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Existence results for hybrid fractional differential equations with Hilfer fractional derivative

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Abstract. This paper investigates the solvability, existence and uniqueness of solutions for a class of nonlinear fractional hybrid differential equations with Hilfer fractional derivative in a weighted normed space. The main result is proved by means of a fixed point theorem due to Dhage. An example to illustrate the results is included.

Keywords: Hybrid fractional differential equations; Initial value problem; Hilfer fractional derivative; Fixed point theorem; Existence and uniqueness.

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1. Introduction

Fractional differential equations (FDEs) have been applied in many fields, such as physics, mechanics, chemistry, engineering etc. There has been a significant progress in ordinary differential equations involving fractional order derivative, see the monographs of Hilfer [13], Kilbas [16] and Podlubny [21]. Especially, numerous works have been devoted to the study of initial value problems, for example, see [2, 11].

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In the recent years, some authors have considered Hilfer fractional derivative see[7, 8, 9, 14, 15, 17] and references therein. R. Hilfer [13] proposed a generalized Riemann-Liouville fractional derivative, for short, Hilfer fractional derivative, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. This operator has appeared in the theoretical simulation of dielectric relaxation in glass forming materials. In [7], Furati et al. considered an initial value problem for a class of nonlinear FDEs involving Hilfer fractional derivative. Recently, Abbas et al. [1] studied the existence and stability results for FDEs with Hilfer fractional derivative.

One of interesting problems in the field of the FDE, which has attracted much attention, is hybrid fractional differential equations (HFDEs). For some works in this topic, one can refer to [3, 4, 18, 19].

The following hybrid differential equation of the first order

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J := [0,T], \\ x(t_0) = x_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

was studied by Dhage et al. [6], under the assumptions $f \in C(J \times \mathbb{R}, \mathbb{R}|\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. In [19], Zhao et al. investigated the fractional version of the problem (1.1), i.e.,

$$\begin{cases} D^{\alpha}_{0^+}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)), \quad t \in J,\ \alpha \in (0,1), \\ x(0) = 0, \end{cases} \tag{1.2}$$

where $f \in C(J\times\mathbb{R}, \mathbb{R}|\{0\})$, and $g \in C(J\times\mathbb{R}, \mathbb{R})$. A fixed point theorem in Banach algebras was the main tool used in this work. Based on the above works, we develop the theory of HFDEs involving Hilfer fractional derivative. In this paper, we consider the following HFDE:

$$D^{\alpha,\beta}_{0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)),\ t \in J := [0,T],\ 0 < \alpha < 1,\ 0 \le \beta \le 1, \tag{1.3}$$

$$I^{1-\gamma}_{0^+}\left(\frac{x(0)}{f(0,x(0))}\right) = \phi, \quad \gamma = \alpha + \beta - \alpha\beta, \tag{1.4}$$

where $T > 0$, $\phi \in \mathbb{R}$, $f \in C(J \times \mathbb{R}, \mathbb{R}|\{0\})$, $g : C(J \times \mathbb{R}, \mathbb{R})$, and $D^{\alpha,\beta}_{0^+}$ is the Hilfer fractional derivative operator of the order α and type β. Moreover, the parameter γ satisfies $0 < \gamma \le 1$, $\gamma \ge \alpha$, $\gamma > \beta$, and $1-\gamma < 1-\beta(1-\alpha)$. Recall that $I^{1-\gamma}_{0^+}$ is the left-sided Riemann-Liouville integral of order $1-\gamma$.

The paper is arranged as follows: In Section 2, we introduce some definitions and some lemmas about Hilfer fractional type differential equations. In Section 3, we estabilish the existence condition about initial value problem (1.3)-(1.4). Subsequently, we discuss the uniqueness condition of the initial value problem (1.3)-(1.4) with $\phi = 0$. In Section 5, conclusion is given.

2. Prerequisites

Recall that $C(J, \mathbb{R})$ is the Banach space of all continuous real-valued functions defined on $J := [0, T]$ with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$. For $t \in J$, we define $x_r(t) = t^r x(t)$, $r \geq 0$. Let $C_r(J, \mathbb{R})$ be the space of all functions x such that $x_r \in C(J, \mathbb{R})$ which is indeed a Banach space endowed with the norm $\|x\|_{C_r} = \sup\{t^r |x(t)| : t \in J\}$.

Let $0 < \gamma \leq 1$ and $C_\gamma(J, \mathbb{R})$ denotes the weighted space of continuous functions defined by

$$C_\gamma(J, \mathbb{R}) = \left\{ g(t) : \ t^\gamma g(t) \in C(J, \mathbb{R}), \quad \|g\|_{C_\gamma} = \|t^\gamma g(t)\| \right\}.$$

Definition 2.1. [16, 17] The left-sided Riemann-Liouville integral of the order $\alpha > 0$ of a function $h \in L^1\{\mathbb{R}_+\}$ is defined by

$$(I_{0^+}^\alpha h)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds, \quad \text{for a.e. } t \in J,$$

where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt; \quad \xi > 0.$$

Notice that for all $\alpha, \alpha_1, \alpha_2 > 0$ and each $h \in C(J, \mathbb{R})$, we have $I_{0^+}^\alpha h \in C(J, \mathbb{R})$, and

$$(I_{0^+}^{\alpha_1} I_{0^+}^{\alpha_2} h)(t) = (I_{0^+}^{\alpha_1+\alpha_2} h)(t); \text{ for a.e. } t \in J. \tag{2.1}$$

Definition 2.2. [16, 17] The Riemann-Liouville fractional derivative of the order $\alpha \in (0, 1]$ of a function $h \in L^1\{\mathbb{R}_+\}$ is defined by

$$\begin{aligned}(D_{0^+}^\alpha h)(t) &= \left(\frac{d}{dt} I_{0^+}^{1-\alpha} h \right)(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s) ds; \quad \text{for a.e. } t \in J.\end{aligned}$$

Definition 2.3. [16, 17] The Caputo fractional derivative of the order $\alpha \in (0, 1]$ of a function $h \in L^1\{\mathbb{R}_+\}$ is defined by

$$\begin{aligned}({}^c D_{0^+}^\alpha h)(t) &= (I_{0^+}^{1-\alpha} \frac{d}{dt} h)(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} h(s) ds; \quad \text{for a.e. } t \in J.\end{aligned}$$

In [13], R. Hilfer studied some applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [14, 10]).

Definition 2.4. (Hilfer derivative). Let $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $h \in L^1\{\mathbb{R}_+\}$, $I_{0^+}^{(1-\alpha)(1-\beta)} \in C_\gamma^1(J, \mathbb{R})$. The Hilfer fractional derivative of the order α and type β of h is defined as

$$(D_{0^+}^{\alpha,\beta} h)(t) = \left(I_{0^+}^{\beta(1-\alpha)} \frac{d}{dt} I_{0^+}^{(1-\alpha)(1-\beta)} h \right)(t); \quad \text{for a.e. } t \in J. \tag{2.2}$$

The generalization (2.2) for $\beta = 0$ coincides with the Riemann-Liouville derivative and for $\beta = 1$ with the Caputo derivative, i.e., $D_{0^+}^{\alpha,0} = D_{0^+}^{\alpha}$, and $D_{0^+}^{\alpha,1} =^c D_{0^+}^{\alpha}$.

The following lemma shows that the solvability of the HFDE (1.3)-(1.4) is equivalent to a Volterra singular integral equation.

Lemma 2.5. *Let $g : J \times \mathbb{R} \to \mathbb{R}$ be a function such that $g(\cdot, x(\cdot)) \in C_\gamma(J, \mathbb{R})$ for any $x \in C_\gamma(J, \mathbb{R})$. A function $x \in C_\gamma(J, \mathbb{R})$ is a solution of the Hilfer fractional initial value problem* (1.3) *and* (1.4)*, if and only if x satisfies the following Volterra integral equation:*

$$x(t) = f(t, x(t)) \left(\frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s)) ds \right).$$

Proof. A simple change variable $(t-s)^\alpha = u$ allows us to prove the following result for any function $h \in L^1\{\mathbb{R}_+\}$

$$\frac{d}{dt}(I_{0^+}^{\alpha} h)(t) = I_{0^+}^{\alpha}(\frac{d}{dt} h(t)) + h(0) \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.3}$$

Defining $y(t) := \frac{x(t)}{f(t,x(t))}$, and then taking the operator $I_{0^+}^{\alpha}$ from both sides of the equation (1.3), we have

$$I_{0^+}^{\alpha}(D_{0^+}^{\alpha,\beta} y)(t) = I_{0^+}^{\alpha} g(t, x(t)),$$

But, applying (2.3), the left side of the above equation can be calculated as

$$\begin{aligned}
I_{0^+}^{\alpha}(D_{0^+}^{\alpha,\beta}y)(t) &= I_{0^+}^{\alpha}\left(I_{0^+}^{\beta(1-\alpha)}\frac{d}{dt}I_{0^+}^{(1-\alpha)(1-\beta)}y\right)(t)\\
&= \left(I_{0^+}^{\alpha+\beta-\alpha\beta}\frac{d}{dt}I_{0^+}^{1-\gamma}y\right)(t)\\
&= \left(I_{0^+}^{\gamma}\frac{d}{dt}I_{0^+}^{1-\gamma}y\right)(t)\\
&= \frac{d}{dt}\left(I_{0^+}^{\gamma}I_{0^+}^{1-\gamma}y\right)(t) - \left((I_{0^+}^{1-\gamma}y)(0)\right)\frac{t^{\gamma-1}}{\Gamma(\gamma)}\\
&= \frac{d}{dt}\left(I_{0^+}^{1}y\right)(t) - \left((I_{0^+}^{1-\gamma}y)(0)\right)\frac{t^{\gamma-1}}{\Gamma(\gamma)}\\
&= y(t) - \left((I_{0^+}^{1-\gamma}y)(0)\right)\frac{t^{\gamma-1}}{\Gamma(\gamma)}.
\end{aligned}$$

Therefore, the HFDE (1.3) under the condition (1.4) can be transformed into the following Volterra singular integral equation

$$x(t) = f(t, x(t))\left(\frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s, x(s))ds\right).$$

□

We also need to introduce a lemma as follows, which will be used in the proof of our main theorem. We recall the Dhage's fixed point point theorem as follows which will be used to obtain the existence and the uniqueness results from the Banach contraction principle.

Lemma 2.6. *(Dhage [5]) Let S be a non-empty, closed convex and bounded subset of the Banach algebra X let $A : X \to X$ and $B : S \to X$ be two operators such that:*

(a) *A is Lipschitzian with a Lipschitz constant k,*
(b) *B is completely continuous,*
(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$, and*
(d) *$Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $x = AxBx$ has a solution.

3. Main results

Before starting and proving the main results, we introduce the following hypotheses.

(H1) The function $f : J \times \mathbb{R} \to \mathbb{R}| \{0\}$ is bounded continuous and there exists a positive bounded function χ with bound $\|\chi\|$ such that

$$|f(t,x(t)) - f(t,y(t))| \leq \chi(t)\,|x(t) - y(t)|\,,$$

for $t \in J$ and for all $x, y \in \mathbb{R}$.

(H2) There exists a function $p \in C(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\Omega : [0,\infty) \to (0,\infty)$ such that

$$|g(t,x(t))| \leq p(t)\Omega(|x|), \quad (t,x) \in J \times \mathbb{R}.$$

(H3) There exists a number $r > 0$ such that

$$r \geq K\left[\frac{|\phi|}{\Gamma(\gamma)} + \frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\,\|p\|\,\Omega(r)\right], \tag{3.1}$$

where $|f(t,x)| \leq K$, for all $(t,x) \in J \times \mathbb{R}$, and

$$\|\chi\|\left[\frac{|\phi|}{\Gamma(\gamma)} + \frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\,\|p\|\,\Omega(r)\right] < 1.$$

(H4) There exists a constant $L_1 > 0$ such that

$$|f(t,x(t)) - f(t,y(t))| \leq L_1\,|x-y|\,, \quad \text{for all } x, y \in \mathbb{R},$$

and set $|f(t,x(t))| \leq M_1$.

(H5) There exists a constant $L_2 > 0$ such that

$$|g(t,x(t)) - g(t,y(t))| \leq L_2\,|x-y|\,, \quad \text{for all } x, y \in \mathbb{R},$$

and set $|g(t,x(t))| \leq M_2$.

Theorem 3.1. *Assume that (H1)-(H3) hold. Then there exists at least a solution of the problem* (1.3)-(1.4) *on J.*

Proof. In the following, we denote $\|x\|_{C_\gamma}$ by $\|x\|_C$. Set $X = C(J,\mathbb{R})$ and define a subset S of X as

$$S = \{x \in X : \ \|x\|_C \leq r\}\,,$$

where r satisfies inequality (3.1).

Clearly, S is non-empty, closed, convex and bounded subset of the Banach algebra X. By Lemma 2.5, the initial value problem (1.3)-(1.4) is equivalent to the integral equation

$$x(t) = f(t,x(t))\left(\frac{\phi}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds\right), \tag{3.2}$$

for all $t \in J$.

Define two operators $A : X \to X$ by

$$Ax(t) = f(t,x(t)), \quad t \in J, \tag{3.3}$$

and $B : S \to X$ by

$$Bx(t) = \frac{\phi}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s)) ds, \quad t \in J. \tag{3.4}$$

Then $x = AxBx$. We shall show that the operators A and B satisfy all the conditions of Lemma 2.6. For the sake of clarity, we split the proof into a sequence of steps.

Claim 1. The operator A is a Lipschitz on X, i.e. (a) of Lemma 2.6 holds.

Let $x, y \in X$. Then by (H1) we have

$$\begin{aligned} \left| t^{1-\gamma} (Ax(t) - Ay(t)) \right| &= t^{1-\gamma} \left| f(t, x(t)) - f(t, y(t)) \right| \\ &\leq \chi(t) t^{1-\gamma} \left| x(t) - y(t) \right| \\ &\leq \|\chi\| \, \|x - y\|_C \end{aligned}$$

for all $t \in J$. Taking the supremum over the interval $[0, T]$, we obtain

$$\|Ax - Ay\|_C \leq \|\chi\| \, \|x - y\|_C$$

for all $x, y \in X$. So A is a Lipschitz on X with Lipschitz constant $\|\chi\|$.

Claim 2. The operator B is completely continuous on S, i.e. (b) of Lemma 2.6 holds.

First we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} t^{1-\gamma} Bx_n(t) &= \lim_{n\to\infty} \left(\frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x_n(s)) ds \right) \\ &= \frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lim_{n\to\infty} g(s, x_n(s)) ds \\ &= \frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s)) ds \\ &= t^{1-\gamma} Bx(t), \end{aligned}$$

for all $t \in J$. This shows that B is continuouss on S. It is sufficient to show that $B(S)$ is uniformly bounded and equicontinuous set in X. First we note that

$$\begin{aligned} t^{1-\gamma} \left| Bx(t) \right| &= \left| \frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s)) ds \right| \\ &\leq \frac{|\phi|}{\Gamma(\gamma)} + \|p\| \, \Omega(r) \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds \\ &\leq \frac{|\phi|}{\Gamma(\gamma)} + \|p\| \, \Omega(r) \frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}, \end{aligned}$$

for all $t \in J$. Taking supremum over the interval J, the above inequality becomes

$$\|Bx\|_C \leq \frac{|\phi|}{\Gamma(\gamma)} + \|p\|\,\Omega(r)\frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)},$$

for all $x \in S$. This shows that B is uniformly bounded on S.

Next, we show that B is an equicontinuous set in X. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in S$. Then we have

$$\begin{aligned}
&\left|t_2^{1-\gamma}(Bx)(t_2) - t_1^{1-\gamma}(Bx)(t_1)\right| \\
&\leq \frac{\left\|p\right\|\Omega(r)}{\Gamma(\alpha)}\Big(\Big|\int_0^{t_1}\Big[t_2^{1-\gamma}(t_2-s)^{\alpha-1} - t_1^{1-\gamma}(t_1-s)^{\alpha-1}\Big]ds\Big| \\
&+ \Big|\int_{t_1}^{t_2} t_2^{1-\gamma}(t_2-s)^{\alpha-1}ds\Big|\Big).
\end{aligned}$$

Obviously the right-hand-side of the above inequality tends to zero independently of $x \in S$ as $t_2 - t_1 \to 0$. Therefore, it follows from the Arzela-Ascoli theorem that B is a completely continuous operator on S.

Claim 3. Next, we show that Hypotheses (c) of Lemma 2.6 is satisfied. Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = AxBy$. Then we have

$$\begin{aligned}
t^{1-\gamma}\,|x(t)| &= t^{1-\gamma}\,|Ax(t)|\,|By(t)| \\
&= |f(t,x(t))|\left|\left(\frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,y(s))ds\right)\right| \\
&\leq K\left|\left(\frac{\phi}{\Gamma(\gamma)} + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,y(s))ds\right)\right| \\
&\leq K\left|\left(\frac{\phi}{\Gamma(\gamma)} + \|p\|\,\Omega(r)\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\right)\right| \\
&\leq K\left[\frac{|\phi|}{\Gamma(\gamma)} + \frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\,\|p\|\,\Omega(r)\right].
\end{aligned}$$

Taking supremum for $t \in J$, we obtain

$$\|x\|_C \leq K\left[\frac{|\phi|}{\Gamma(\gamma)} + \frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\,\|p\|\,\Omega(r)\right] \leq r,$$

that is, $x \in S$.

Claim 4. Now we show that $Mk < 1$, that is, (d) of Lemma 2.6 holds.

This is obvious by (H3), since we have $M = \|B(s)\| = \sup\{\|Bx\| : \ x \in S\} \leq \frac{|\phi|}{\Gamma(\gamma)} + \frac{T^{1-\gamma+\alpha}}{\Gamma(\alpha+1)}\,\|p\|\,\Omega(r)$ and $k = \|\chi\|$.

Thus all the conditions of Lemma 2.6 are satisfied and hence the operator equation $x = AxBx$ has a solution in S. Consequently, the problem (1.3)-(1.4) has a solution on J. This completes the proof. $\square$

Remark 3.2. Note that the involvement of the term $\frac{\phi}{\Gamma(\gamma)}t^{\gamma-1}$ in the integral solution (3.2) of the problem (1.3)-(1.4) makes it unbounded. In this scenario, Banach's fixed point theorem cannot be used in the weighted normed space.

Motivated by Remark 3.2 above, we should explore other sufficient conditions for uniqueness. In fact, we can adopt a slightly different set of assumptions which allow us to derive the uniqueness result.

Theorem 3.3. *Assume that (H4)-(H5) hold. If*

$$\frac{T^{\alpha}}{\Gamma(\alpha+1)}\left(L_1M_2+M_1L_2\right)<1 \tag{3.5}$$

then, the problem (1.3)-(1.4) *with* $\phi=0$ *has a unique solution on* J.

Proof. Transform the problem (1.3)-(1.4) into a fixed point problem. By Remark 3.2, if we take $\phi = 0$, then we can obtain the integral equation $x = Fx$.
Consider the operator $F : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$(Fx)(t)=\frac{1}{\Gamma(\alpha)}f(t,x(t))\int_0^t(t-s)^{\alpha-1}g(s,x(s))ds. \tag{3.6}$$

Clearly, the fixed point of the operator F are solutions of the problem (1.3)-(1.4). We shall use the Banach contraction principle to prove that F defined by (3.6) has a fixed point. We shall show that F is a contraction.

Let $x, y \in C(J,\mathbb{R})$. Then, for each $t \in J$ we have

$$\begin{aligned}
&\Big|(Fx)(t)-(Fy)(t)\Big|\\
&=\frac{1}{\Gamma(\alpha)}\Big|f(t,x(t))\int_0^t(t-s)^{\alpha-1}g(s,x(s))ds\\
&\ -f(t,y(t))\int_0^t(t-s)^{\alpha-1}g(s,y(s))ds\Big|\\
&=\frac{1}{\Gamma(\alpha)}\Big|\Big[f(t,x(t))-f(t,y(t))\Big]\int_0^t(t-s)^{\alpha-1}g(s,x(s))ds\\
&+f(t,y(t))\int_0^t(t-s)^{\alpha-1}\Big[g(s,x(s))-g(s,y(s))\Big]ds\Big|\\
&\le\max_{t\in J}\left[\frac{1}{\Gamma(\alpha)}\Big\{\Big(L_1M_2+M_2L_2\Big)\int_0^t(t-s)^{\alpha-1}ds\Big\}\right]\Big\|x-y\Big\|\\
&\le\frac{T^{\alpha}}{\Gamma(\alpha+1)}\Big(L_1M_2+M_1L_2\Big)\Big\|x-y\Big\|.
\end{aligned}$$

From (3.5), it follows that F has a unique fixed point which is the solution of the problem (1.3)-(1.4) with $\phi = 0$. ☐

Next, we present an example illustrating Theorem 3.1.

4. An example

Consider the following Hilfer type HFDE

$$D_{0^+}^{\alpha,\beta}\left(\frac{x(t)}{f(t,x(t))}\right)=g(t,x),\quad t\in[0,1], \tag{4.1}$$

$$I_{0^+}^{1-\gamma}\left(\frac{x(0)}{f(0,x(0))}\right)=1, \tag{4.2}$$

where

$$f(t,x)=\frac{1}{5\sqrt{\pi}}\left(\sin t\tan^{-1}x+\frac{\pi}{2}\right),$$

$$g(t,x)=\frac{1}{10}\left(\frac{1}{6}|x|+\frac{1}{8}\cos x+\frac{|x|}{4(1+|x|)}+\frac{1}{16}\right).$$

Denote $\alpha=\frac{2}{3}$, $\beta=\frac{1}{2}$ and choose $\gamma=\frac{5}{6}$. Obviously, $|f(t,x)|\le\frac{\sqrt{\pi}}{5}=K$, $\chi(t)=\frac{1}{5\sqrt{\pi}}$, $|g(t,x)|\le\frac{1}{10}\left(\frac{1}{6}|x|+\frac{7}{16}\right)$.

We choose $\|p\|=\frac{1}{10}$, $\Omega(r)=\frac{1}{6}r+\frac{7}{16}$. Clearly all the conditions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, it follows that problem (4.1)-(4.2) has a solution.

5. Conclusion

The paper is concerned with existence of solutions of HFDEs with Hilfer fractional derivative which generalizes the famous Riemann-Liouville fractional derivative. By the fixed point theorem due to Dhage, we obtained some sufficient conditions to ensure the existence of solution. Finally, we discussed important observations on uniqueness result.

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