Weighted slant Toep-Hank operators

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Abstract. A weighted slant Toep-Hank operator $L^\beta_\phi$ with symbol $\phi \in L^\infty(\beta)$ is an operator on $L^2(\beta)$ whose representing matrix consists of all even (odd) columns from a weighted slant Hankel (slant weighted Toeplitz) matrix, $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence of positive numbers with $\beta_0 = 1$. A matrix characterization for an operator to be weighted slant Toep-Hank operator is also obtained.

Keywords: Weighted slant Hankel operators, Slant weighted Toeplitz operators, slant Toep-Hank operator.


1. Introduction

Let $\mathbb{C}$ and $\mathbb{Z}$ denote the set of all complex numbers and integers respectively. Throughout this paper, the spaces are considered, unless otherwise stated, under the assumption that the sequence $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ is a semi-dual sequence of positive numbers (that is $\beta_n = \beta_{-n}$ for each $n \in \mathbb{Z}$) with $\beta_0 = 1$ and $r \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$, for some $r > 0$.

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Consider the spaces
\[ L^2(\beta) = \{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, \quad \|f\|^2_\beta = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \} \]
and
\[ H^2(\beta) = \{ f(z) = \sum_{n=0}^{\infty} a_n z^n \mid a_n \in \mathbb{C}, \quad \|f\|^2_\beta = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \}. \]

The space \((L^2(\beta), \| \cdot \|_\beta)\) is a Hilbert space with the inner product defined by
\[ \langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \rangle = \sum_{n=-\infty}^{\infty} a_n \overline{b}_n \beta_n^2. \]

The set \(\{ e_n : e_n(z) = z^n/\beta_n \}_{n \in \mathbb{Z}}\) forms an orthonormal basis for the space \(L^2(\beta)\) and \(H^2(\beta)\) is a subspace of \(L^2(\beta)\).

Let \(L^\infty(\beta)\) denote the set of formal Laurent series \(\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n\) such that \(\phi L^2(\beta) \subseteq L^2(\beta)\) and there exists some \(c > 0\) satisfying \(\|\phi f\|_\beta \leq c\|f\|_\beta\) for each \(f \in L^2(\beta)\). For \(\phi \in L^\infty(\beta)\), define the norm \(\|\phi\|_\infty\) as
\[ \|\phi\|_\infty = \inf \{ c > 0 : \|\phi f\|_\beta \leq c\|f\|_\beta \text{ for each } f \in L^2(\beta) \}. \]

\(L^\infty(\beta)\) is a Banach space with respect to \(\| \cdot \|_\infty\). Also, \(L^\infty(\beta) \subseteq L^2(\beta)\). \(H^\infty(\beta)\) denotes the set of formal power series \(\phi\) such that \(\phi H^2(\beta) \subseteq H^2(\beta)\). These weighted sequence spaces cover Bergman, Hardy, Dirichlet and Fischer spaces for specifically designed sequences \(\beta = \{ \beta_n \}\) and thus become more demanding. For the detailed study of these spaces, we refer [11] and the references therein. If \(\phi \in L^\infty(\beta)\), then the weighted Laurent operator \(M_\phi^\beta\) on \(L^2(\beta)\) is given by
\[ M_\phi^\beta e_k(z) = \frac{1}{\beta_k} \sum_{n=-\infty}^{\infty} a_n \beta_{n+k} e_{n+k}(z). \]

If we put \(\phi(z) = z\), then the operator \(M_z^\beta e_k(z) = \frac{\beta_{k+1}}{\beta_k} e_{k+1}(z)\), for all \(k \in \mathbb{Z}\), and is known as a weighted shift [11].

The Hankel and Toeplitz operators arise in plenty of applications like stationary processes, perturbation theory, wavelet analysis and many more. For the detailed study of these operators and their applications, we refer [5, 6, 7, 10] and the references therein. Over the years, many generalizations of these operators also came up including slant Hankel [1] and slant Toeplitz [8] operators on the space \(L^2(\mathbb{T})\), \(\mathbb{T}\) being the unit circle. Meanwhile, the weighted sequence spaces \(L^2(\beta), H^2(\beta)\) and their generalizations came up and gained popularity.
Motivated by all these developments and the study of a slant Toep-Hank operator $L_\phi$ on $L^2(\mathbb{T})$ discussed in [4] (whose matrix representation provides a slant Hankel (slant Toeplitz) matrix if only even (odd) columns are considered), we now introduce and study the notion of a weighted slant Toep-Hank operator $L^\beta_\phi$ on the space $L^2(\beta)$. In the second section, we obtain various characterizations for an operator to be weighted slant Toep-Hank operator. In the third section, we obtain symbols for any weighted slant Hankel or slant weighted Toeplitz operator to be a weighted slant Toep-Hank operator.

2. Main Result

The main aim of this section is to find characterizations for weighted slant Toep-Hank operators in terms of matrices and operator equations. We begin with the following definitions of operators frequently used in the paper.

**Definition 2.1.** [2] For $\phi \in L^\infty(\beta)$, a slant weighted Toeplitz operator $U^\beta_\phi$ on the space $L^2(\beta)$ is an operator given by $U^\beta_\phi = W^\beta M^\beta_\phi$, where $W^\beta$ be the operator on $L^2(\beta)$ given by

$$W^\beta e_n(z) = \begin{cases} \frac{\beta_m}{\beta^{2m}} e_m(z) & \text{if } n=2m \text{ for some } m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

**Definition 2.2.** [3] A weighted slant Hankel operator $K^\beta_\phi$ induced by $\phi$ in $L^\infty(\beta)$ is an operator on $L^2(\beta)$ given by $K^\beta_\phi = J^\beta W^\beta M^\beta_\phi$, where $J^\beta$ is the reflection operator on $L^2(\beta)$ given by $J^\beta(e_n) = e_{-n}$ for $n \in \mathbb{Z}$.

It is known [3] that if the sequence $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ is semi-dual, then the expression $\bar{\phi}(z) = \sum_{n=-\infty}^{\infty} a-n z^n$ is in $L^\infty(\beta)$ for each $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ in $L^\infty(\beta)$.

For $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, the operators $U^\beta_\phi$ and $K^\beta_\phi$ satisfy that

$$U^\beta_\phi e_j(z) = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{2n-j} \beta_n e_n(z)$$

and

$$K^\beta_\phi e_j(z) = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{-2n-j} \beta_{-n} e_n(z)$$
for each $j \in \mathbb{Z}$.

**Definition 2.3.** [4] Let $\phi \in L^\infty(\mathbb{T})$. A **slant Toeplitz-Hankel** operator $L_\phi$ on $L^2(\mathbb{T})$ induced by $\phi$ is given by $L_\phi = K_\phi W + U_{z^\phi} M$, where $W$ and $M$ are operators on $L^2(\mathbb{T})$ given by $W(e_{2n}) = e_n$, otherwise zero and $M(e_{2n+1}) = e_n$, otherwise zero.

We now extend the notion of **slant Toeplitz-Hankel** operator to $L^2(\beta)$ as follows.

**Definition 2.4.** Let $\phi \in L^\infty(\beta)$. A **weighted slant Toeplitz-Hankel** operator $L_\phi^\beta$ on $L^2(\beta)$ is given by $L_\phi^\beta = K_\phi^\beta \hat{W}^\beta + U_{z^\phi}^\beta \hat{M}^\beta$, where we define operators $\hat{M}^\beta$ and $\hat{W}^\beta$ on $L^2(\beta)$ as

$$
\hat{M}^\beta e_n = \begin{cases} e_{\frac{n-1}{4}} & \text{if } n \text{ is odd}; \\
0 & \text{if } n \text{ is even}, \end{cases}
$$

and

$$
\hat{W}^\beta e_n = \begin{cases} e_{\frac{n}{4}} & \text{if } n \text{ is even}; \\
0 & \text{if } n \text{ is odd}, \end{cases}
$$

for each $n \in \mathbb{Z}$.

It is worth noticing that weighted slant Hankel and slant weighted Toeplitz operators are linear with respect to their symbols. Thus, the class $\{L_\phi^\beta | \phi \in L^\infty(\beta)\}$ is a linear subspace of $\mathcal{B}(L^2(\beta))$, the space of all bounded operators on $L^2(\beta)$, that is, $L_\phi^\beta + L_\psi^\beta = L_{\phi+\psi}^\beta$ and $\alpha L_\phi^\beta = L_{\alpha \phi}^\beta$.

Furthermore, $\|L_\phi^\beta\| \leq 2\|\phi\|_{\infty}$ and $L_\phi^\beta = 0$ if and only if $\phi = 0$. One can also observe that the correspondence $\phi \rightarrow L_\phi^\beta$ is an injective linear mapping from $L^\infty(\beta)$ into $\mathcal{B}(L^2(\beta))$. The matrix of $L_\phi^\beta$ with respect to the orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ of $L^2(\beta)$ is of the form
As is observed in the case of slant Toe-Hank operators, the matrix of weighted slant Toe-Hank operator \( L_\phi^\beta \) provides the matrix of weighted slant Hankel operator \( K_\phi^\beta \) if only even columns are considered and the matrix of slant weighted Toeplitz operator \( U_z^\beta \) if only odd columns are considered. Further, if \( \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \) is the Fourier expansion of \( \phi \) and \( \{\alpha_{i,j}\}_{i,j \in \mathbb{Z}} \) denotes the matrix of the operator \( L_\phi^\beta \), then the \((i,j)^{th}\) entry is given by \( \langle \alpha_{i,j} \rangle = \langle a_{-2i-2n+1} \frac{\beta_k}{\beta_n^{1/2}} \rangle \), if \( j = 2n \) and \( \langle \alpha_{i,j} \rangle = \langle a_{-2i+n+1} \frac{\beta_k}{\beta_n^{1/2}} \rangle \), if \( j = 2n + 1, n \in \mathbb{Z} \). Clearly, \( \{\alpha_{i,j}\}_{i,j \in \mathbb{Z}} \) satisfies the following relations:

\[
\begin{aligned}
\beta_{2j-1} &\cdot \alpha_{k+j,4j-1} = \beta_{2j-1} \cdot \alpha_{k,j,4j} = \beta_{2j} \cdot \alpha_{k,0} \quad \text{for } k,j \in \mathbb{Z} \\
\beta_{2j} &\cdot \alpha_{k,j,4j+1} = \beta_{2j} \cdot \alpha_{k,j,4j-2} = \beta_{2j} \cdot \alpha_{k,1} \quad \text{for } k,j \in \mathbb{Z}.
\end{aligned}
\] (2.1)

In [2] and [3], the matrix characterizations for slant weighted Toeplitz and weighted slant Hankel operators are obtained. Similarly, one can expect a matrix characterization for weighted slant Toe-Hank operators. For that purpose, we introduce the following notion.

A doubly infinite matrix \( \{\alpha_{i,j}\}_{i,j \in \mathbb{Z}} \) is said to be a weighted slant Toe-Hank matrix if it satisfies the relation (2.1).

We begin with the result which serves as a great tool for our study.

**Lemma 2.5.** If \( A \) is any bounded linear operator on \( L^2(\beta) \) such that its matrix \( \{\alpha_{i,j}\}_{i,j \in \mathbb{Z}} \) is a weighted slant Toe-Hank matrix, then the following holds:

\[
\begin{align*}
(1) \quad \frac{\beta_{i+k}}{\beta_{i-k}} \alpha_{i-k,2j+4k} &= \frac{\beta_{i-k}}{\beta_{i-k}} \alpha_{i,j} \quad \text{for } i,j,k \in \mathbb{Z}. \\
(2) \quad \frac{\beta_{i+2}}{\beta_{i+1}} \alpha_{i+2,2j+5} &= \frac{\beta_{i+2}}{\beta_{i+1}} \alpha_{i,j+1} \quad \text{for } i,j,k \in \mathbb{Z}.
\end{align*}
\]

**Proof.** We first prove (1). Let \( i', k' \) and \( j' \) be any integers. First we consider the case when \( j' \) is an odd integer. As \( \{\alpha_{i,j}\}_{i,j \in \mathbb{Z}} \) is a weighted slant Toe-Hank matrix, it satisfies the relation (2.1). Hence, for each \( k,j \in \mathbb{Z} \),

\[
\frac{\beta_{2j-1}}{\beta_{2j-1}} \alpha_{k,j-4j-2} = \frac{\beta_{0}}{\beta_{2j-1}} \alpha_{k,1}.
\]

On substituting \( k = i' + \frac{j'+1}{2}, j = i' + \frac{j'+2}{2} \) and \( k = i' + \frac{j'+1}{2}, j = i' + \frac{j'+2}{2} \) successively in the above equation, we get

\[
\frac{\beta_{2j+2}}{\beta_{2j-1}} \alpha_{i' - k',2j+4k} = \frac{\beta_{0}}{\beta_{2j+1}} \alpha_{i'+(L'+1)} = \frac{\beta_{2j+2}}{\beta_{-i'}} \alpha_{i'+2j'}.
\]
Now consider the case when \( j' \) is an even integer. Again from relation (2.1), for each \( k, j \in \mathbb{Z} \),

\[
\frac{\beta_{2j}}{\beta_{-(k-j)}} \alpha_{k-j,4j} = \frac{\beta_0}{\beta_{-k}} \alpha_{k,0}.
\]

This equation, on substituting \( k = i' + \left( \frac{j'}{2} \right), j = \frac{j'+2k'}{2} \) and \( k = i' + \left( \frac{j'}{2} \right), j = j' + 2k' \) successively, gives that

\[
\frac{\beta_{j'+2k'}}{\beta_{-(i'-k')}} \alpha_{i'-k',2j'+4k'} = \frac{\beta_0}{\beta_{-(i'+j'+2k')}} \alpha_{i'+j'+2k',0} = \frac{\beta_{j'}}{\beta_{-i'}} \alpha_{i'+j'}.
\]

This completes the proof of (1). Similarly, (2) can be obtained using the equations

\[
\frac{\beta_{2j}}{\beta_{k+j}} \alpha_{k+j,4j+1} = \frac{\beta_0}{\beta_k} \alpha_{k,1} \quad \text{and} \quad \frac{\beta_{2j-1}}{\beta_{k+j}} \alpha_{k+j,4j-1} = \frac{\beta_0}{\beta_{-k}} \alpha_{k,0}
\]

from (2.1).

In [12], Zorboska discussed the notion of composition operator \( C_{\beta}^\psi \) (\( f \to f \circ \phi \)) on the weighted sequence spaces. It is evident from here that if the sequence \( \{\beta_n\}_{n \in \mathbb{Z}} \) be such that the sequence \( \{\frac{\beta_{2n}}{\beta_n}\}_{n \in \mathbb{Z}} \) is bounded, then the composition operator \( C_{\beta}^\psi \) is a bounded operator on \( L^2(\beta) \). For \( \beta_n = 1 \) for each \( n \), the operator \( C_{\beta}^\psi \) coincides with the composition operator \( C_{\beta}^\psi \) on \( L^2(\mathbb{T}) \). Further, it is proved in [4] that \( AC_{\beta}^\psi \) is a slant Hankel operator and \( AM_{\beta}^\psi C_{\beta}^\psi \) is a slant Toeplitz operator for every slant Toep-Hank operator \( A \) on \( L^2(\mathbb{T}) \). However, we will see that this is not the situation in case of weighted slant Toe-Hank operator.

It is known that corresponding to the weight sequence \( \{\beta_n\}_{n \in \mathbb{Z}} \) of positive real numbers, a doubly infinite matrix \( \{\lambda_{ij}\}_{i,j \in \mathbb{Z}} \) is called

(1) slant weighted Toeplitz matrix if \( \frac{\beta_{i}}{\beta_{k}} \lambda_{i,j} = \frac{\beta_{i+k}}{\beta_{i+1}} \lambda_{i+1,j+2} \) for each \( i, j \in \mathbb{Z} \).

(2) weighted slant Hankel matrix if \( \frac{\beta_{i}}{\beta_{-i}} \lambda_{i,j} = \frac{\beta_{i+k}}{\beta_{-i-k}} \lambda_{i-k,j+2k} \) for each \( i, j, k \in \mathbb{Z} \).

Under the assumptions of \( \beta = \{\beta_n\}_{n \in \mathbb{Z}} \) being semi-dual and \( \{\frac{\beta_{2n}}{\beta_n}\}_{n \in \mathbb{Z}} \) being bounded, it is shown in [2] ([3]) that an operator on \( L^2(\beta) \) is slant weighted Toeplitz (weighted slant Hankel) operator if and only if its matrix is a slant weighted Toeplitz (weighted slant Hankel) matrix.

We show through next example that for a weighted slant Toe-Hank operator \( A \) on \( L^2(\beta) \), \( AC_{\beta}^\psi \) and \( AM_{\beta}^\psi C_{\beta}^\psi \) need not be weighted slant Hankel and slant weighted Toeplitz operator respectively.
Example 2.6. Let $\phi(z) = z^{-3} + 1$ and $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be defined as

$$
\beta_n = \begin{cases} 
1 & \text{if } n = 0, 1, -1 \\
2 & \text{otherwise}
\end{cases}
$$

Then $\{\beta_n\}$ is a bounded semi-dual sequence such that $\frac{1}{2} \leq \frac{\beta_n}{\beta_{n+1}} \leq 1$ for $n \geq 0$ and $\{\beta_{2n}\}_{n \in \mathbb{Z}}$ is bounded. We see that $\phi \in L^\infty(\beta)$. Consider the weighted slant Toeplitz operator $A$ ($= L^\beta_\phi$) on $L^2(\beta)$. Let $\{\alpha_{i,j}\}_{i,j \in \mathbb{Z}}$, $\{\lambda_{i,j}\}_{i,j \in \mathbb{Z}}$ and $\{\gamma_{i,j}\}_{i,j \in \mathbb{Z}}$ be the matrices of $A$, $AC_z^\beta$ and $AM_z^\beta C_z^\beta$, respectively with respect to the orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ of $L^2(\beta)$. Now using Lemma 2.5, we find that the matrix $\{\lambda_{i,j}\}_{i,j \in \mathbb{Z}}$ of $AC_z^\beta$ satisfies

$$
\frac{\beta_{j+2k}}{\beta_{-i-k}} \lambda_{i-k,j+2k} = \frac{\beta_{j+2k}}{\beta_{-i-k}} \langle AC_z^\beta e_{j+2k}, e_{i-k} \rangle = \frac{\beta_{j+2k}}{\beta_{-i-k}} \langle \beta_{2j+4k} A e_{2j+4k}, e_{i-k} \rangle = \frac{\beta_{2j+4k}}{\beta_{-i-k}} \alpha_{i-k,2j} \beta_{j+2k} \beta_{-i-k} \alpha_{i,2j}
$$

and $\frac{\beta_{i+2k}}{\beta_{i+1}} \lambda_{i-1,j+2} = \frac{\beta_{i+2k}}{\beta_{i+1}} \langle AM_z^\beta C_z^\beta e_{j+2}, e_{i+1} \rangle = \frac{\beta_{2j+5}}{\beta_{j+2}} \frac{\beta_{i}}{\beta_{i+1}} \alpha_{i,2j+1}$ for each $i, j, k \in \mathbb{Z}$. Thus for $i = j = k = 1$, we find that $\frac{\beta_{j+2k}}{\beta_{i-k}} \lambda_{i-1,j+2} = 1 \neq 2 = \frac{\beta_{i}}{\beta_{i+1}} \alpha_{i,2j}$. Hence, $AC_z^\beta$ can not be a weighted slant Hankel operator.

On the similar lines of computation, we obtain that

$$
\frac{\beta_{j+2}}{\beta_{i+1}} \gamma_{i+1,j+2} = \frac{\beta_{j+2}}{\beta_{i+1}} \langle AM_z^\beta C_z^\beta e_{j+2}, e_{i+1} \rangle = \frac{\beta_{2j+5}}{\beta_{j+2}} \frac{\beta_{i}}{\beta_{i+1}} \alpha_{i,2j+1}
$$

and $\frac{\beta_{i+2}}{\beta_{i+1}} \gamma_{i,j} = \frac{\beta_{i+2}}{\beta_{i+1}} \alpha_{i,2j+1}$ for each $i, j \in \mathbb{Z}$. These for $i = j = 1$ show that $\frac{\beta_{i+2}}{\beta_{i+1}} \gamma_{i+1,j+2} = 1 \neq 2 = \frac{\beta_{i}}{\beta_{i+1}} \gamma_{i,j}$. This shows that $AM_z^\beta C_z^\beta$ can not be a slant weighted Toeplitz operator.

In order to derive a weighted slant Hankel operator (slant weighted Toeplitz operator) from a given weighted slant Toeplitz operator, we proceed to define the following operators.

**Definition 2.7.** Consider the following operators defined for $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^2(\beta)$,

1. An operator $\hat{C}_z^\beta$ on $L^2(\beta)$ is defined as

$$
\hat{C}_z^\beta(f(z)) = \sum_{n=-\infty}^{\infty} a_n \frac{\beta_n}{\beta_{2n}} z^{2n}.
$$
An operator $\tilde{M}_\beta^z$ on $L^2(\beta)$ is defined as
\[
\tilde{M}_\beta^z(f(z)) = \sum_{n=-\infty}^\infty \frac{\beta_n}{\beta_{n+1}} a_n z^{n+1}.
\]
Clearly $\hat{C}_\beta^z$ and $\tilde{M}_\beta^z$ are bounded linear operators on $L^2(\beta)$. Further, $\hat{C}_\beta^z(e_n) = \hat{C}_\beta^z(\frac{\beta_n}{\beta_{n+1}}) = \frac{e_{2n}}{\beta_{2n}} = e_{2n}$ and $\tilde{M}_\beta^z(e_n) = e_{n+1}$ for each $n \in \mathbb{Z}$.

The following result is now immediate.

**Proposition 2.8.** Let $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence such that $\{\frac{\beta_{2n}}{\beta_{2n+1}}\}_{n \in \mathbb{Z}}$ is bounded. If matrix of any bounded linear operator $A$ defined on $L^2(\beta)$ is a weighted slant Toeplitz operator, then $A\hat{C}_\beta^z$ is a weighted slant Hankel operator and $A\tilde{M}_\beta^z \hat{C}_\beta^z$ is a slant weighted Toeplitz operator on $L^2(\beta)$.

It is clear that the matrix representation of a weighted slant Toeplitz operator on $L^2(\beta)$ is always (without $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ being semi-dual or $\{\frac{\beta_{2n}}{\beta_{2n+1}}\}_{n \in \mathbb{Z}}$ is bounded) a weighted slant Toeplitz matrix. However, these additional assumptions on the sequence $\{\beta_n\}_{n \in \mathbb{Z}}$ along with Proposition 2.8 help us to prove the main result of this section as follows.

**Theorem 2.9.** Let $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be a sequence such that $\{\frac{\beta_{2n}}{\beta_{2n+1}}\}_{n \in \mathbb{Z}}$ is bounded. Then for a bounded operator $A$ on $L^2(\beta)$, the following are equivalent.

1. $A$ is a weighted slant Toeplitz operator.
2. Matrix of $A$ with respect to the orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ of $L^2(\beta)$ is a weighted slant Toeplitz matrix.
3. $A$ satisfies the following equations:
   \begin{align*}
   (a) & \quad M_{z^{-1}}^z A \hat{C}_\beta^z = A \hat{C}^z_{z^{-1}} M_{z^2}^\beta, \\
   (b) & \quad M_z^z A \tilde{M}_z^\beta \hat{C}_\beta^z = A \tilde{M}_z^z \hat{C}_\beta^z M_{z^2}^\beta, \\
   (c) & \quad M_{z^{-1}}^z A M_{z^{-1}}^z \hat{C}_\beta^z e_1 = A M_{z^2}^\beta e_1, \\
   (d) & \quad A M_{z^2}^z \hat{C}_\beta^z e_1 = M_z^z A M_{z^{-1}}^z e_1.
   \end{align*}

**Proof.** Let $\{\alpha_{i,j}\}_{i,j \in \mathbb{Z}}$ denotes the matrix of operator $A$ with respect to the orthonormal basis $\{e_n\}_{n \in \mathbb{Z}}$ of $L^2(\beta)$.

(1) implies (2) is obvious. For the reverse, assume that the matrix $\{\alpha_{i,j}\}_{i,j \in \mathbb{Z}}$ of $A$ is a weighted slant Toeplitz matrix and hence satisfies the relation (2.1). Using Proposition 2.8, $A\hat{C}_\beta^z$ is a weighted slant Hankel operator and $A\tilde{M}_\beta^z \hat{C}_\beta^z$ is a slant weighted Toeplitz operator on $L^2(\beta)$. Let $A\hat{C}_\beta^z = U_\psi^\beta$ and $A\tilde{M}_\beta^z \hat{C}_\beta^z = K_\zeta^\beta$ for some $\psi(z) = \sum_{n=-\infty}^{\infty} b_n z^n$.
and $\zeta(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ in $L^\infty(\beta)$. Let $\{\gamma_{i,j}\}_{i,j \in \mathbb{Z}}$ and $\{\lambda_{i,j}\}_{i,j \in \mathbb{Z}}$ be the matrices of $A M^\beta z \hat{C}^\beta z^2$ and $A \hat{C}^\beta z^2$ respectively. Then using the definition of slant weighted Toeplitz operator, we have $\gamma_{i,j} = \frac{\beta}{\beta_j} b_{2i-j}$ for $i,j \in \mathbb{Z}$. This fact along with the equations in (2.1) yields that for each $n \in \mathbb{Z}$

\[
\langle \psi, e_n \rangle = \begin{cases} 
\frac{\beta_0}{\beta_k} \gamma_{0,0} = \frac{\beta_0}{\beta_k} \langle A M^\beta z \hat{C}^\beta z^2 e_0, e_k \rangle & \text{if } n = 2k \\
\frac{\beta_1}{\beta_k} \gamma_{1,1} = \frac{\beta_1}{\beta_k} \langle A M^\beta z \hat{C}^\beta z^2 e_1, e_k \rangle & \text{if } n = 2k - 1 
\end{cases}
\]

\[
= \begin{cases} 
\frac{\beta_0}{\beta_k} \alpha_{k,1} & \text{if } n = 2k \\
\frac{\beta_1}{\beta_k} \alpha_{k,2} = \frac{\beta_0}{\beta_{-k-1}} \alpha_{k+1,0} & \text{if } n = 2k - 1
\end{cases}
\]

Define a complex valued function $\phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, where $a_n = b_{-n+1}$ for $n \in \mathbb{Z}$. Then $\psi = z \phi$ so that $\phi \in L^\infty(\beta)$ (since $\{\beta_n\}_{n \in \mathbb{Z}}$ is semi-dual). Now by the definition of weighted slant Hankel operator, we have $\lambda_{i,j} = \frac{\beta}{\beta_j} c_{-2i-j} = \frac{\beta}{\beta_j} c_{-2i-j}$ for $i,j \in \mathbb{Z}$. This gives that for $n \in \mathbb{Z}$

\[
c_n = \langle \zeta, e_n \rangle = \begin{cases} 
\frac{\beta_0}{\beta_{k+1}} \lambda_{k,0} = \frac{\beta_0}{\beta_{k+1}} \langle A \hat{C}^\beta z^2 e_0, e_k \rangle & \text{if } n = -2k \\
\frac{\beta_1}{\beta_k} \lambda_{k,1} = \frac{\beta_1}{\beta_k} \langle A \hat{C}^\beta z^2 e_1, e_k \rangle & \text{if } n = -2k - 1 
\end{cases}
\]

\[
= \begin{cases} 
\frac{\beta_0}{\beta_{k+1}} \alpha_{k,0} = b_{2k+1} & \text{if } n = -2k \\
\frac{\beta_1}{\beta_k} \alpha_{k,2} = \frac{\beta_0}{\beta_{k+1}} \alpha_{k+1,1} & \text{if } n = -2k - 1
\end{cases} = a_n.
\]

This provides that $\zeta = \phi$. Hence, $A \hat{C}^\beta z^2$ is the weighted slant Hankel operator $K^\beta_\phi$ and $A M^\beta z \hat{C}^\beta z^2$ is the slant weighted Toeplitz operator $U^\beta_{z \phi}$.

Now each $h(z) \in L^2(\beta)$ can be written as $h(z) = h_1(z^2) + z h_2(z^2)$ with $h_1, h_2 \in L^2(\beta)$. Say, $h_1(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ and $h_2(z) = \sum_{n=-\infty}^{\infty} b_n z^n$. Then, $h_1(z^2) = \sum_{n=-\infty}^{\infty} a_n z^{2n}, \; z h_2(z^2) = \sum_{n=-\infty}^{\infty} b_n z^{2n+1}$ and

\[
L^\beta_\phi h(z) = (K^\beta_\phi \hat{W}^\beta + U^\beta_{z \phi} \hat{M}^\beta)(h_1(z^2) + z h_2(z^2))
\]

\[
= K^\beta_\phi (\sum_{n=-\infty}^{\infty} a_n \beta_{2n} e_n) + U^\beta_{z \phi} (\sum_{n=-\infty}^{\infty} b_n \beta_{2n+1} e_n)
\]

\[
= A \hat{C}^\beta z^2 (\sum_{n=-\infty}^{\infty} a_n \beta_{2n} e_n) + A M^\beta z \hat{C}^\beta z^2 (\sum_{n=-\infty}^{\infty} b_n \beta_{2n+1} e_n)
\]
2.1

operator. The equations (c) and (d) respectively give slant Hankel operator and slant weighted Toeplitz if and only if

\[ A \text{ is a weighted slant } \text{Toeplitz} \] if and only if \( (1) \), we obtain that (a) and (b) using Proposition 2.8.

From (2.1), \( \frac{\partial_{j-1}}{\partial_{j-k}} \alpha_{k,j} = \frac{\partial_{j}}{\partial_{k}} \alpha_{k,1} \) for each \( k, j \in \mathbb{Z} \). It provides on replacing \( k \) by \( k+1 \) and \( j \) by 1, that \( \frac{\partial_{j}}{\partial_{k}} \alpha_{k,2} = \frac{\partial_{j}}{\partial_{k+1}} \alpha_{k+1,1} \) for each \( k \in \mathbb{Z} \). Then \( (M_{z}^\alpha B \hat{M}_{z}^\beta) \) is the slant weighted Toeplitz operator and \( \hat{M}_{z}^\beta \) is the slant weighted Toeplitz operator. The equations (c) and (d) respectively gives \( \frac{\partial_{j}}{\partial_{k}} \alpha_{k,2} = \frac{\partial_{j}}{\partial_{k+1}} \alpha_{k+1,1} \) and \( \frac{\partial_{j}}{\partial_{k}} \alpha_{k,3} = \frac{\partial_{j}}{\partial_{k-1}} \alpha_{k-1,0} \) for each \( k \in \mathbb{Z} \).

On using these facts and applying the arguments as earlier, we obtain (d).

Now on replacing \( k \) by \( k+1 \) and \( j \) by 1 in the equation \( \frac{\partial_{j}}{\partial_{k}} \alpha_{k,0} \) of (2.1) and applying the same arguments as earlier, we obtain (d).

In order to obtain (1) from (3), suppose that \( A \) satisfies (a), (b), (c) and (d). Then, (a) and (b) respectively provide that \( A \hat{C}_{z}^\beta \) is the slant weighted Hankel operator and \( \hat{A} \hat{C}_{z}^\beta \) is the slant weighted Toeplitz operator. This completes the proof.

The adjoint \( L_\phi^* \) of a weighted slant Toeplitz operator \( L_\phi^\beta \) is nothing but an operator on \( L^2(\beta) \) satisfying \( L_\phi^* = \hat{W}^{\beta} K_\phi^\beta + \hat{M}^{\beta} U_{z}^{\beta} \), where \( \hat{W}^{\beta} \) and \( \hat{M}^{\beta} \) on \( L^2(\beta) \) are defined as \( \hat{W}^{\beta} (e_n) = e_{2n} \) and \( \hat{M}^{\beta} (e_n) = e_{2n+1} \) for \( n \in \mathbb{Z} \). Further, if \( \phi(z) = \sum_{n=-m}^{\infty} a_n z^n \in L^\infty(\beta) \), then for each \( j \in \mathbb{Z} \)

\[ K_\beta^\beta \ e_j = \beta_{-j} \sum_{n=-\infty}^{\infty} \frac{\partial_{2j-n}}{\beta_n} e_n \] and \( U_{z}^{\beta} \ e_j = \beta_{j} \sum_{n=-\infty}^{\infty} \frac{\partial_{2j+n+1}}{\beta_n} e_n \].
It is now natural from Theorem 2.9 that the adjoint of a weighted slant Toep-Hank operator need not be a weighted slant Toep-Hank operator on $L^2(\beta)$. This can be verified from the following example.

**Example 2.10.** Let $\phi(z) = z^{-1}$ and $\beta = \{\beta_n\}_{n \in \mathbb{Z}}$ be the sequence as in Example 2.6. Clearly, $\phi \in L^\infty(\beta)$. But the matrix of $L^\beta_{\phi^*}$ is not a weighted slant Toep-Hank matrix as, if we take $k = 2$ and $j = 2$ then the condition

$$\frac{\beta_{2j-1}}{\beta_{k+j}} \langle L^\beta_{\phi^*} e_{4j-1}, e_{k+j} \rangle = \frac{\beta_0}{\beta_{-k}} \langle L^\beta_{\phi^*} e_0, e_k \rangle$$

implies that $0 = \frac{1}{2}$. Hence $L^\beta_{\phi^*}$ is not a weighted slant Toep-Hank operator on $L^2(\beta)$.

However if $L^\beta_{\phi^*}$, for $\phi \in L^\infty(\beta)$, is a weighted slant Toep-Hank operator on $L^2(\beta)$, then from relation (2.1), for each $k, j \in \mathbb{Z},$

$$\frac{\beta_{2j}}{\beta_{-(k-j)}} \langle L^\beta_{\phi^*} e_{4j}, e_{k-j} \rangle = \frac{\beta_0}{\beta_{-k}} \langle L^\beta_{\phi^*} e_0, e_k \rangle.$$

In particular, for $k = j = 2l$ ($l \in \mathbb{Z}$),

$$\frac{\beta_{4l}}{\beta_0} \langle e_{8l}, K^\beta_{\phi} e_0 \rangle = \frac{\beta_0}{\beta_{-2l}} \langle e_0, K^\beta_{\phi} e_l \rangle.$$

This implies that $a_{-l} = \beta_l \beta_{-2l} \beta_{4l} \beta_{-8l} a_{-16l}$ for each $l \in \mathbb{Z}$. From here we observe that if the sequence $\{\beta_n\}_{n \in \mathbb{Z}}$ is such that $\beta_n = 1$ for each $n$, then using the fact that $\lim_{n \to \infty} a_n = 0$, we have $a_n = 0$ for all $n \in \mathbb{Z}$. So $\phi = 0$.

The above observation can be summed up in the following form.

**Remark 2.11.** The only self-adjoint slant Toep-Hank operator on $L^2(\mathbb{T})$ is the zero operator.

### 3. Connection among various classes

Proposition 2.8 helps us to obtain a weighted slant Hankel operator or slant weighted Toeplitz operator from a given weighted slant Toep-Hank operator. In the present section, our aim is to compute the intersection of the class of weighted slant Toep-Hank operators with the classes of weighted slant Hankel and slant weighted Toeplitz operators. Let the classes of weighted slant Toep-Hank, weighted slant Hankel and slant weighted Toeplitz operators be denoted by $C_{wst-h}$, $C_{wsh}$ and $C_{swt}$ respectively. Now we have the following.

**Theorem 3.1.** A weighted slant Hankel operator on $L^2(\beta)$ is a weighted slant Toep-Hank operator if and only if it is a zero operator.
Proof. Let a weighted slant Hankel operator \( A \) be a weighted slant Toep-Hank operator on \( L^2(\beta) \), say \( A = L^\beta_\phi \) for some \( \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta) \). Thus, its matrix satisfies equations in (2.1). The operator \( A \), being a weighted slant Hankel operator, satisfies

\[
A e_j(z) = L^\beta_\phi e_j(z) = \frac{1}{\beta_j} \sum_{n=-\infty}^{\infty} a_{-2n-j} \beta_n e_n(z).
\]

Now using relation (2.1), for all \( k, j \in \mathbb{Z} \), we have

\[
\frac{\beta_{2j}}{\beta_{-(k-j)}} \langle L^\beta_\phi e_{4j}, e_{k-j} \rangle = \frac{\beta_{0}}{\beta_{-k}} \langle L^\beta_\phi e_0, e_k \rangle.
\]

Hence

\[
\frac{\beta_{2j}}{\beta_{-(k-j)}} \left( \frac{1}{\beta_{4j}} \sum_{n=-\infty}^{\infty} a_{-2n-4j} \beta_n e_n, e_{k-j} \right) = \frac{\beta_{0}}{\beta_{-k}} \left( \frac{1}{\beta_{0}} \sum_{n=-\infty}^{\infty} a_{-2n} \beta_n e_n, e_k \right).
\]

Therefore \( a_{-2k-2j} = a_{-2k} \frac{\beta_{4j}}{\beta_{2j}} \) for all \( k, j \in \mathbb{Z} \). Putting \( k = 0 \), we get \( a_{-2j} = a_0 \frac{\beta_{4j}}{\beta_{2j}} \) for all \( j \in \mathbb{Z} \).

Again from (2.1) for \( k, j \in \mathbb{Z} \),

\[
\frac{\beta_{2j-1}}{\beta_{k+j}} \langle L^\beta_\phi e_{4j-1}, e_{k+j} \rangle = \frac{\beta_{0}}{\beta_{-k}} \langle L^\beta_\phi e_0, e_k \rangle.
\]

Using the same computations as above, we get \( a_{-2j+1} = a_0 \frac{\beta_{4j}}{\beta_{2j}} \) for all \( j \in \mathbb{Z} \). Now, since

\[
|a_0|^2 \sum_{n=-\infty}^{\infty} (1)^n \leq \sum_{n=-\infty}^{\infty} \frac{\beta_{4n}}{\beta_{2n}} |a_0|^2 \beta_n^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty,
\]

hence we must have \( a_0 = 0 \) so that \( \phi = 0 \). This provides that \( A \) is a zero operator. The converse is obvious. Hence the theorem. □

The next result, proof of which follows almost along the same arguments as made in Theorem 3.1, is the following.

Theorem 3.2. A non-zero slant weighted Toeplitz operator cannot be a weighted slant Toep-Hank operator on \( L^2(\beta) \).

From Theorem 3.1 and Theorem 3.2, we conclude that

\[
C_{wst-h} \cap C_{wsh} = \{0\} = C_{wst-h} \cap C_{swt}.
\]

Our next result discusses the product of weighted slant Toep-Hank operator with the operator \( W^\beta \) as well as with the weighted Laurent operator. We recall that the adjoint of \( W^\beta \) is given by \( W^{\beta^*} e_n(z) = \frac{\beta_n}{\beta_{2n}} e_{2n}(z), n \in \mathbb{Z} \).

Theorem 3.3. For \( \phi, \psi \) in \( L^\infty(\beta) \),
For the necessary part of (1), let \( M \) be the matrix representation of that operator and \( \beta_k \) be its matrix. Then relation (2.1) gives that for each \( k, j \in \mathbb{Z} \),

\[
\frac{\beta_{2j-1}}{\beta_{k+j}} \alpha_{k+j, 4j-1} = \frac{\beta_0}{\beta_{-k}} \alpha_{k, 0},
\]

which implies that

\[
\frac{\beta_{2j-1}}{\beta_{k+j}} \langle W^\beta L^\beta_\phi a_{4j-1}, e_{k+j} \rangle = \frac{\beta_0}{\beta_{-k}} \langle W^\beta L^\beta_\phi a_0, e_k \rangle.
\]

Hence,

\[
\frac{\beta_{2j-1}}{\beta_{k+j}} \langle U^\beta e_{2j-1}, W^\beta e_k \rangle = \frac{\beta_0}{\beta_{-k}} \langle K^\beta_\phi e_0, W^\beta e_k \rangle.
\]

This equality for \( k = 0 \) provides that \( a_0 = a_{-2j} \) for all \( j \in \mathbb{Z} \). Similarly, using the equality \( \frac{\beta_{2j}}{\beta_{k+j}} \alpha_{k+j, 4j+1} = \frac{\beta_0}{\beta_{-k}} \alpha_{k, 1} \) of (2.1), we obtain that \( a_{-2} = a_{-2j+1} \) for \( j \in \mathbb{Z} \). As \( \phi \in L^\infty(\beta) \subseteq L^2(\beta) \), we must have \( a_0 = a_{-2} = 0 \) so that \( \phi = 0 \). Sufficient part is obvious. Hence the result.

Now we prove (2). Let \( \psi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in L^\infty(\beta) \) and \( \{\alpha_{i,j}\}_{i,j} \in \mathbb{Z} \) be the matrix representation of \( M^\beta_{z} L^\beta_\psi \). Without much computations, we obtain that

\[
\frac{\beta_{2j-1}}{\beta_{k+j}} \alpha_{k+j, 4j-1} = \frac{\beta_{2j}}{\beta_{-k}} \alpha_{k-j, 4j} = \frac{\beta_0}{\beta_{-k}} \alpha_{k, 0} = a_{-2k+2m}
\]

and

\[
\frac{\beta_{2j}}{\beta_{k+j}} \alpha_{k+j, 4j+1} = \frac{\beta_{2j-1}}{\beta_{-k}} \alpha_{k-j, 4j-2} = \frac{\beta_0}{\beta_{-k}} \alpha_{k, 1} = a_{-2k+2m+1}
\]

for each \( k, j \in \mathbb{Z} \). Thus \( \{\alpha_{i,j}\}_{i,j} \in \mathbb{Z} \) is a weighted slant Toeplitz matrix.

Similarly, if \( \{\gamma_{i,j}\}_{i,j} \in \mathbb{Z} \) denotes the matrix of \( M^\beta_\psi L^\beta_{z} \), \( \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n \), then for each \( k, j \in \mathbb{Z} \)

\[
\frac{\beta_{2j-1}}{\beta_{k+j}} \gamma_{k+j, 4j-1} = \frac{\beta_{2j}}{\beta_{-k}} \gamma_{k-j, 4j} = \frac{\beta_0}{\beta_{-k}} \gamma_{k, 0} = \begin{cases} \overline{a}_{k+m} & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}
\]
and
\[
\beta_{2j} \gamma_{k+j,4j+1} = \beta_{2j-1} \gamma_{k-j,4j-2} = \beta_0 \gamma_{j,1} = \begin{cases} 
0 & \text{if } m \text{ is even} \\
\pi_{k+\left(\frac{m-1}{2}\right)} & \text{if } m \text{ is odd}.
\end{cases}
\]

This proves that \( \{ \gamma_{i,j} \}_{i,j \in \mathbb{Z}} \) is a weighted slant Toep-Hank matrix. This completes the proof.

\[\square\]

References