φ for which the composition operator $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is a bounded or compact operator.

Keywords: Composition operator, Growth space, Circular domain.

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1. Introduction

Let X be a complex Banach space. We recall that $\Omega_X \subset X$ is a circular domain if Ω_X is an open set such that for $x \in \Omega_X$, and any real number θ, we have $e^{i\theta}x \in \Omega_X$. A domain $\Omega_X \subset X$ is said to be complete circular if $\lambda x \in \Omega_X$ for any $|\lambda| \leqslant 1$ and $x \in \Omega_X$. A set $\Omega_X \subset X$ is strictly convex

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if it is convex and contains the open line segment (x_1, x_2) for each pair of boundary points $x_1, x_2 \in \partial\Omega_X$. A convex set is strictly convex iff its boundary does not contain a line segment, [10]. The Minkowski functional r_{Ω_X}, associated to nonempty, bounded, circular and strictly convex domain Ω_X is defined by

$$r_{\Omega_X}(x) = \inf\{\lambda > 0 : \lambda^{-1}x \in \Omega_X\}.$$

For all $x \in X - \{0\}$, we have $(r_{\Omega_X}(x))^{-1}x \in \partial\Omega_X$ and $r_{\Omega_X}(x) = 1$ if and only if $x \in \partial\Omega_X$, [7]. Moreover by [11], r_{Ω_X} is a continuous seminorm such that

(i) $m||x|| \leq r_{\Omega_X}(x) \leq M||x||$ for all $x \in X$ and fixed $m, M > 0$,
(ii) $\Omega_X = \{x \in X : r_{\Omega_X}(x) < 1\}$.

Let $H(\Omega_X)$ be the space of all holomorphic functions from the nonempty, bounded, circular and strictly convex domain Ω_X in a complex Banach space X to $\mathbb{C}$. We recall that a mapping $f : \Omega_X \to \mathbb{C}$ is said to be holomorphic or analytic if for each $a \in \Omega_X$, there exists a ball $B(a, r) \subset \Omega_X$ and a sequence of continuous m-homogeneous polynomials $P_m : \Omega_X \to \mathbb{C}$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - a),$$

uniformly for $x \in B(a, r)$. Whence, $P_m : \Omega_X \to \mathbb{C}$ is said to be an m-homogeneous polynomial if there exists an m-linear mapping $A : \Omega_X^m \to \mathbb{C}$ such that $P_m(x) = A(x, ..., x)$.

Let $\nu : [0, 1) \to (0, \infty)$ be a weight, that is, ν is a non-decreasing, continuous and unbounded function.

The growth space $\mathcal{A}^\nu(\Omega_X)$ consists of those functions $f \in H(\Omega_X)$ for which

$$|f(x)| \leq C\nu(r_{\Omega_X}(x)), \quad x \in \Omega_X,$$

for some constant $C > 0$. The space $\mathcal{A}^\nu(\Omega_X)$ is a Banach space with the norm

$$\|f\|_{\mathcal{A}^\nu(\Omega_X)} := \sup_{x\in\Omega_X} \frac{|f(x)|}{\nu(r_{\Omega_X}(x))}.$$

For a weight ν, the associated weight $\tilde{\nu}$ on $r_{\Omega_X}(\Omega_X)$ is defined by

$$\tilde{\nu}(r_{\Omega_X}(x)) := \|\delta_x\| = sup\{|f(x)| : \|f\|_{\mathcal{A}^\nu(\Omega_X)} \leqslant 1\}, \quad x \in \Omega_X, \quad (1.1)$$

where δ_x denotes the point evaluation in $x \in X$. Since ν is nondecreasing thus $\frac{1}{\nu(r_{\Omega_X}(x))} \leqslant \frac{1}{\nu(0)}$ hence the norm topology on $\mathcal{A}^\nu(\Omega_X)$ is stronger than the pointwise convergence topology so $\delta_x \in (\mathcal{A}^\nu(\Omega_X))^*$ and the norm $\|\cdot\|$ is taken in $(\mathcal{A}^\nu(\Omega_X))^*$. Each $f \in \bar{\mathbb{B}}_{\mathcal{A}^\nu(\Omega_X)}$, the closed unit ball of $\mathcal{A}^\nu(\Omega_X)$, satisfies $|f(x)| \leq \nu(r_{\Omega_X}(x))$ on Ω_X. Thus we have $\tilde{\nu} \leqslant \nu$ on $r_{\Omega_X}(\Omega_X)$. Similar to [2, Proposition 1.2], since $\tilde{\nu} \circ r_{\Omega_X}$ is bounded on each compact subset of Ω_X, Montel's theorem [5, Proposition 9.16], implies that $\bar{\mathbb{B}}_{\mathcal{A}^\nu(\Omega_X)}$ is compact in $(H(\Omega_X), co)$. Using this fact and the continuity of point evaluations immediately yields that the sup in the definition of $\tilde{\nu}$, must be maximum. Thus for each $x \in \Omega_X$ there is $f_x \in \bar{\mathbb{B}}_{\mathcal{A}^\nu(\Omega_X)}$ with $|f_x(x)| = \tilde{\nu}(r_{\Omega_X}(x))$. Also, Ascoli's theorem [5, Theorem 9.12], implies that $\bar{\mathbb{B}}_{\mathcal{A}^\nu(\Omega_X)}$ is equicontinuous, and thus $\tilde{\nu}$, must be continuous.

Let X, Y be two Banach spaces, $\varphi : \Omega_X \to \Omega_Y$ be a holomorphic mapping. Then the composition operator $C_\varphi : H(\Omega_Y) \to H(\Omega_X)$ is defined by $C_\varphi(f) = f \circ \varphi$.

Let ν, ω be two weights and X, Y be two Banach spaces, then the norm topologies on $\mathcal{A}^\nu(\Omega_X)$ and $\mathcal{A}^\omega(\Omega_Y)$ are stronger than the pointwise convergence topology. If $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is well defined, then by closed graph theorem, C_φ is bounded. As a consequence, to fined out if the composition operator C_φ is bounded it is enough to fined out if C_φ is well defined.

Dubtsov gave characterization of the bounded (compact) weighted composition operators on growth spaces in the special case $\Omega_X = \mathbb{C}^n$, $\Omega_Y = \mathbb{C}^m$ and $\nu(t)$, $\omega(t)$ are equal to $\frac{1}{(1-t)^\alpha}, \alpha > 0$ or $\log \frac{e}{1-t}$ in [3]. Abakumov and Doubtsov characterized the boundedness and compactness problem for the weighted composition operators between growth spaces in the case $X = Y = \mathbb{C}^n$ by the reverse estimate in growth space in [1]. We applied the reverse estimate in the growth space $\mathcal{A}^\omega(\Omega_{\mathbb{C}^n})$ and determined conditions under which $C_\varphi : \mathcal{A}^\omega(\Omega_{\mathbb{C}^n}) \to \mathcal{A}^\nu(\Omega_X)$ be a bounded or compact operator in [9]. The purpose of this paper, is to generalize the results of [9] for $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$.

Throughout the remainder of this paper X, Y are Banach spaces, Ω_X, Ω_Y are the nonempty, bounded, circular and strictly convex domains in X, Y respectively, $\varphi : \Omega_X \to \Omega_Y$ is a nonzero holomorphic map and the open unit ball of a given Banach space X is denoted by $\mathbb{B}_X$. Constants are denoted by C, they are positive and not necessarily the same in each occurrence.

To prove the main results of this paper, we need the following lemma which is modification of the compactness criterion in [4]. Hence we omit its proof.

Lemma 1.1. *The following statements are equivalent;*

(i) $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ *is compact.*

(ii) C_φ *is bounded and* $\|C_\varphi f_\alpha\|_{\mathcal{A}^\nu(\Omega_X)}$ *converges to zero for any bounded net* $(f_\alpha)_{\alpha \in A}$ *in* $\mathcal{A}^\omega(\Omega_Y)$ *such that* $\{f_\alpha : \alpha \in A\}$ *is a countable set which converges to zero uniformly on compact subsets of* Ω_Y*.*

If, furthermore, Y *is separable, then (i) and (ii) are equivalent to*

(iii) *C_φ is bounded and $\|C_\varphi f_n\|_{\mathcal{A}^\nu(\Omega_X)}$ converges to zero for any bounded sequence $(f_n)_{n\in\mathbb{N}}$in $\mathcal{A}^\omega(\Omega_Y)$ which converges to zero uniformly on compact subsets of Ω_Y.*

2. Main result

In this section, we consider the boundedness and compactness of the operator $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$. It is easy to see that if φ is a linear mapping then $r_{\Omega_Y}(\varphi(x)) = r_{\Omega_X}(x)$. This implies that for linear mapping φ we get

$$\begin{aligned}\|C_\varphi f\|_{\mathcal{A}^\nu(\Omega_X)} = \sup_{x\in\Omega_X} \frac{|f(\varphi(x))|}{\nu(r_{\Omega_X}(x))} &= \sup_{x\in\Omega_X} \frac{|f(\varphi(x))|}{\nu(r_{\Omega_Y}(\varphi(x)))}\\ &\leqslant \|f\|_{\mathcal{A}^\nu(\Omega_Y)},\end{aligned}$$

therefore $C_\varphi(\mathcal{A}^\nu(\Omega_Y)) \subset \mathcal{A}^\nu(\Omega_X)$, hence $C_\varphi : \mathcal{A}^\nu(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is bounded. In addition for linear mapping φ if we assume that $\omega \leqslant \nu$ we have $C_\varphi(\mathcal{A}^\omega(\Omega_Y)) \subset \mathcal{A}^\nu(\Omega_X)$, which implies that C_φ is a bounded operator from $\mathcal{A}^\omega(\Omega_Y)$ to $\mathcal{A}^\nu(\Omega_X)$.

Using the following lemma one see that if Ω_X and Ω'_X are nonempty, bounded, complete circular and strictly convex domains in the complex Banach space X and $\varphi(0) = 0$, then $C_\varphi(\mathcal{A}^\nu(\Omega'_X)) \subset \mathcal{A}^\nu(\Omega_X)$.

Lemma 2.1. [7] *Let G, G' be complete circular domains in a complex Banach space X which G' is pseudoconvex. If $\varphi : G \to G'$ is a holomorphic map with $\varphi(0) = 0$, then $r_{G'}(\varphi(x)) \leqslant r_G(x)$ for all $x \in G$ and if $r_{G'}(\varphi(x)) = r_G(x)$ for some $x \in G$, then, for all $\lambda \in \mathbb{C}$ with $|\lambda| < (r_G(x))^{-1}$, we have $r_{G'}(\varphi(\lambda x)) = r_G(\lambda x) = |\lambda| r_G(x)$.*

In following, we give some characterizations for the boundedness and compactness of composition operators. We show that, if $\varphi(\Omega_X) \subseteq r_0\Omega_Y$ for some $0 < r_0 < 1$, then $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is bounded. For showing this, since $r_{\Omega_Y}(y) < 1$ for all $y \in \Omega_Y$, and ω is nondecreasing, then for every $f \in \mathbb{B}_{\mathcal{A}^\omega(\Omega_Y)}$ we have

$$\begin{aligned}
\|C_\varphi f\|_{\mathcal{A}^\nu(\Omega_X)} &= \sup_{x\in\Omega_X} \frac{|(C_\varphi f)(x)|}{\nu(r_{\Omega_X}(x))} \\
&\leqslant \frac{1}{\nu(0)} \sup_{x\in\Omega_X} |f(\varphi(x))| \\
&\leqslant \frac{C}{\nu(0)} \sup_{x\in\Omega_X} \omega(r_{\Omega_Y}(\varphi(x))) \\
&\leqslant \frac{C}{\nu(0)} \sup_{y\in\Omega_Y} \omega(r_{\Omega_Y}(r_0 y)) \\
&\leqslant \frac{C}{\nu(0)} \sup_{y\in\Omega_Y} \omega(r_0 r_{\Omega_Y}(y)) \\
&\leqslant C\frac{\omega(r_0)}{\nu(0)}.
\end{aligned}$$

Theorem 2.2. *Assume that ν and ω be two weights and $\varphi : \Omega_X \to \Omega_Y$ be a holomorphic mapping. Then the following statements are equivalent:*

(i) *The operator $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is bounded.*

(ii) $\displaystyle\sup_{x\in\Omega_X} \frac{\widetilde{\omega}(r_{\Omega_Y}(\varphi(x)))}{\nu(r_{\Omega_X}(x))} < \infty.$

If in addition ω is an analytic function, then (i) and (ii) are equivalent to

(iii) $\displaystyle\sup_{x\in\Omega_X} \frac{\omega(r_{\Omega_Y}(\varphi(x)))}{\nu(r_{\Omega_X}(x))} < \infty.$

Proof. Suppose that $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ be bounded. If (ii) does not hold, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in Ω_X such that

$$\lim_{n\to\infty} \frac{\tilde{\omega}(r_{\Omega_Y}(\varphi(x_n)))}{\nu(r_{\Omega_X}(x_n))} = \infty.$$

For each $n \in \mathbb{N}$, we can take $f_n \in \bar{\mathbb{B}}_{\mathcal{A}^\omega(\Omega_Y)}$ such that $|f_n(\varphi(x_n))| = \tilde{\omega}(r_{\Omega_Y}(\varphi(x_n)))$. Hence

$$\|C_\varphi f_n\|_{\mathcal{A}^\nu(\Omega_X)} = \sup_{x\in\Omega_X} \frac{|f_n(\varphi(x))|}{\nu(r_{\Omega_X}(x))} \geqslant \frac{\tilde{\omega}(r_{\Omega_Y}(\varphi(x_n)))}{\nu(r_{\Omega_X}(x_n))},$$

which is a contradiction with the fact that C_φ is bounded. If (ii) holds, for every $f \in \mathcal{A}^\omega(\Omega_Y)$ with $\|f\|_{\mathcal{A}^\omega(\Omega_Y)} \leqslant 1$, we have

$$\begin{aligned}\|C_\varphi f\|_{\mathcal{A}^\nu(\Omega_X)} &= \sup_{x\in\Omega_X} \frac{|(C_\varphi f)(x)|}{\nu(r_{\Omega_X}(x))}\\ &= \sup_{x\in\Omega_X} \frac{|f(\varphi(x))|}{\nu(r_{\Omega_X}(x))}\\ &\leq \sup_{x\in\Omega_X} \frac{|\widetilde{\omega}(r_{\Omega_Y}(\varphi(x)))|}{\nu(r_{\Omega_X}(x))}.\end{aligned}$$

The last inequality comes from the definition of $\widetilde{\omega}$. The implication (iii)⇒(ii) is clear because $\widetilde{\omega} \leq \omega$ on $r_{\Omega_Y}(\Omega_Y)$. Now suppose that ω is an analytic weight function and (i) holds. Define the function f as follows

$$f(y) = \omega(r_{\Omega_Y}(y)).$$

Then $f \in \mathcal{A}^\omega(\Omega_Y)$ and the boundedness of C_φ implies that

$$\infty > \|C_\varphi\| \geq \|C_\varphi f\|_{\mathcal{A}^\nu(\Omega_X)} = \sup_{x\in\Omega_X} \frac{|f(\varphi(x))|}{\nu(r_{\Omega_X}(x))} = \sup_{x\in\Omega_X} \frac{|\omega(r_{\Omega_Y}(\varphi(x)))|}{\nu(r_{\Omega_X}(x))},$$

and the condition (iii) is obtained. □

Now we are going to prove the necessary and sufficient conditions for the compactness of $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$. Similar to [8] we have the following definition.

Definition 2.3. A weight ν is said to be essential if there exists $C > 0$ such that $\tilde{\nu}(x) \leqslant \nu(x) \leqslant C\tilde{\nu}(x)$ for all $x \in \Omega_X$.

Theorem 2.4. *Let $C_\varphi(\mathcal{A}^\omega(\Omega_Y)) \subset \mathcal{A}^\nu(\Omega_X)$ and ω be an essential weight. Then $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is compact if and only if*

$$\lim_{r_{\Omega_Y}(\varphi(x))\to 1} \frac{\tilde{\omega}(r_{\Omega_Y}(\varphi(x)))}{\nu(r_{\Omega_X}(x))} = 0. \tag{2.1}$$

Proof. If C_φ does not compact by Lemma 1.1, there is a bounded net $(g_\alpha)_{\alpha\in A}$ in $\mathcal{A}^\omega(\Omega_Y)$ with $\{g_\alpha : \alpha \in A\}$ countable which converges to zero uniformly on compact subsets of Ω_Y such that $(\|C_\varphi g_\alpha\|_{\mathcal{A}^\nu(\Omega_X)})_{\alpha\in A}$ does not converge to zero. Therefore, there is $C > 0$ and a subnet $(g_\beta)_\beta$ so that $\|C_\varphi g_\beta\|_{\mathcal{A}^\nu(\Omega_X)} > C$ for all β. Note that $(g_\beta)_\beta$ is countable. Let us write $(g_\beta)_\beta = (f_n)_{n\in\mathbb{N}}$. Then we have $(f_n)_{n\in\mathbb{N}}$ bounded and in particular $\|C_\varphi f_n\|_{\mathcal{A}^\nu(\Omega_X)} > C$ for every $n \in \mathbb{N}$. Hence there exists $(x_n)_{n\in\mathbb{N}} \subset \Omega_X$ such that

$$\frac{|f_n(\varphi(x_n))|}{\nu(r_{\Omega_X}(x_n))} \geqslant C,$$

for each $n \in \mathbb{N}$. Since $(f_n) \subset \mathcal{A}^\omega(\Omega_Y)$ is bounded then we can assume that $\sup_n \|f_n\|_{\mathcal{A}^\omega(\Omega_Y)} \leq 1$, the definition $\mathcal{A}^\omega(\Omega_Y)$ and the condition ω is essential implies

$$\frac{C\tilde{\omega}(r_{\Omega_Y}(\varphi(x_n)))}{\nu(r_{\Omega_X}(x_n))} \geqslant \frac{\omega(r_{\Omega_Y}(\varphi(x_n)))}{\nu(r_{\Omega_X}(x_n))} \geq \frac{|f_n(\varphi(x_n))|}{\nu(r_{\Omega_X}(x_n))} \geq C.$$

Therefore the limit in (2.1) can not be zero and we obtain the result. Now suppose that C_φ is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Ω_X such that $r_{\Omega_Y}(\varphi(x_n)) \to 1$ as $n \to \infty$. There exist functions $f_n \in \bar{\mathbb{B}}_{\mathcal{A}^\omega(\Omega_Y)}$ such that $|f_n(\varphi(x_n))| = \tilde{\omega}(r_{\Omega_Y}(\varphi(x_n)))$. We choose $y_n^* \in Y^*$ with $\|y_n^*\| = 1$ such that $|y_n^*(\varphi(x_n))| = \|\varphi(x_n)\|$. Define $g_n(y) = \frac{1}{\|y\|_Y^n}(y_n^*(y))^n f_n(y) r_{\Omega_Y}^n(y)$ for all $y \in \Omega_Y$. We show that g_n is a bounded sequence in $\mathcal{A}^\omega(\Omega_Y)$ which converges to zero on any compact subset of Ω_Y. We have

$$\begin{aligned}\sup_{y\in\Omega_Y} \frac{|g_n(y)|}{\omega(r_{\Omega_Y}(y))} &= \sup_{y\in\Omega_Y} \frac{|y_n^*(y)|^n |f_n(y)| r_{\Omega_Y}^n(y)}{\|y\|_Y^n \omega(r_{\Omega_Y}(y))} \\ &\leqslant \sup_{y\in\Omega_Y} \frac{1}{\|y\|_Y^n} \|y\|_Y^n \|f_n\|_{\mathcal{A}^\omega(\Omega_Y)} r_{\Omega_Y}^n(y) < \infty.\end{aligned}$$

Given any compact set $K \subset Y$, since $\|f_n\|_{\mathcal{A}^\omega(\Omega_Y)} \leqslant 1$ and $\omega \circ r_{\Omega_Y}$ is continuous, there exists $C > 0$ such that

$$\sup_{y\in K} |f_n(y)| \leqslant \sup_{y\in K} \omega(r_{\Omega_Y}(y)) \leqslant C,$$

for all $n \in \mathbb{N}$. Also the compactness of K implies that

$$|g_n(y)| = \frac{1}{\|y\|_Y^n} |y_n^*(y)|^n |f_n(y)| r_{\Omega_Y}^n(y) \leqslant C r_{\Omega_Y}^n(y) \leqslant C r_0^n,$$

for some $0 < r_0 < 1$. It follows that, g_n is a bounded sequence in $\mathcal{A}^\omega(\Omega_Y)$ which converges to zero on any compact subset of Ω_Y. Using Lemma 1.1, we have $\|C_\varphi g_n\|_{\mathcal{A}^\nu(\Omega_X)} \to 0$ as $n \to \infty$. Observe that for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\|C_\varphi g_n\|_{\mathcal{A}^\nu(\Omega_X)} &= \sup_{x\in\Omega_X} \frac{|y_n^*(\varphi(x))|^n |f_n(\varphi(x))|}{\|\varphi(x)\|^n \nu(r_{\Omega_X}(x))} r_{\Omega_Y}^n(\varphi(x)) \\ &\geqslant \frac{|y_n^*(\varphi(x_n))|^n |f_n(\varphi(x_n))|}{\|\varphi(x_n)\|^n \nu(r_{\Omega_X}(x_n))} r_{\Omega_Y}^n(\varphi(x_n)) \\ &= \frac{\tilde{\omega}(r_{\Omega_Y}(\varphi(x_n)))}{\nu(r_{\Omega_X}(x_n))} r_{\Omega_Y}^n(\varphi(x_n))\end{aligned}$$

Using this inequality along with $r_{\Omega_Y}(\varphi(x_n)) \to 1$ as $n \to \infty$, we have

$$\lim_{r_{\Omega_Y}(\varphi(x))\to 1} \frac{\tilde{\omega}(r_{\Omega_Y}(\varphi(x)))}{\nu(r_{\Omega_X}(x))} = 0.$$

□

Theorem 2.5. *Let $C_\varphi(\mathcal{A}^\omega(\Omega_Y)) \subset \mathcal{A}^\nu(\Omega_X)$. Then $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is compact if and only if $\varphi(\Omega_X)$ is a relatively compact subset of Ω_Y.*

Proof. First suppose that C_φ is compact. We apply the same techniques used in [4]. If $\varphi(\Omega_X)$ is not relatively compact, by assumption, there is $\varepsilon > 0$ and a sequence $(x_n)_{n\in\mathbb{N}} \subset \Omega_X$ such that $\|\varphi(x_n) - \varphi(x_m)\|_Y > \varepsilon$ for every $n \neq m$. Thus for each pair (n, m), $n \neq m$, we can choose $T_{mn} \in Y^*$ with $\|T_{mn}\| = 1$ such that

$$|T_{mn}(\varphi(x_n)) - T_{mn}(\varphi(x_m))| \geqslant \varepsilon. \tag{2.2}$$

Since Ω_Y is bounded, thus

$$\sup_{y\in\Omega_Y} \frac{|T_{mn}(y)|}{\omega(r_{\Omega_Y}(y))} \leqslant \sup_{y\in\Omega_Y} \frac{\|T_{mn}\|\|y\|_Y}{\omega(r_{\Omega_Y}(y))} \leqslant \frac{C}{\omega(0)},$$

which implies that T_{mn} is a bounded subset of $\mathcal{A}^\omega(\Omega_Y)$ for all $n \neq m$. The compactness of C_φ implies that the adjoint operator of $(C_\varphi)^* : (\mathcal{A}^\nu(\Omega_X))^* \to (\mathcal{A}^\omega(\Omega_Y))^*$ is also compact. Also

$$\|\delta_x\| = \tilde{\nu}(r_{\Omega_X}(x)) \quad x \in \Omega_X,$$

and since $\tilde{\nu}$ is bounded, so $\{\delta_x : x \in \Omega_X\}$ is bounded in $(\mathcal{A}^\nu(\Omega_X))^*$. Hence $\{(C_\varphi)^*(\delta_x) : x \in \Omega_X\} = \{\delta_{\varphi(x)} : x \in \Omega_X\}$ is relatively compact in $(\mathcal{A}^\omega(\Omega_Y))^*$. On the other hand by (2.2),

$$\varepsilon \leqslant |\delta_{\varphi(x_n)}(T_{mn}) - \delta_{\varphi(x_m)}(T_{mn})| \leqslant \|\delta_{\varphi(x_n)} - \delta_{\varphi(x_m)}\|\|T_{mn}\|,$$

for every $n \neq m$. Hence

$$\|\delta_{\varphi(x_n)} - \delta_{\varphi(x_m)}\| \geqslant \varepsilon,$$

for all $n \neq m$. This contradicts the fact that $\{\delta_{\varphi(x)} : x \in \Omega_X\}$ is relatively compact. Therefore $\varphi(\Omega_X)$ is relatively compact.

Now suppose that $\varphi(\Omega_X)$ is a relatively compact subset of Ω_Y and $(g_\alpha)_{\alpha\in A}$ is a net in $\mathbb{B}_{\mathcal{A}^\omega(\Omega_Y)}$ with $\{g_\alpha : \alpha \in A\}$ countable, such that g_α converges to zero uniformly on compact subsets of Ω_Y. Let us write $(g_\alpha)_{\alpha\in A} = (f_n)_{n\in\mathbb{N}}$. In particular if $K \subseteq \Omega_Y$ is compact, then $\sup_{y\in K} |f_n(y)| \to 0$ as $n \to \infty$. It follows that, for every $\varepsilon > 0$ there is an integer $N > 1$ such that

$$\sup_{y\in\varphi(\Omega_X)} |f_n(y)| \leq \sup_{y\in\overline{\varphi(\Omega_X)}} |f_n(y)| < \varepsilon,$$

for all $n \geqslant N$. Thus

$$\|C_\varphi f_n\|_{\mathcal{A}^\nu(\Omega_X)} = \sup_{x\in\Omega_X} \frac{|f_n(\varphi(x))|}{\nu(r_{\Omega_X}(x))} \leqslant \frac{\varepsilon}{\nu(0)},$$

for all $n \geqslant N$. It follows that $\|C_\varphi f_n\|_{\mathcal{A}^\nu(\Omega_X)} \to 0$ as $n \to \infty$. By Lemma 1.1, $C_\varphi : \mathcal{A}^\omega(\Omega_Y) \to \mathcal{A}^\nu(\Omega_X)$ is compact. □

We finalize the paper by a question:

Question Are there any conditions on which the statements of Theorem 2.1 are equivalent to the following?

$$\sup_{x\in\Omega_X} \frac{\widetilde{\omega}(r_{\Omega_Y}(\varphi(x)))}{\widetilde{\nu}(r_{\Omega_X}(x))} < \infty \quad \textit{or} \quad \sup_{x\in\Omega_X} \frac{\omega(r_{\Omega_Y}(\varphi(x)))}{\widetilde{\nu}(r_{\Omega_X}(x))} < \infty.$$

The similar question can be stated for the compactness.

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