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Inverse Scattering Problem for the Impulsive Schrödinger Equation with a Polynomial Spectral Dependence in the Potential

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Abstract. In the present work, under some differentiability conditions on the potential functions , we first reduce the inverse scattering problem (ISP) for the polynomial pencil of the Schrödinger equation to the corresponding ISP for the generalized matrix Schrödinger equation. Then ISP will be solved in analogy of the Marchenko method. We aim to establish an effective algorithm for uniquely reconstructing of the potential functions of the equation in that case when there is no a discrete spectrum.

Keywords: One dimensional Schrödinger equation, scattering data, Jost solution, integral representation, direct problem, inverse scattering problem, main integral equation.

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1. Introduction

Consider the Schrödinger equation

$$y'' + \left[E - \sum_{m=0}^{n} \left(E^{\frac{1}{2n}} \right)^m q_m(x) \right] y = 0, \ x \in \mathbb{R}^* = (-\infty, +\infty) \backslash \{a\} \quad (1)$$

with the discontinuity conditions

$$y(a-0) = \alpha y(a+0), \ y'(a-0) = \alpha^{-1} y'(a+0) \quad (2)$$

Here E is a complex parameter, $q_m(x) \left(m = \overline{0,n}, \ n > 1 \right)$ are supposed to be sufficiently regular real functions decreasing fast enough as $x \to \pm\infty$, $a \in (-\infty, +\infty)$, $1 \neq \alpha > 0$.

An equation of type (1) is has a great extent on its connection with the nonlinear evolution equations constructed in [1](see also [2]) where a family of nonlinear Hamiltonian equations has obtained and solved by the method of the inverse scattering transform provided that the inverse scattering problem (ISP) for (1) can be solved. For further discussion of the inverse scattering theory of Schrödinger type operators we refer to monographs [16, 21, 17, 18, 19, 20].

The ISP for (1) was first considered by Jaulent and Jean [5]. Under some regularity conditions on the "potentials" $q_0, q_{1,\dots,} q_n$, they solved the ISP for (1) by reduction this problem to the ISP for generalized matrix-Schrödinger equation. Since in Eq.(1) the $(2n)^{th}$ root $E^{\frac{1}{2n}}$ is an analytic function on a Riemann's $2n$ sheet surface, it is convenient to set $E = \lambda^{2n}\,(\lambda \in C)$ and to represent $E^{\frac{1}{2n}}$ by $\lambda e^{\frac{il\pi}{n}}\,(l = 0, 1, ..., 2n-1)$. Equation (1) is then represented by the $2n$ scalar Sturm-Liouville equations

$$y_l'' + \left[\lambda^{2n} - \sum_{m=0}^{n} \lambda^m e^{\frac{iml\pi}{n}} q_m(x) \right] y_l = 0, \ x \in \mathbb{R}^* \quad (3)$$

and the jump conditions take the form

$$y_l(a-0) = \alpha y_l(a+0), \ y_l'(a-0) = \alpha^{-1} y_l'(a+0). \quad (4)$$

Remark that in case $n = 1$, on a Riemann's two sheet surface the equation (1) is represented by the equations

$$y^{\pm''} + \left[\lambda^2 \mp \lambda p(x) - q(x) \right] y^{\pm} = 0 \quad (5)$$

In the case when the potential functions are real valued differentiable functions belonging to the spaces of integrable functions together with derivatives the full-line inverse scattering problem for (3) without discrete spectrum has been studied in [3],[4]. The direct and inverse scattering problems, also some inverse problems of the spectral analysis for Eq. (3) in various statements were studied in details by many authors. We refer for further discussion to articles [6, 7, 8, 9, 10, 11, 12, 13]

In this work the inverse scattering problem is investigated by reduction it to the corresponding inverse scattering problem for the energy dependent matrix Schrödinger equation with the related discontinuity conditions at a real point a. We aim to establish an effective algorithm for uniquely reconstructing of the potential functions $q_m(x)\,(m=0,1,...,\ n)$ of the equation (1).

2. Equivalent Representations Of Equations

Let us define the vector functions

$$Y^+=(y_0,y_2,...,y_{2n-2})^T,\ Y^-=(y_1,y_3,...,y_{2n-1})^T$$

where $y_l\,(l=0,1,...,2n-1)$ satisfies the equation (3). Then the vector function $Y^{\pm}$ will satisfy the matrix Schrödinger equation

$$Y^{\pm\prime\prime}+\left[\lambda^{2n}I-V^{\pm}(\lambda,x)\right]Y^{\pm}=0,\ x\in\mathbb{R}^*, \tag{6}$$

where

$$V^{\pm}(\lambda,x)=\sum_{m=0}^{n}\lambda^m q_m(x)\,S_{\pm}^m,$$

$$S_+=diag\left(1,\varepsilon,...,\varepsilon^{n-1}\right),\ \varepsilon=e^{\frac{2i\pi}{n}},S_-=e^{\frac{i\pi}{n}}S_+$$

Note that the discontinuity conditions can be written as

$$Y^{\pm}(a-0)=\alpha Y^{\pm}(a+0),\ Y^{\pm\prime}(a-0)=\alpha^{-1}Y^{\pm\prime}(a+0) \tag{7}$$

Now consider the $n\times n$ matrix $P^+(\lambda)=\left(p_{ij}^+(\lambda)\right)$, where

$$p_{ij}^+(\lambda)=\varepsilon^{(i-1)(j-1)}\lambda^{j-1},\ i,j=1,2,..,n$$

Additionally let

$$q_{ij}^+(\lambda)=\frac{1}{n\varepsilon^{(i-1)(j-1)}\lambda^{i-1}},\ i,j=1,2,..,n$$

Since

$$\sum_{k=1}^{n}p_{ik}^+(\lambda)\,q_{kj}^+(\lambda)=\frac{1}{n}\sum_{k=1}^{n}\varepsilon^{(i-j)(k-1)}\text{ and }1+\varepsilon+...+\varepsilon^{n-1}=0$$

we have

$$\sum_{k=1}^{n}p_{ik}^+(\lambda)\,q_{kj}^+(\lambda)=\delta_{ij}=\begin{cases}1,if\ i=j\\0,if\ i\neq j\end{cases}$$

Therefore, we obtain

$$\left[P^+(\lambda)\right]^{-1}=\left(q_{ij}^+(\lambda)\right)_{i,j=1}^{n}. \tag{8}$$

Analogously, if we define

$$P^-(\lambda)=P^+(\lambda e^{\frac{i\pi}{n}})$$

we can find

$$\left[P^{-}(\lambda)\right]^{-1}=\left(q_{ij}^{-}(\lambda)\right)_{i,j=1}^{n} \tag{9}$$

where

$$q_{ij}^{-}(\lambda)=\frac{1}{n\varepsilon^{(i-1)(j-1)}e^{\frac{i\pi(i-1)}{n}}\lambda^{i-1}},\ i,j=1,2,..,n$$

Let $Y^{\pm}$ be any solution of (6).with the jump conditions (7) . We define the vector function

$$\widetilde{Y^{\pm}}=\left[P^{\pm}(\lambda)\right]^{-1}Y^{\pm} \tag{10}$$

and easily obtain that $\widetilde{Y^{\pm}}$ satisfies the equation

$$\widetilde{Y^{\pm}}''+\lambda^{2n}\widetilde{Y^{\pm}}=\left[\left[P^{\pm}(\lambda)\right]^{-1}V^{\pm}(\lambda,x)P^{\pm}(\lambda)\right]\widetilde{Y^{\pm}},\ x\in\mathbb{R}^{*}$$

Now using (8) , (9) , (10) we compute that

$$\left[P^{\pm}(\lambda)\right]^{-1}V^{\pm}(\lambda,x)P^{\pm}(\lambda)=U(x)\pm\lambda^{n}Q(x)$$

Hence we transform whole system of $(2n)$ equations (3) to a pair of the generalized matrix Schrödinger equations

$$\widetilde{Y^{\pm}}''+\left[k^{2}I-\widetilde{V^{\pm}}(k,x)\right]\widetilde{Y^{\pm}}=0,x\in\mathbb{R}^{*} \tag{11}$$

with the jump conditions

$$\widetilde{Y^{\pm}}(a-0)=\alpha\widetilde{Y^{\pm}}(a+0),\widetilde{Y^{\pm}}\,'(a-0)=\alpha^{-1}\widetilde{Y^{\pm}}'(a+0) \tag{12}$$

which can be viewed as another 'representation' of (3) ,where I is the $n\times n$ identity matrix,

$$\begin{aligned}\widetilde{V^{\pm}}(k,x)&=\left[P^{\pm}(\lambda)\right]^{-1}V^{\pm}(\lambda,x)P^{\pm}(\lambda)\\&=U(x)\pm kQ(x),\ k=\lambda^{n},\end{aligned} \tag{13}$$

$$U(x)=\begin{pmatrix}q_0 & 0 & ... & 0 & 0\\ q_1 & q_0 & ... & 0 & 0\\ q_2 & q_1 & q_0 & ... & 0\\ . & . & . & . & .\\ q_{n-1} & . & . & q_1 & q_0\end{pmatrix},$$

$$Q(x)=\begin{pmatrix}q_n & q_{n-1} & q_{n-2} & \cdots & q_1\\ 0 & q_n & q_{n-1} & \cdots & q_2\\ . & . & . & . & .\\ 0 & 0 & 0 & q_n & q_{n-1}\\ 0 & 0 & 0 & ... & q_n\end{pmatrix}$$

To sum up we have obtained three equivalent "representations" (3) , (6) , (11) for Eq.(1)

3. Comparison Between The Scattering Functions For Equivalent Equations

In this section we define the scattering functions of problems (3), (6), (11) and find the connection between the scattering functions. We suppose that potentials $q_i\left(i=\overline{0,n}\right)$ satisfy the following conditions:

(a) For $i=\overline{0,n-1}$ is continuously differentiable, and $xq_i(x), q_i'(x)$ are integrable on $\mathbb{R}$.

(b) $q_n(x)\ (x\in\mathbb{R})$ is twice continuously differentiable, and $q_n(x), q_n'(x), q_n''(x)$ are integrable on $\mathbb{R}$.

The right and left Jost solutions $f_l(\lambda,x)$ and $g_l(\lambda,x)$ of (3), respectively, $F^{\pm}(\lambda,x)$ and $G^{\pm}(\lambda,x)$ of (6), respectively, $\widetilde{F^{\pm}}(k,x)$ and $\widetilde{G^{\pm}}(k,x)$ of (11), are defined as follows:

$$
\begin{aligned}
&f_l(\lambda,x) \underset{x\to+\infty}{\sim} e^{i\lambda^n x},\ g_l(\lambda,x) \underset{x\to-\infty}{\sim} e^{-i\lambda^n x},\\
&F^{\pm}(\lambda,x) \underset{x\to+\infty}{\sim} e^{i\lambda^n x}(1,1,...,1)^T,\ G^{\pm}(\lambda,x) \underset{x\to-\infty}{\sim} e^{-i\lambda^n x}(1,1,...,1)^T,\\
&\widetilde{F^{\pm}}(k,x) \underset{x\to+\infty}{\sim} e^{ikx}V,\ \widetilde{G^{\pm}}(k,x) \underset{x\to-\infty}{\sim} e^{-ikx}V,
\end{aligned}
$$

where T means "transposed", and $V=(1,0,...,0)^T$. $f_l(\lambda,x)$ and $g_l(\lambda,x)$ are defined equivalently as the solution of the following integral equations:

$$
f_l(\lambda,x)=e_0^+(x,\lambda^n)+\int_x^{+\infty} S_0^+(x,t,\lambda)\sum_{m=0}^{n}\lambda^m e^{\frac{iml\pi}{n}}q_m(t)\,f_l(\lambda,t)dt, \tag{14}
$$

$$
g_l(\lambda,x)=e_0^-(x,\lambda^n)+\int_{-\infty}^{x} S_0^-(x,t,\lambda)\sum_{m=0}^{n}\lambda^m e^{\frac{iml\pi}{n}}q_m(t)\,g_l(\lambda,t)dt, \tag{15}
$$

where

$$
e_0^{\pm}(x,\lambda^n)=\begin{cases} e^{\pm i\lambda^n x} & ,\ \pm x>\pm a \\ A^+e^{\pm i\lambda^n x}\pm A^- e^{\pm i\lambda^n(2a-x)} & ,\ \pm x<\pm a \end{cases}, \tag{16}
$$

$$
S_0^{\pm}(x,t,\lambda^n)=\begin{cases} \frac{\pm\sin\lambda^n(t-x)}{\lambda^n}, \pm a<\pm x<\pm t & \text{or } \pm x<\pm t<\pm a \\ \frac{\pm A^+\sin\lambda^n(t-x)}{\lambda^n}+\frac{A^-\sin\lambda^n(t-2a+x)}{\lambda^n} & ,\ \pm x<\pm a<\pm t \end{cases}, \tag{17}
$$

$A^{\pm}=\frac{1}{2}\left(\alpha\pm\frac{1}{\alpha}\right)$. For fixed x, $f_l(\lambda,x)$ and $g_l(\lambda,x)$ are continuous for $0\le\arg\lambda\le\frac{\pi}{n}$, analytic for $0<\arg\lambda<\frac{\pi}{n}$ and obey estimate

$$
|f_1(\lambda,x)|\le Ce^{-bx}e^{d_+(x)},\ 0\le\arg\lambda\le\frac{\pi}{n},\ b=Im\lambda^n, \tag{18}
$$

$$
|g_l(\lambda,x)|\le Ce^{bx}e^{d_-(x)},\ 0\le\arg\lambda\le\frac{\pi}{n},\ b=Im\lambda^n \tag{19}
$$

where

$$d_{\pm}(x)=\pm 2\int_{x}^{\pm\infty}\left(|t-x|+1\right)\sum_{m=0}^{n}\left|q_{m}\left(t\right)\right|dt \tag{20}$$

and $C>0$ is a constant. It is clear that $F^{\pm}(\lambda,x)$ and $G^{\pm}(\lambda,x)$ are also defined and continuous for $0\leq\arg\lambda\leq\frac{\pi}{n}$, analytic for $0<\arg\lambda<\frac{\pi}{n}$ and verify

$$F^{\pm}(\lambda,x)=e_{0}^{+}(x,\lambda^{n})\left(1,1,...,1\right)^{T}+\int_{x}^{+\infty}S_{0}^{+}(x,t,\lambda^{n})V^{\pm}\left(\lambda,t\right)F^{\pm}(\lambda,t)dt, \tag{21}$$

$$G^{\pm}(\lambda,x)=e_{0}^{-}(x,\lambda^{n})\left(1,1,...,1\right)^{T}+\int_{-\infty}^{x}S_{0}^{-}(x,t,\lambda^{n})V^{\pm}\left(\lambda,t\right)G^{\pm}(\lambda,t)dt \tag{22}$$

Using $(8),(9)$ and (11) we have

$$\widetilde{F^{\pm}}(k,x)=e_{0}^{+}(x,k)V+\int_{x}^{+\infty}S_{0}^{+}(x,t,k)\widetilde{V^{\pm}}\left(k,t\right)\widetilde{F^{\pm}}(k,t)dt, \tag{23}$$

$$\widetilde{G}^{\pm}(k,x)=e_{0}^{-}(x,k)V+\int_{-\infty}^{x}S_{0}^{-}(x,t,k)\widetilde{V^{\pm}}\left(k,t\right)\widetilde{G}^{\pm}(k,t)dt. \tag{24}$$

It is not difficult to prove that $\widetilde{F^{\pm}}(k,x)$ and $\widetilde{G}^{\pm}(k,x)$ are defined and continuous for $Imk\geq 0$, analytic for $Imk>0$ and admits the following inequalities:

$$\left\|\widetilde{F^{\pm}}(k,x)\right\|\leq Ce^{-bx}e^{h_{+}(x)},x\in\mathbb{R},b=Imk\geq 0, \tag{25}$$

$$\left\|\widetilde{G}^{\pm}(k,x)\right\|\leq Ce^{bx}e^{h_{-}(x)},x\in\mathbb{R},b=Imk\geq 0, \tag{26}$$

where

$$h_{\pm}(x)=\pm 2\int_{x}^{\pm\infty}\left[|y-x|\sum_{m=0}^{n-1}\left|q_{m}\left(y\right)\right|+\sum_{m=1}^{n}\left|q_{m}\left(y\right)\right|\right]dy \tag{27}$$

$$\left\|(\alpha_{1},\alpha_{2},...,\alpha_{n})\right\|=\max_{1\leq i\leq n}\left|\alpha_{i}\right|,\left(\alpha_{1},\alpha_{2},...,\alpha_{n}\right)^{T}\in\mathbb{R}^{n}$$

For $\lambda>0$, $f_{l}(\lambda,x)$ and $f_{l-1}(\lambda e^{\frac{i\pi}{n}},x)$ form a fundamental system of solutions of $(3)_{l}$. For all l and with the convention $f_{-1}=f_{2n-1}$, we have the relation

$$g_{l}(\lambda,x)=b_{l}\left(\lambda\right)f_{l}(\lambda,x)+a_{l}\left(\lambda\right)f_{l-1}(\lambda e^{\frac{i\pi}{n}},x),\lambda>0 \tag{28}$$

where

$$a_{l}\left(\lambda\right)=\frac{1}{2i\lambda^{n}}W\left[g_{l}(\lambda,x),f_{l}(\lambda,x)\right], \tag{29}$$

$$b_{l}\left(\lambda\right)=-\frac{1}{2i\lambda^{n}}W\left[g_{l}(\lambda,x),f_{l-1}(\lambda e^{\frac{i\pi}{n}},x)\right] \tag{30}$$

and $W[f,g]$ is the Wronskian of f and g. We have from the formula (29) that the function $a_l(\lambda)$ admits a unique continuous extension $a_l(\lambda)$ $(0 \le \arg\lambda \le \frac{\pi}{n})$ which is analytic for $0 < \arg\lambda < \frac{\pi}{n}$. Because of the convention $f_{-1} = f_{2n-1}$, we can write

$$[f_{2n-1}(\lambda,x), f_1(\lambda,x), ..., f_{2n-2}(\lambda,x)]^T = M\,[f_1(\lambda,x), f_2(\lambda,x), ..., f_{2n-1}(\lambda,x)]^T, \tag{31}$$

where

$$M = \begin{pmatrix} 0 & 0 & ... & 0 & 1 \\ 1 & 0 & ... & 0 & 0 \\ 0 & 1 & ... & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & ... & 1 & 0 \end{pmatrix}.$$

It follows from (29) and (31) that there exist diagonal matrices $A^{\pm}(\lambda)$ and $B^{\pm}(\lambda)$ such that

$$G^{+}(\lambda,x) = B^{+}(\lambda)\,F^{+}(\lambda,x) + A^{+}(\lambda)MF^{-}(\lambda e^{\frac{i\pi}{n}},x),\ \ \lambda > 0, \tag{32}$$

$$G^{-}(\lambda,x) = B^{-}(\lambda)\,F^{-}(\lambda,x) + A^{-}(\lambda)F^{+}(\lambda e^{\frac{i\pi}{n}},x),\ \ \lambda > 0, \tag{33}$$

where

$$\begin{aligned} A^{+}(\lambda) &= diag\,(a_0(\lambda), a_2(\lambda), ..., a_{2n-2}(\lambda)), \\ B^{+}(\lambda) &= diag\,(b_0(\lambda), b_2(\lambda), ..., b_{2n-2}(\lambda)), \\ A^{-}(\lambda) &= diag\,(a_1(\lambda), a_3(\lambda), ..., a_{2n-1}(\lambda)), \\ B^{-}(\lambda) &= diag\,(b_1(\lambda), b_3(\lambda), ..., b_{2n-1}(\lambda)) \end{aligned}$$

Clearly, $A^{\pm}(\lambda)$ are continuous for $0 \le \arg\lambda \le \frac{\pi}{n}$ and analytic for $0 < \arg\lambda < \frac{\pi}{n}$. From relations (32) and (33), taking into account the formula (10) and the equality

$$\left[P^{+}(\lambda)\right]^{-1} M = \left[P^{-}\left(\lambda e^{\frac{i\pi}{n}}\right)\right]^{-1}$$

(see [5]) we obtain

$$\widetilde{G}^{\pm}(k,x) = \widetilde{B}^{\pm}(k)\widetilde{F}^{\pm}(k,x) + \widetilde{A}^{\pm}(k)\widetilde{F}^{\mp}(-k,x), k = \lambda^n, k \in \mathbb{R}, \tag{34}$$

where

$$\widetilde{B}^{\pm}(k) = \left[P^{\pm}(\lambda)\right]^{-1} B^{\pm}(\lambda)P^{\pm}(\lambda),\ \ k = \lambda^n \tag{35}$$
$$\widetilde{A}^{\pm}(k) = \left[P^{\pm}(\lambda)\right]^{-1} A^{\pm}(\lambda)P^{\pm}(\lambda),\ \ k = \lambda^n.$$

Since the function $\lambda = k^{\frac{1}{n}}$ is continuous for $0 \le \arg k \le \pi$, analytic for $0 < \arg k < \pi$ we have that $\widetilde{A}^{\pm}(k)$ is continuous for $Imk \ge 0$ and analytic for $Imk > 0$.

Now we define reflection coefficients $r_l(\lambda)$, $R^{\pm}(\lambda)$ and $\widetilde{R}^{\pm}(k)$ for the problems (3), (6) and (11), respectively, as follows:

$$r_l(\lambda) = \frac{b(\lambda)}{a(\lambda)}, \lambda > 0, \tag{36}$$

$$R^{\pm}(\lambda) = \left[A^{\pm}(\lambda)\right]^{-1} B^{\pm}(\lambda),\ \lambda > 0, \tag{37}$$

$$\begin{aligned}\widetilde{R}^{\pm}(k) &= \left[\widetilde{A}^{\pm}(k)\right]^{-1} \widetilde{B}^{\pm}(k) \\ &= \left[P^{\pm}(\lambda)\right]^{-1} R^{\pm}(\lambda) P^{\pm}(\lambda),\ k = \lambda^n,\ k \in R.\end{aligned} \tag{38}$$

As in the classical case(see [22]) it can be shown that $r_l(\lambda)$ and $R^{\pm}(\lambda)$ are continuous for $\lambda > 0$. This implies $\widetilde{R}^{\pm}(k)$ is continuous for $k \in \mathbb{R}$. We impose the following condition (c) :

(c) The zeros λ_{lj} of $a_l(\lambda)$ are simple, in finite number N_l,

$$0 < \arg\lambda_{lj} < \frac{\pi}{n}, \lambda_{lj} \neq \lambda_{l'j}$$

if $l \neq l'$, have the same parity.

The square integrable solutions of (3),(4) corresponds to the zeros $\lambda_{lj}\,(j = 1, 2, ...N_l)$ of $a_l(\lambda)$. Similarly, the square integrable solutions of (6), (7) corresponds to the zeros $\lambda_{m^{\pm}}(m^{\pm} = 1, 2, ..., M^{\pm})$ of $\det A^{\pm}(\lambda)$ and the square integrable solutions of (11), (12) corresponds to the zeros $k_{m^{\pm}} = (\lambda_{m^{\pm}})^n$ $(m^{\pm} = 1, 2, ..., M^{\pm})$ of $\det \widetilde{A^{\pm}}(k)$. Moreover, it is clear that

$$\left\{\lambda_{m^+}, m^+ = 1, 2, ..., M^+\right\} = \left\{\lambda_{2lj},\ j = 1, 2, ...N_l,\ l = 0, 1, ..., n-1\right\}, \tag{39}$$

$$\left\{\lambda_{m^-}, m^- = 1, 2, ..., M^-\right\} = \left\{\lambda_{2l+1j},\ j = 1, 2, ...N_l,\ l = 0, 1, ..., n-1\right\}. \tag{40}$$

To each zero λ_{lj} of $a_l(\lambda)$ we associate a constant scalar c_{lj} defined as

$$c_{lj} = \lim_{\lambda \to \lambda_{lj}} (\lambda - \lambda_{lj}) \frac{b(\lambda)}{a(\lambda)}. \tag{41}$$

Similarly to the zeros $\lambda_{m^{\pm}}$ and $k_{m^{\pm}}$ we correspond the matrices

$$C_{m^+} = n(\lambda_{m^+})^{n-1} diag(0, 0, ..., c_{2lj}, 0, ..., 0) \text{ if } \lambda_{m^+} = \lambda_{2lj}, \tag{42}$$

$$C_{m^-} = n(\lambda_{m^-})^{n-1} diag(0, 0, ..., c_{2l+1j}, 0, ..., 0) \text{ if } \lambda_{m^+} = \lambda_{2l+1j}, \tag{43}$$

$$\widetilde{C}_{m^{\pm}} = \left[P^{\pm}(\lambda_{m^{\pm}})\right]^{-1} C_{m^{\pm}} P^{\pm}(\lambda_{m^{\pm}}),\ k_{m^{\pm}} = (\lambda_{m^{\pm}})^n.$$

We define the scattering data $s, L, \widetilde{L}$ of $(3)-(4)$, $(5)-(6)$, $(11)-(12)$ respectively by

$$S = \{r_l(\lambda)(\lambda > 0); \lambda_{lj}; c_{lj}\,(j = 1, 2, ..., N_l,\ l = 0, 1, ..., 2n-1)\}, \tag{44}$$

$$L = \left\{R^{\pm}(\lambda)(\lambda > 0); \lambda_{m^{\pm}}; C_{m^{\pm}}\left(m^{\pm} = 1, 2, ..., M^{\pm}\right)\right\}, \tag{45}$$

$$\widetilde{L}=\left\{\widetilde{R}^{\pm}(k)\,(k\in\mathbb{R})\,;k_{m^{\pm}};\widetilde{C}_{m^{\pm}}\left(m^{\pm}=1,2,...,M^{\pm}\right)\right\}. \tag{46}$$

The scattering data S, L and $\widetilde{L}$ are equivalent. Therefore, later on we just consider the ISP for $(11)-(12)$ .It also follows that the functions $\det\left[A^{\pm}(\lambda)\right]$(respectively $\det\left[\widetilde{A}^{\pm}(k)\right]$) have not any zero on the corresponding regions.

4. Integral Representations of Solutions

In this section similarly to the scalar case [3, 4] we have the integral representations for the solutions $\widetilde{F}^{+}(k,x)$, $\widetilde{G^{\pm}}(k,x)$ and prove the following assertions. Let

$$\sigma^{\pm}(x)=\pm\int_{x}^{\pm\infty}\left\{(1+|t|)\left\|U(t)V\right\|+\left\|Q(t)V\right\|\right\}dt.$$

Lemma 4.1. *If the condition* (a) *is satisfied then the solutions* $\widetilde{F}^{+}(k,x)$ *and* $\widetilde{G^{\pm}}(k,x)$ *can be expressed as*

$$\widetilde{F}^{\pm}(k,x)=f_1^{\pm}(x)Ve^{ikx}+f_2^{\pm}(x)Ve^{ik(2a-x)}$$
$$+_{x}^{\infty}K^{\pm}(x,t)e^{ikt}dt,\ Imk\geq 0,\ x\in\mathbb{R}, \tag{47}$$

$$\widetilde{G}^{\pm}(k,x)=g_1^{\pm}(x)Ve^{-ikx}+g_2^{\pm}(x)Ve^{-ik(2a-x)}$$
$$+_{-\infty}^{x}H^{\pm}(x,t)e^{-ikt}dt,\ Imk\geq 0,\ x\in\mathbb{R} \tag{48}$$

respectively, where

$$f_1^{\pm}(x)=\exp\left(\pm\frac{i}{2}\int_{x}^{+\infty}q_n(t)dt\right)\left\{\begin{array}{c}1, x>a\\ A^{+},\ x<a\end{array}\right.$$

$$g_1^{\pm}(x)=\exp\left(\pm\frac{i}{2}\int_{-\infty}^{x}q_n(t)dt\right)\left\{\begin{array}{c}1, x<a\\ A^{+},\ x>a\end{array}\right. ,$$

$$f_2^{\pm}(x)\ =\ A^{-}\exp\left(\mp\frac{i}{2}\int\limits_{x}^{+\infty}q_n(s)ds\pm i\int\limits_{a}^{+\infty}q_n(s)ds\right)\ \textit{for } x<a,$$

$$g_2^{\pm}(x)\ =\ -A^{-}\exp\left(\mp\frac{i}{2}\int\limits_{-\infty}^{x}q_n(s)ds\pm i\int\limits_{-\infty}^{a}q_n(s)ds\right)\ \textit{for } x>a,$$

$$f_2^{\pm}(x)=0,\ \textit{for } x>a\ \textit{and } g_2^{\pm}(x)=0,\ \textit{for } x<a$$

and the kernels $K^{\pm}(x,t)$ *and* $H^{\pm}(x,t)$ *which are real vector functions defined on* $x\leq t<\infty$ *and* $-\infty<t\leq x$ *respectively, satisfy the inequalities*

$$\int_x^{\infty} \left\|K^{\pm}(x,t)\right\| dt \leq Ce^{\sigma^+(x)}, \quad \int_{-\infty}^{x} \left\|H^{\pm}(x,t)\right\| dt \leq Ce^{\sigma^-(x)} \tag{49}$$

for every real x and for some constant $C > 0$.

Lemma 4.2. *If the conditions (a) and (b) are satisfied then $K^{\pm}(x,t)$ ($x \leq t < \infty$) and $H^{\pm}(x,t)$ ($-\infty < t \leq x$) are continuous vectors at $t \neq 2a-x,\ x \neq a$ for which the inequalities (41) are satisfied. Moreover, the functions $K^{\pm}(x,t)$ and $H^{\pm}(x,t)$ have the following properties:*

$$2K^{\pm}(x,x) = \left(\int_x^{+\infty} \left[U(s) + \frac{1}{4}Q^2(s)\right] V ds \mp \frac{i}{2}Q(x)V\right) f_1^{\pm}(x), \tag{50}$$

$$2K^{\pm}(x,2a-x+0) - 2K^{\pm}(x,2a-x-0) =$$
$$\left(\mp\frac{i}{2}Q(x)V + \int\limits_a^{+\infty} (U(s) + \frac{1}{4}Q^2(s))V ds - \int\limits_x^{a} (U(s) + \frac{1}{4}Q^2(s))V ds\right) f_2^{\pm}(x), x < a, \tag{51}$$

$$2H^{\pm}(x,x) = \left(\int_{-\infty}^{x} \left[U(s) + \frac{1}{4}Q^2(s)\right] V ds \mp \frac{i}{2}Q(x)V\right) g_1^{\pm}(x), \tag{52}$$

$$2H^{\pm}(x,2a-x-0) - 2H^{\pm}(x,2a-x+0) =$$
$$\left(\mp\frac{i}{2}Q(x)V + \int\limits_{-\infty}^{a} (U(s) + \frac{1}{4}Q^2(s))V ds - \int\limits_a^{x} (U(s) + \frac{1}{4}Q^2(s))V ds\right) f_2^{\pm}(x),\ x > a \tag{53}$$

respectively.

Lemma 4.3. *The matrices $\widetilde{A}^{\pm}(k)$ and $\widetilde{B}^{\pm}(k)$ can be expressed as*

$$\widetilde{A}^{\pm}(k)V = A^{+}\left[1 - \frac{1}{2ik}{}_{-\infty}^{+\infty}(U(t) \pm kQ(t)) \exp\left(\pm\frac{i}{2}{}_{-\infty}^{t} q_n(s)ds\right) dt\right] V$$
$$+ \frac{1}{2ik}{}_{-\infty}^{0} G^{\pm}(s)Ve^{-iks}ds, \tag{54}$$

$$\widetilde{B}^{\pm}(k)V = -A^{-}e^{-2ika}V$$
$$+ \frac{A^{-}e^{-2ika}}{2ik}\left({}_{-\infty}^{a}(U(t) \pm kQ(t)) \exp\left(\pm\frac{i}{2}{}_{-\infty}^{t} q_n(s)ds\right) dt\right) V \tag{55}$$
$$-\frac{A^{-}e^{-2ika}}{2ik}\left({}_{a}^{+\infty}(U(t) \pm kQ(t)) \exp\left(\mp\frac{i}{2}{}_{-\infty}^{t} q_n(s)ds \pm i{}_{-\infty}^{a} q_n(s)ds\right) dt\right) V$$
$$+\frac{1}{2ik}{}_{-\infty}^{\infty} L^{\pm}(s)Ve^{-iks}ds,$$

where${}_{-\infty}^{0} \|G^{\pm}(s)V\| ds < \infty$ and ${}_{-\infty}^{\infty} \|L^{\pm}(s)V\| ds < \infty$.

Proof. By equation (24) it follows that, for real $k \neq 0$,

$$\widetilde{G}^{\pm}(k,x) = e^{-ikx}\left[A^{+}V -{}_{-\infty}^{a} \frac{A^{+}e^{ikt} + A^{-}e^{ik(2a-t)}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt\right.$$

$$\left. -{}_{a}^{+\infty}\frac{e^{ikt}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt\right]$$

$$+\, e^{ikx}\left[{}_{-\infty}^{a}\frac{A^{+}e^{-ikt} + A^{-}e^{-ik(2a-t)}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt\right.$$

$$\left. +{}_{a}^{+\infty}\frac{e^{-ikt}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt\right] + o(1), x \to +\infty.$$

On the other hand by (34) , we have

$$\widetilde{G}^{\pm}(k,x) = \widetilde{B}^{\pm}(k)e^{ikx} + \widetilde{A}^{\pm}(k)e^{-ikx} + o(1), x \to +\infty.$$

A comparision of corresponding terms shows that

$$\widetilde{A}^{\pm}(k)V = A^{+}V - \int_{-\infty}^{a}\frac{A^{+}e^{ikt} + A^{-}e^{ik(2a-t)}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt$$

$$- \int_{a}^{+\infty}\frac{e^{ikt}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt, \qquad (56)$$

$$\widetilde{B}^{\pm}(k)V = \int_{-\infty}^{a}\frac{A^{+}e^{-ikt} + A^{-}e^{-ik(2a-t)}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt +$$

$$\int_{a}^{+\infty}\frac{e^{-ikt}}{2ik}\widetilde{V^{\pm}}(k,t)\,\widetilde{G}^{\pm}(k,t)dt. \qquad (57)$$

for real $k \neq 0$. Now,the formulas (54) , (55) easily obtained from the representation (48) of the solution $\widetilde{G}^{\pm}(k,x)$. □

From the above lemma we have

$$\widetilde{A}^{\pm}(k)V = A^{+}\exp\left(\pm\frac{i}{2}\int_{-\infty}^{+\infty}q_n(s)ds\right)V - \frac{A^{+}}{2ik}W_A + \frac{1}{2ik}\int_{-\infty}^{0}G^{\pm}(s)Ve^{-iks}ds, \qquad (58)$$

$$\widetilde{B}^{\pm}(k)V = -A^{-}e^{-2ika\mp ip_0}V + \frac{A^{-}e^{-2ika}}{2ik}W_B + \frac{1}{2ik}\int_{-\infty}^{\infty}L^{\pm}(s)Ve^{-iks}ds, \qquad (59)$$

where W_A, W_B are constant vectors in $\mathbb{R}^n$,

$$p_0 = \frac{1}{2}{}_{-\infty}^{\infty}q_n(t)sgn(t-a)dt,$$

$$\alpha_0 = \frac{1}{2}{}_{-\infty}^{\infty}q_n(t)dt. \qquad (60)$$

Using the formulas (54), (55), the matricial representation of $\widetilde{A}^{\pm}(k)$, $\widetilde{B}^{\pm}(k)$ (see [5]) and the Wiener-Levy theorem we can prove

$$\widetilde{A}^{\pm}(k) = A^{+}e^{\pm i\alpha_0}I + \frac{1}{2ik}\int_{-\infty}^{0}\widetilde{G}^{\pm}(t)e^{-ikt}dt, \tag{61}$$

$$\widetilde{B}^{\pm}(k) = -A^{-}e^{-2ika\mp ip_0}I + \frac{1}{2ik}\int_{-\infty}^{\infty}\widetilde{L}^{\pm}(t)e^{-ikt}dt, \tag{62}$$

where $\int_{-\infty}^{0}\left\|\widetilde{G}^{\pm}(t)V\right\|dt < \infty$,$\int_{-\infty}^{\infty}\left\|\widetilde{L}^{\pm}(t)V\right\|dt < \infty$ and I is $n\times n$ unit matrix.The following lemma is a direct result of (61) and (61).

Lemma 4.4. *The reflection coefficient* $\widetilde{R}^{\pm}(k)$ *is expressed as*

$$\widetilde{R}^{\pm}(k) - \widetilde{R}_0^{\pm}(k) = \int_{-\infty}^{\infty}R^{\pm}(s)e^{-iks}ds, \tag{63}$$

where

$$\widetilde{R}_0^{\pm}(k) = -\frac{A^{-}}{A^{+}}e^{-2ika\mp i\gamma^{+}}I,\ \gamma^{+} = \int_{a}^{+\infty}q_n(s)ds \tag{64}$$

and $\int_{-\infty}^{\infty}\|R^{\pm}(s)V\| < \infty$.

5. Study the ISP

We recall that $\widetilde{F}^{+}(k,x)$ is defined equivalently as the solution in the class of continuous functions for $x\geq 0$ of the equation (23) and, for fixed x, $\widetilde{F}^{+}(k,x)$ is continuous for $Imk\geq 0$ and analytic for $Imk>0$. By applying the successive approximation method to (23) we can find the behavior for large values of $|k|$ of $\widetilde{F}^{+}(k,x)$:

$$\widetilde{F}^{\pm}(k,x) = \left(f_1^{\pm}(x)e^{ikx} + f_2^{\pm}(x)e^{ik(2a-x)}\right)V + \frac{e^{ikx}}{k}W_1(x) + \frac{e^{ik(2a-x)}}{k}W_2(x) + O\left(\frac{1}{k^2}\right), Imk\geq 0, |k|\to\infty, \tag{65}$$

$$\widetilde{G}^{\pm}(k,x) = \left(g_1^{\pm}(x)e^{-ikx} + g_2^{\pm}(x)e^{-ik(2a-x)}\right)V + \frac{e^{-ikx}}{k}U_1(x) + \frac{e^{-ik(2a-x)}}{k}U_2(x) + O\left(\frac{1}{k^2}\right), Imk\geq 0, |k|\to\infty, \tag{66}$$

where $W_j(x)$ and $U_j(x)\,(j=1,2)$ are vectors in $\mathbb{R}^n$ such that $W_2(x)$ and $U_2(x)$ are zero vectors for $x>a$ and $x<a$ respectively. Consequently, $\widetilde{F}^{\pm}(k,x) - \left(f_1^{\pm}(x)e^{ikx} + f_2^{\pm}(x)e^{ik(2a-x)}\right)V$, for fixed x, belongs to $L_2(R)$

and admits a Fourier transform. In fact, similarly to the scaler case [23], $\widetilde{F}^{\pm}(k,x)$ has the following representation :

$$\widetilde{F}^{\pm}(k,x) = \left(f_1^{\pm}(x)e^{ikx} + f_2^{\pm}(x)e^{ik(2a-x)}\right)V$$

$$+_x^{+\infty} K^{\pm}(x,t)e^{ikt}dt,\ Imk \geq 0,\ x \in \mathbb{R} \tag{67}$$

Here $K^{\pm}(x,t) = \left(K_0^{\pm}(x,t),...,K_{n-1}^{\pm}(x,t)\right)$ is the $\mathbb{R}^n$-valued function solution of the PDE system

$$\left(D_{xx}^2 - D_{tt}^2 - U(x) \mp iQ(x)D_t\right)K^{\pm}(x,t) = 0,\ t > x, \tag{68}$$

with additional conditions

$$f_1^{\pm}(x)'' V - 2\frac{d}{dx}K^{\pm}(x,x) \mp iQ(x)K^{\pm}(x,x) - U(x) f_1^{\pm}(x) V = 0,\ x > a \tag{69}$$

$$f_2^{\pm}(x)'' V - 2\frac{d}{dx}\left[K^{\pm}(x,2a-x-0) - K^{\pm}(x,2a-x+0)\right]$$
$$\mp iQ(x)\left[K^{\pm}(x,2a-x-0) - K^{\pm}(x,2a-x+0)\right]$$
$$- U(x) f_2^{\pm}(x) V = 0,\ x < a \tag{70}$$

and $K^{\pm}(x,+\infty) = 0$. It is important to remark that, if we seek U and Q in the form given by (13) we can construct them from f_1^+, f_1^-, f_2^+, f_2^-, K^+ and K^-. Using the (13) form in Eg. (68) and taking into account the relations from Lemma1 we obtain the triangular system with $(n+1)$ equation and $(n+1)$ unknown values $q_0, q_1, ..., q_n$:

$$\pm i\sum_{j=0}^{n-1} q_{n-j}(x) K_j^{\pm}(x,x) - q_0(x) f_1^{\pm}(x) = 2\frac{d}{dx}K_0^{\pm}(x,x) + f_1^{\pm}(x)'',$$

$$\pm i\sum_{j=0}^{n-m-1} q_{n-j}(x) K_{j+m}^{\pm}(x,x) - q_m(x) f_1^{\pm}(x) = 2\frac{d}{dx}K_m^{\pm}(x,x), m = 1,...,n-1,$$

$$q_n(x) = \pm 2i\frac{f_1^{\pm}(x)''}{f_1^{\pm}(x)} \text{ if } x > a \tag{71}$$

$$\pm i\sum_{j=0}^{n-1} q_{n-j}(x) K_j^{\pm}(x,x) - q_0(x) f_2^{\pm}(x) = 2\frac{d}{dx}K_0^{\pm}(x,x) + f_2^{\pm}(x)'',$$

$$\pm i\sum_{j=0}^{n-m-1} q_{n-j}(x) K_{j+m}^{\pm}(x,x) - q_m(x) f_2^{\pm}(x) = 2\frac{d}{dx}K_m^{\pm}(x,x), m = 1,...,n-1,$$

$$q_n(x) = \pm 2i\frac{f_2^{\pm}(x)''}{f_2^{\pm}(x)} \text{ if } x < a. \tag{72}$$

Clearly, $q_0, q_1, ..., q_n$ are uniquely determined by the system (71) and (72).

Using Eq. (58), (59) by integration in parts we have the following estimations for $\widetilde{A}^{\pm}(k)$ and $\widetilde{B}^{\pm}(k)$ as $|k| \to \infty$:

$$\widetilde{A}^{\pm}(k)V = A^{+}e^{\pm i\alpha_0}V + \frac{W_1}{k} + O\left(\frac{1}{k^2}\right), |k| \to \infty, Imk \geq 0$$

$$\widetilde{B}^{\pm}(k)V = A^{-}e^{2ika+ip_0}V + \frac{W_2}{k} + O\left(\frac{1}{k^2}\right), |k| \to \infty, k \in \mathbb{R}$$

where

$$\alpha_0 = \frac{1}{2}\int_{-\infty}^{+\infty} q_n(x)dx,\ p_0 = \frac{1}{2}\int_{a}^{+\infty} q_n(x)dx - \frac{1}{2}\int_{-\infty}^{a} q_n(x)dx$$

and W_1, W_2 are constant vectors in $\mathbb{R}^n$. Using the matricial representations of $\widetilde{A}^{\pm}(k)$ and $\widetilde{B}^{\pm}(k)$ it is easy to obtain

$$\widetilde{A}^{\pm}(k) = A^{+}e^{\pm i\alpha_0}I + T + O\left(\frac{1}{k}\right), Imk \geq 0, \tag{73}$$

$$\left[\widetilde{A}^{\pm}(k)\right]^{-1} = \frac{1}{A^{+}}e^{\mp i\alpha_0}I + T' + O\left(\frac{1}{k}\right), Imk \geq 0, k \neq k_m, \tag{74}$$

where T and T' are constant superior triangular matrices with zeros on the diagonal, and

$$\widetilde{B}^{\pm}(k) = -A^{-}e^{-2ika\mp ip_0}V + O\left(\frac{1}{k}\right), k \in \mathbb{R}, \tag{75}$$

$$\widetilde{R}^{\pm}(k) = \widetilde{R}_0^{\pm}(k) + O\left(\frac{1}{k}\right),\ k \in \mathbb{R}, \tag{76}$$

$$\widetilde{R}_0^{\pm}(k) = \mp\frac{A^{-}}{A^{+}}e^{\mp 2i(ka\pm\gamma^{\pm})}I,\ \gamma^{\pm} = \frac{1}{2}\int_{a}^{\pm\infty} q_n(s)ds. \tag{77}$$

$$\det \widetilde{A}^{\pm}(k) = \left[A^{+}e^{\pm i\alpha_0}\right]^{n} + O\left(\frac{1}{k}\right), Imk \geq 0, \tag{78}$$

$$\det\left[\widetilde{A}^{\pm}(k)\right]^{-1} = \left[A^{+}e^{\pm i\alpha_0}\right]^{-n} + O\left(\frac{1}{k}\right), Imk \geq 0, k \neq k_m. \tag{79}$$

Note that $\widetilde{R}^{\pm}(k) - \widetilde{R}_0^{\pm}(k)$ has a Fourier transform in $L_2(\mathbb{R})$.

In order to establish the main integral equations of the scattering problem we start from the formula (34) written in the form

$$\left[\widetilde{A}^{\pm}(k)\right]^{-1}\widetilde{G}^{\pm}(k,x) - \left[\widetilde{A}^{\pm}(k)\right]^{-1}\widetilde{B}^{\pm}(k)\widetilde{F}^{\pm}(k,x) = \widetilde{F}^{\mp}(-k,x),\ k \in \mathbb{R} \tag{80}$$

and in the equivalent form for fixed x,

$$G_x^{\pm}(k) - H_x^{\pm}(k) = \widetilde{F}^{\mp}(-k,x) - \left(f_1^{\mp}(x)e^{-ikx} + f_2^{\mp}(x)e^{-ik(2a-x)}\right)V$$
$$=_x^{+\infty} A^{\pm}(x,t)e^{-ikt}dt, \tag{81}$$

where

$$G_x^{\pm}(k) = \left[\widetilde{A}^{\pm}(k)\right]^{-1}\widetilde{G}^{\pm}(k,x) - \left(f_1^{\mp}(x)e^{-ikx} + f_2^{\mp}(x)e^{-ik(2a-x)}\right)V, \tag{82}$$

$$H_x^{\pm}(k) = \left[\widetilde{A}^{\pm}(k)\right]^{-1}\widetilde{B}^{\pm}(k)\widetilde{F}^{\pm}(k,x) = \widetilde{R}^{\pm}(k)\widetilde{F}^{\pm}(k,x). \tag{83}$$

Let us compute the Fourier transform of functions $G_x^{\pm}(k)$ and $H_x^{\pm}(k)$. The function $G_x^{\pm}(k)$ is continuous for $Imk \geq 0$, $k \neq k_m$, and analytic for $Imk > 0$, $k \neq k_m$. Since

$$G_x^{\pm}(k) = \left[\left[\widetilde{A}^{\pm}(k)\right]^{-1} - \frac{1}{A^+}e^{\mp i\alpha_0}I\right]\left[\widetilde{G}^{\pm}(k,x) - \left(g_1^{\pm}(x)e^{-ikx} + g_2^{\pm}(x)e^{-ik(2a-x)}\right)V\right]$$
$$+\left[\left[\widetilde{A}^{\pm}(k)\right]^{-1} - \frac{1}{A^+}e^{\mp i\alpha_0}I\right]\left[\left(g_1^{\pm}(x)e^{-ikx} + g_2^{\pm}(x)e^{-ik(2a-x)}\right)V\right]$$
$$+\frac{1}{A^+}e^{\mp i\alpha_0}\left[\widetilde{F}^{\pm}(k,x) - \left(g_1^{\pm}(x)e^{-ikx} + g_2^{\pm}(x)e^{-ik(2a-x)}\right)V\right] \tag{84}$$

using formulas (66) and (74) we have

$$G_x^{\pm}(k) = e^{-ikx}O(\frac{1}{k}),\ Imk \geq 0, k \neq k_m. \tag{85}$$

From the estimation (85) and the formulas (35), (42), (43) we can easily obtain that

$$\int_{-\infty}^{+\infty} G_x^{\pm}(k)e^{ikt}dk = 2\pi K^{\mp}(x,t) = \sum_{m^{\pm}=1}^{M\pm}\widetilde{C}_{m^{\pm}}\widetilde{F}^{\pm}(k_{m^{\pm}},x)e^{i\left(k_{m^{\pm}}\right)t}. \tag{86}$$

To obtain the Fourier transform for $H_x^{\pm}(k)$, we write thus

$$H_x^{\pm}(k) = \left[\widetilde{R}^{\pm}(k) - \widetilde{R}_0^{\pm}(k)\right]\left[\widetilde{F}^{\pm}(k,x) - \left(f_1^{\pm}(x)e^{ikx} + f_2^{\pm}(x)e^{ik(2a-x)}\right)V\right]$$
$$+\left[\widetilde{R}^{\pm}(k) - \widetilde{R}_0^{\pm}(k)\right]\left(f_1^{\pm}(x)e^{ikx} + f_2^{\pm}(x)e^{ik(2a-x)}\right)V \tag{87}$$

Now recalling the formulas (76) and taking into account the result (86) we obtain the main integral equations

$$K^{\pm}(x,t) \mp \frac{A^-}{A^+}e^{\mp 2i\gamma^+}K^{\pm}(x,2a-t) = f_1^{\mp}(x)F_0^{\mp}(x+t) + f_2^{\mp}(x)F_0^{\mp}(2a-x+t)$$
$$+\int_x^{+\infty} F_0^{\mp}(t+y)K^{\mp}(x,t)dt,\ t > x, \tag{88}$$

where

$$F_0^{\pm}(x) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\widetilde{R}^{\pm}(k) - \widetilde{R}_0^{\pm}(k)\right]e^{ikx}dk + \sum_{m^{\pm}=1}^{M\pm}\widetilde{C}_{m^{\pm}}e^{i\left(k_{m^{\pm}}\right)x}. \tag{89}$$

Using (69) and (70) , to the system (88) we add the coupling conditions

$$f_1^{-}(x)K_{n-1}^{+}(x,x) = f_1^{+}(x)K_{n-1}^{-}(x,x), n > 1,\ x > a, \tag{90}$$

$$f_2^{-}(x)\left[K_{n-1}^{+}(x,2a-x-0) - K_{n-1}^{+}(x,2a-x+0)\right] =$$
$$f_2^{+}(x)\left[K_{n-1}^{-}(x,2a-x-0) - K_{n-1}^{-}(x,2a-x+0)\right], n > 1, x < a. \tag{91}$$

Note that the uniqueness property and the solution of the inverse problem can be proved by the same arguments as in [23].

Theorem 2 If the conditions (**a**), (**b**), (**c**) are satisfied then equations (88) have the unique solutions $K^{+}(x,.) \in L_1(x,\infty)$ and $K^{-}(x,.) \in L_1(-\infty,x)$ for each fixed $x > -\infty$ and $x < +\infty$ respectively.

Proof: For each fixed $x > -\infty$ consider the operator (see [24])

$$(M_x^{+}f)y = \begin{cases} \overline{f(y)} & , \ x > a \\ \overline{f(y)} - \frac{A^{-}}{A^{+}}e^{-2i\gamma^{+}}f(2a-y) & , \ x < a \end{cases}$$

acting in the space $L_1(x,\infty)$ (and also $L_2(x,\infty)$). It is easy to show that the operator M_x^{+} is invertible. Using this operator the main equation (88) can be rewritten as

$$\overline{K^{+}(x,y)} + (M_x^{+})^{-1}F^{+}(x,y) + (M_x^{+})^{-1}\phi^{+}K^{+}(x,.)(y) = 0 \ , \ y > x \tag{92}$$

where the operator ϕ^{+} is defined as

$$\phi^{+}f(y) = \int_x^{+\infty} F_0^{+}(t+y)f(t)dt,\ y > x \tag{93}$$

for each fixed $x > -\infty$.

It is known that (see [16]) the operator ϕ^{+} is a compact operator in the space $L_1(x,\infty)$ (also in $L_2(x,\infty)$). By the boundness of the operator M_x^{-1} we have that the operator $M_x^{-1}\phi^{+}$ is also a compact operator. Therefore, to prove the theorem, it is sufficient to show that the homogeneous equation

$$\overline{h_x(y)} - \frac{A^{-}}{A^{+}}e^{-2i\gamma^{+}}h_x(2a-y) + \int_x^{+\infty} h_x(t)F_0^{+}(t+y)dx = 0,\ y > x \tag{94}$$

has only the trivial solution $h_x(y) \in L_1(x,\infty)$. By our assumptions the function $F_0^{+}(y)$ and the corresponding solution $h_x(y)$ are bounded in the half axis $x \leq y < +\infty$. Therefore $h_x(\cdot) \in L_2(x,\infty)$. Consequently, we have

$$0 = \int_x^{+\infty} \|h_x(y)\|^2\, dy - \frac{A^-}{A^+} e^{-2i\gamma^+} \int_x^{+\infty} h_x(2a-y) h_x(y) dy$$
$$+ \int_x^{+\infty}\int_x^{+\infty} h_x(t) h_x(y) F_0^+(t+y) dt dy \tag{95}$$

Using the formula (95) and the Parseval's identities

$$\int_x^{+\infty} \|h_x(y)\|^2\, dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\|\widetilde{h}(\lambda)\right\|^2 d\lambda,$$

$$-\frac{A^-}{A^+} e^{-2i\gamma^+} \int_x^{+\infty} h_x(y) h_x(2a-y) dy = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{R}_0^+(\lambda)\, \widetilde{h}^2(\lambda) d\lambda,$$

where $\widetilde{h}(\lambda) = \int\limits_x^{+\infty} h_x(t) e^{-i\lambda t} dt$, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\|\widetilde{h}(\lambda)\right\|^2 d\lambda + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\widetilde{R}^+(\lambda) - \widetilde{R}_0^+(\lambda)\right) \widetilde{h}^2(\lambda) d\lambda + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{h}^2(\lambda) d\lambda = 0$$

i.e.

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \left\|\widetilde{h}(\lambda)\right\|^2 d\lambda = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \widetilde{R}^+(\lambda)\, \widetilde{h}^2(\lambda) d\lambda.$$

Therefore

$$\int_{-\infty}^{+\infty} \left\|\widetilde{h}(\lambda)\right\|^2 d\lambda = \left\| -\int_{-\infty}^{+\infty} \widetilde{R}^+(\lambda)\, \widetilde{h}^2(\lambda) d\lambda \right\| \leq \int_{-\infty}^{+\infty} \left\|\widetilde{R}^+(\lambda)\right\| \left\|\widetilde{h}(\lambda)\right\|^2 d\lambda,$$

that is

$$\int_{-\infty}^{+\infty} (1 - \left\|\widetilde{R}^+(\lambda)\right\|) \left\|\widetilde{h}(\lambda)\right\|^2 d\lambda \leq 0. \tag{96}$$

Since $\left\|\widetilde{R}^+(\lambda)\right\| < 1$ for $\lambda \neq 0$, (96) implies that $\widetilde{h}(\lambda) \equiv 0$. Consequently the equation (94) has a unique solution.

This theorem implies that the potential functions $q_0(x), ..., q_n(x)$ in problem $(1)-(2)$ without discrete spectrum are uniquely defined by the right(left) reflection coefficient.

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