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Berezin number inequalities involving superquadratic functions

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Abstract. We consider superquadratic functions f and define selfadjoint operators $f(A)$ from some selfadjoint operators A on a reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(Q)$. We estimate the so-called Berezin number of operator $f(A)$.

Keywords: Convex function, Superquadratic function, Berezin symbol, Berezin number, Positive operator, Selfadjoint operator.

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1. Introduction

A function $f : [0, \infty) \to \mathbb{R}$ is said to be superquadratic provided that for all $s \geq 0$ there exists a constant $C_s \in \mathbb{R}$ such that

$$f(t) \geq f(s) + C_s(t - s) + f(|t - s|), \tag{1.1}$$

for all $t \geq 0$. This notion was introduced by Abramovich, Jameson and Sinnamon in their paper [1].

In this paper we give some inequalities for the Berezin number of some operator classes. Our arguments based on superquadratic functions and

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operators from such functions. For more definition and fact about superquadratic functions and their applications, we refer to Abramovich, Jameson and Sinnamon [1], Agarwal and Dragomir [2] and Furuta, Hot, Pečarić and Seo [12].

Recall that the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(Q)$ (shortly, RKHS) is the Hilbert space of complex-valued functions on some set Q such that the evaluation functional $f \to f(\lambda)$ is bounded on $\mathcal{H}$ for every $\lambda \in Q$. Then, by Riesz representation theorem there exists a unique vector k_λ in $\mathcal{H}$ such that $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The normalized reproducing kernel is defined by $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathcal{H}}}$. For a bounded linear operator A acting in $\mathcal{H}$, its Berezin symbol (see Berezin [6, 7]) is defined by

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \ (\lambda \in Q).$$

Berezin set and Berezin number of operator A is defined respectively by

$$\operatorname{Ber}(A) := \operatorname{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in Q\right\}$$

and

$$\operatorname{ber}(A) := \sup_{\lambda \in Q} \left|\widetilde{A}(\lambda)\right|.$$

It is clear from definitions that $\widetilde{A}$ is a bounded function, $\operatorname{Ber}(A)$ lies in the numerical range $W(A)$, and so $\operatorname{ber}(A)$ does not exceed the numerical radius $w(A)$ of operator A. Recall that the numerical range and the numerical radius are defined, respectively, by

$$W(A) := \{\langle Ax, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$$

and

$$w(A) := \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [16]. For the basic properties and facts on these new concepts, see [3, 4, 5, 10, 13, 14, 15, 17, 18, 19, 20, 23, 26, 27, 28, 29, 30, 31].

In the present paper we consider superquadratic functions and define continuous functional calculus for some selfadjoint operators, including positive operators, and prove new inequalities for the Berezin number of such operators. Our arguments mainly use ideas of papers [21, 22], while in these papers the estimation of the Berezin number is not considered.

2. Some facts for superquadratic functions and functions of selfadjoint operators

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$ and I denote the identity operator. If $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with the matrix algebra $\mathcal{M}_n$ of all $n \times n$ matrices with complex entries. We denote by $S(J)$ the set of all selfadjoint operators in $\mathcal{B}(\mathcal{H})$ whose spectra lie in an interval $J \subseteq \mathbb{R} = (-\infty, +\infty)$. Let $f : J \to \mathbb{R}$ be a continuous real function. For $A \in S(J)$, we mean by $f(A)$ the continuous functional calculus at A. Let $A \in S([m, M])$ and $\{E_t\}$ be its spectral family. Then, $f(A)$ can be represented via the well-known spectral representation as

$$f(A) = \int_{m-0}^{M} f(t)\, dE_t, \tag{2.1}$$

in which the integral is in terms of the Riemann-Stieltjes integral. If $x, y \in \mathcal{H}$, then

$$\langle f(A)x, y\rangle = \int_{m-0}^{M} f(t)\, d\langle E_t x, y\rangle .$$

It was shown in [1] that:

Lemma 2.1. *If f is a superquadratic function with C_s as in* (2.1)*, then*
(i) $f(0) \le 0$;
(ii) If $f(0) = f'(0) = 0$ and f is differentiable at s, then $C_s = f'(s)$;
(iii) If $f \ge 0$, then f is convex and $f(0) = f'(0) = 0$.

Recall that a function $f : J \to \mathbb{R}$ is called convex if and only if

$$f(t\alpha + (1-t)\beta) \le tf(\alpha) + (1-t) f(\beta),$$

for all points $\alpha, \beta \in J$ and all $t \in [0, 1]$.

Mond and Pečarić [25] showed that if $f : J \to \mathbb{R}$ is a convex function, then

$$f(\langle Ax, x\rangle) \le \langle f(A)x, x\rangle, \tag{2.2}$$

for all $A \in S(J)$ and all unit vectors $x \in \mathcal{H}$.

Regarding the possible refinement of (2.2), Dragomir [9] proved the following result.

Lemma 2.2. *Let $f : J \to \mathbb{R}$ be a convex and differentiable function on the interior J^0 of J, whose derivative f' is continuous on J^0. Then*

$$0 \le \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \le \left\langle f'(A)Ax, x\right\rangle - \langle Ax, x\rangle \left\langle f'(A)x, x\right\rangle,$$

for every $A \in S(J)$ and every unit vector $x \in \mathcal{H}$.

Recall that a linear map is defined to be $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ which preserves additivity and homogeneity, i.e., $\Phi(\alpha A+\beta B)=\alpha\Phi(A)+\beta\Phi(B)$, for any $\alpha,\beta\in\mathbb{C}$ and $A,B\in\mathcal{B}(\mathcal{H})$. We say that the linear map $\Phi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{K})$ is positive if it preserves the operator order, that is, if $A\in\mathcal{B}^{+}(\mathcal{H})$ then $\Phi(A)\in\mathcal{B}^{+}(\mathcal{K})$. Here $\mathcal{B}^{+}(\mathcal{H})$ denotes the convex cone of all positive operators on $\mathcal{H}$.

Obviously, a positive linear map Φ preserves the order relation, namely, $A\le B\Rightarrow\Phi(A)\le\Phi(B)$ and preserves the adjoint operation $\Phi(A^{*})=\Phi(A)^{*}$. Moreover, Φ is said to be normalized (unital) if it preserves the identity operator, i.e., $\Phi(I_{\mathcal{H}})=I_{\mathcal{K}}$.

Recall also that a bounded linear operator A on $\mathcal{H}$ is selfadjoint (i.e., $A^{*}=A$) if and only if $\langle Ax,x\rangle\in\mathbb{R}$, for all $x\in\mathcal{H}$. For two selfadjoint operators $A,B\in\mathcal{B}(\mathcal{H})$, we write $A\le B$ if $\langle Ax,x\rangle\le\langle Bx,x\rangle$, for all $x\in\mathcal{H}$.

In [24], the authors proved the following similar inequality to (2.2) for positive linear mappings.

Lemma 2.3. *If $f:J\to\mathbb{R}$ is a convex function with $f(0)\le 0$ and A is a Hermitian matrix, then for every vector $x\in\mathcal{H}$ with $\|x\|\le 1$ and every positive linear map $\Phi:\mathcal{M}_n(\mathbb{C})\to\mathcal{M}_m(\mathbb{C})$ with $0\le\Phi(I)\le I$, the inequality*

$$f(\langle\Phi(A)x,x\rangle)\le\langle\Phi(f(A))x,x\rangle \tag{2.3}$$

holds true.

Kian [21] proved a Jensen operator inequality for superquadratic functions.

Lemma 2.4. *([21]) If $f:[0,+\infty)\to\mathbb{R}$ is a continuous superquadratic function, then*

$$f(\langle Ax,x\rangle)\le\langle f(A)x,x\rangle-\langle f(|A-\langle Ax,x\rangle|)x,x\rangle, \tag{2.4}$$

for any positive operator A and any unit vector $x\in\mathcal{H}$.

This inequality improves (2.2) for some convex functions.

3. The Berezin number inequalities

In this section, we prove some operator inequalities in order to estimate Berezin number of some operators on the RKHS $\mathcal{H}=\mathcal{H}(Q)$.

Theorem 3.1. *If $f:[0,\infty)\to\mathbb{R}$ is a nonnegative continuous superquadratic function, then*

$$\sup_{\lambda\in Q} f\left(\widetilde{A}(\lambda)\right)\le \operatorname{ber}(f(A)),$$

for any positive operator $A\in\mathcal{B}(\mathcal{H}(Q))$.

Proof. Let $A \geq 0$, i.e., $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$. For each $s \geq 0$ it follows from (2.1) that

$$f(A) \geq f(s) I + C_s A - C_s s I + f(|A - sI|).$$

So, for every $\lambda \in Q$ we have that

$$\left\langle f(A) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \geq f(s) + C_s \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle - C_s s + \left\langle f(|A - sI|) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle . \tag{3.1}$$

Applying (3.1) with $s = \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle$ and to use that $f\left(\left|A - \widetilde{A}(\lambda) I_{\mathcal{H}}\right|\right) \geq 0$, we get

$$f\left(\widetilde{A}(\lambda)\right) \leq \widetilde{f(A)}(\lambda) - f\left(\left|A - \widetilde{\widetilde{A}(\lambda) I}\right|\right)(\lambda) \leq \widetilde{f(A)}(\lambda) \tag{3.2}$$

for all $\lambda \in Q$. Taking module and supremum from the both side of the inequality, we have that

$$\sup_{\lambda \in Q} f\left(\widetilde{A}(\lambda)\right) \leq \operatorname{ber}(f(A))$$

as desired. □

We need some auxiliary lemmas to prove our next result (see [11, 22]).

Lemma 3.2. *Every unital positive map on a commutative C^*-algebra is completely positive.*

Theorem 3.3. *([8]) Let Φ be a unital completely positive linear map from a C^*-subalgebra $\mathcal{A}$ of $\mathcal{M}_n(\mathbb{C})$ into $\mathcal{M}_n(\mathbb{C})$. Then, there exists a Hilbert space $\mathcal{K}$, an isometry $V : \mathbb{C}^m \to \mathcal{K}$ and a unital $*$-homomorphism π from $\mathcal{A}$ into the C^*-algebra $\mathcal{B}(\mathcal{K})$ such that $\Phi(A) = V^* \pi(A) V$.*

Our next result extends (2.2) for superquadratic functions.

Theorem 3.4. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous superquadratic function and let $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ be a unital positive linear map. Then,*

$$f\left(\widetilde{\Phi(A)}(\lambda)\right) \leq \Phi\widetilde{(f(A))}(\lambda) - \left(\Phi\left(f\left(\left|A - \widetilde{\Phi(A)}(\lambda) I_n\right|\right)\right)\right)^{\sim}(\lambda), \tag{3.3}$$

for every positive matrix $A \in \mathcal{M}_n(\mathbb{C})$ and every $\lambda \in Q$.

Proof. Let $A \in \mathcal{M}_n(\mathbb{C})$ be positive. Suppose that $\mathcal{A} \subset \mathcal{M}_n(\mathbb{C})$ is the C^*-subalgebra generated by A and I. We may assume without loss of generality that Φ is defined on $\mathcal{A}$. It follows from Lemma 3.2 that Φ is completely positive. Hence, by Theorem 3.3, there exists a RKHS Hilbert space $\mathcal{K} = \mathcal{K}(\Omega)$, an isometry $V : \mathbb{C}^m \to \mathcal{K}$ and a unital $*$-homomorphism π from $\mathcal{A}$ into the C^*-algebra $\mathcal{B}(\mathcal{K})$ such that $\Phi(A) =$

$V^*\pi(A)V$. Obviously, $f(\pi(A)) = \pi(f(A))$. Moreover, for any $\alpha \in \mathbb{C}$, it is easy to see that

$$f(|\pi(A-\alpha I)|) = \pi(f(|A-\alpha I|)). \tag{3.4}$$

Since $\left\|V\widehat{k}_\lambda\right\| = 1$, for all $\lambda \in Q$, we have,

$$\begin{aligned}
f\left(\left\langle \Phi(A)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right) &= f\left(\left\langle V^*\pi(A)V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right) = f\left(\left\langle \pi(A)V\widehat{k}_\lambda, V\widehat{k}_\lambda\right\rangle\right)\\
&\le \left\langle f(\pi(A))V\widehat{k}_\lambda, V\widehat{k}_\lambda\right\rangle -\\
&- \left\langle f\left(\left|\pi(A) - \left\langle \pi(A)V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)V\widehat{k}_\lambda, V\widehat{k}_\lambda\right\rangle \text{ (by Lemma 2.3)}\\
&= \left\langle f(\pi(A))V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle -\\
&- \left\langle \pi\left(f\left(\left|A - \left\langle \pi(A)V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|\right)\right)V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \text{ (By (3.4))}\\
&= \left\langle V^*\pi(f(A))V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle -\\
&- \left\langle V^*\pi\left(f\left(\left|A - V^*\pi(A)V\widehat{k}_\lambda, \widehat{k}_\lambda\right|\right)\right)V\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\\
&= \left\langle \Phi(f(A))\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle -\\
&- \left\langle \Phi\left(f\left(\left|A - \left\langle \Phi(A)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle I\right|\right)\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\\
&= \widetilde{\Phi(f(A))}(\lambda) - \left(\Phi\left(f\left(\left|A - \widetilde{\Phi(A)}(\lambda)\right|\right)\right)\right)^{\sim}(\lambda).
\end{aligned}$$

Hence

$$f\left(\widetilde{\Phi(A)}(\lambda)\right) \le \widetilde{\Phi(f(A))}(\lambda) - \left(\Phi\left(f\left(\left|A - \widetilde{\Phi(A)}(\lambda)\right|\right)\right)\right)^{\sim}(\lambda)$$

which proves (3.3). □

Corollary 3.5. *If f is non-negative, then*

$$\sup_{\lambda\in Q} f\left(\widetilde{\Phi(A)}(\lambda)\right) \le \operatorname{ber}(\Phi(f(A))).$$

4. Some reverse inequalities

In this section, we give some reverse inequalities for Berezin symbols and Berezin number.

Lemma 2.1 can be improved for non-negative superquadratic functions. First, we prove a reverse inequality for (3.3).

Theorem 4.1. *Let $f : [0,\infty) \to [0,\infty)$ be a differentiable superquadratic function whose derivative f' is continuous. If $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is* *a unital positive linear map, then*

$$0 \leq \widetilde{\Phi (f (A))} (\lambda) - f \left(\widetilde{\Phi (A)} (\lambda) \right)$$
$$\leq \widetilde{\Phi (f' (A) A)} (\lambda) - \widetilde{\Phi (A)} (\lambda) \widetilde{\Phi (f' (A))} (\lambda) -$$
$$- \left(\Phi \left(f \left(\left| A - \widetilde{\Phi (A)} (\lambda) I \right| \right) \right) \right)^{\sim} (\lambda) ,$$

for every positive matrix $A \in \mathcal{M}_n (\mathbb{C})$ and every $\lambda \in Q$; here and in what follows, Q is the set over which $\mathbb{C}^n$ is the RKHS, i.e., $\mathbb{C}^n = \mathbb{C}^n (\Omega)$ as the RKHS.

Proof. Let $s \geq 0$ be an arbitrary fixed number. Since f is superquadratic, there is $C_s \in \mathbb{R}$ such that

$$f (t) \geq f (s) + C_s (t - s) + f (|t - s|) , \tag{4.1}$$

for every $t \geq 0$. As $f \geq 0$, it follows from Lemma C of the paper [22] that f is convex and $C_s = f' (s)$. So, the first inequality follows from (2.3) by putting $x = \widehat{k}_{\lambda,n}$, where $\widehat{k}_{\lambda,n}$ is the normalized reproducing kernel of the space $\mathbb{C}^n$. Now, let $\widehat{k}_{\lambda,m}$ denote the normalized reproducing kernel of the space $\mathbb{C}^m$. Assume now that $x \in \mathbb{C}^m$ with $x = \widehat{k}_{\lambda,m}$ and $A \geq 0$. Using the functional calculus for (4.1) with $s = A$ and $t = \widetilde{\Phi (A)} (\lambda)$, we obtain

$$f \left(\widetilde{\Phi (A)} (\lambda) \right) \geq f (A) + f' (A) \widetilde{\Phi (A)} (\lambda) - f' (A) A + f \left(\left| A - \widetilde{\Phi (A)} (\lambda) I_n \right| \right) .$$

Applying the positive linear map Φ to both sides of the last inequality, we get

$$f \left(\widetilde{\Phi (A)} (\lambda) \right) \geq \Phi (f (A)) + \Phi \left(f' (A) \right) \widetilde{\Phi (A)} (\lambda) - \Phi \left(f' (A) A \right) +$$
$$+ \Phi \left(f \left(\left| A - \widetilde{\Phi (A)} (\lambda) I_n \right| \right) \right) ,$$

which gives us the desired result. □

For the case $\Phi (A) = A$, the last theorem gives an improvement of Lemma 2.1. Namely, let f be as in Theorem 3.4. Then, we have

$$0 \leq \widetilde{f (A)} (\lambda) - f \left(\widetilde{A} (\lambda) \right)$$
$$\leq \widetilde{f' (A) A} (\lambda) - \widetilde{A} (\lambda) \widetilde{f' (A)} (\lambda) - \left(f \left(\left| A - \widetilde{A} (\lambda) I_n \right| \right) \right)^{\sim} (\lambda) , \tag{4.2}$$

for every positive operator A and all $\lambda \in Q$.

Since $\left(f \left(\left| A - \widetilde{A} (\lambda) I_n \right| \right) \right)^{\sim} (\lambda) \geq 0$, for all $\lambda \in Q$,an immediate corollary of (4.2) is the following.

Corollary 4.2. *We have :*

(i) $\operatorname{ber}(f(A)) \geq \sup_{\lambda\in Q}\left(f\left(\widetilde{A}(\lambda)\right)\right)$;

(ii) $$\begin{aligned}\operatorname{ber}(f(A)) &\leq \sup_{\lambda\in Q}\left(f\left(\widetilde{A}(\lambda)\right)\right)+\operatorname{ber}(f'(A)A)+\operatorname{ber}(A)\operatorname{ber}(f'(A))\\ &\leq f(\operatorname{ber}(A))+\operatorname{ber}(f'(A)A)+\operatorname{ber}(A)\operatorname{ber}(f'(A)),\end{aligned}$$

since $\widetilde{A}(\lambda) \leq \operatorname{ber}(A)$ *for all* λ *and every non-negative superquadratic function is non-decreasing.*

Example 4.3. If $r \geq 2$, then $f(t) = t^r$ is a non-negative superquadratic function on $[0,\infty)$. If $A \geq 0$ and $\lambda \in Q$, then applying Corollary 4.2, we get

$$\begin{aligned} 0 &\leq \widetilde{A^r}(\lambda) - \widetilde{A}(\lambda)^r \\ &\leq r\widetilde{A^r}(\lambda) - r\widetilde{A}(\lambda)\widetilde{A^{r-1}} - \left(\left|A-\widetilde{A}(\lambda)I_n\right|^r\right)^{\sim}(\lambda). \end{aligned}$$

This implies,in particular, that :

(i) $\operatorname{ber}(A)^r \leq \operatorname{ber}(A^r)$

(ii) $r\left[\widetilde{A}(\lambda)\widetilde{A^{r-1}}(\lambda)-\widetilde{A^r}(\lambda)\right] \leq \widetilde{A}(\lambda)^r+(r-1)\widetilde{A^r}(\lambda)$, which implies that

$$\sup_{\lambda\in Q}\left[\widetilde{A}(\lambda)\widetilde{A^{r-1}}(\lambda)-\widetilde{A^r}(\lambda)\right] \leq \frac{\operatorname{ber}(A)^r}{r}+\frac{r-1}{r}\operatorname{ber}(A^r).$$

It is necessary to note that, in general, the Berezin symbol is not multiplicative, i.e., $\widetilde{AB} \neq \widetilde{A}\widetilde{B}$ (see Kılıç [23]).

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