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A Novel Successive Approximation Method for Solving a Class of Optimal Control Problems

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Abstract. This paper presents a successive approximation method (SAM) for solving a large class of optimal control problems. The proposed analytical-approximate method successively solves the Two-Point Boundary Value Problem (TPBVP), obtained from the Pontryagin's Maximum Principle (PMP). The convergence of this method is proved and a control design algorithm with low computational complexity is presented. Through the finite number of algorithm iterations, a suboptimal control law is obtained for the optimal control problem. An illustrative example is given to show the efficiency of the proposed method.

Keywords:Optimal control problem, Successive approximation method, Pontryagin's maximum principle, Suboptimal control.

2000 Mathematics subject classification: 49K15; Secondary 49M05.

1. Introduction

In the control theory, a major importance is conferred to optimal control problems. This interest is justified by a great number of practical applications in physics, economy, aerospace, chemical engineering, robotics, etc. [5, 15, 14, 20, 7]. For the general optimal control problem

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(OCP), however, an analytical solution does not exist. This has inspired researchers to propose approaches which obtain an approximate solution for it.

It is well-known that the OCP leads to a TPBVP obtained from the PMP. Many recent approaches have been devoted to solving this problem. One of these approaches is the Successive Approximation Approach (SAA) which designs a suboptimal controller for a class of nonlinear systems with a quadratic performance index. In this approach, a sequence of nonhomogeneous linear time-varying TPBVP's is solved to produce a finite-step iteration of the nonlinear compensation sequence obtaining the suboptimal control law [19]. However, SAA needs to solve a linear time-varying TPBVP which cannot be solved easily and thus, reduces the efficiency of this method.

In [11], a novel method that implements Modal series to solve a class of nonlinear OCP's with quadratic performance index has been proposed. This method which requires solving a sequence of linear time-invariant TPBVP's has less efficiency for large-scale problems.

Recently, a growing interest has been appeared toward the application of approximate analytical techniques in solving the TPBVP obtained from the PMP. In [22], the authors used He's variational iteration method (VIM) for linear quadratic OCP's. They transfer the linear TPBVP obtained from PMP to an initial value problem and then implement the VIM to get a feedback controller. Although their proposed method is important for its analytical approximation solutions, it is not applicable for nonlinear OCP's.

In [4], the authors give an analytical approximate solution for linear and non-linear quadratic OCP's using the homotopy perturbation method (HPM). Applying the HPM, the associated TPBVP is solved recursively and gets a suboptimal control law. Also, in [18], a basic and a modified VIM are successfully applied to the TPBVP, obtained from nonlinear quadratic OCP's. The authors combined the basic ideas of the shooting method to VIM and get the solutions consecutively. Though both of these two methods give accurate results, they suffer from a root-finding subroutine and then solving a system of algebraic equations which decreases the efficiency of the proposed methods.

Recently, in [1], a hybrid technique based on homotopy analysis and parametrization methods is presented. The authors applied an appropriate parametrization of control and computed the states using the homotopy analysis method (HAM). Other available methods are optimal homotopy analysis method [12], modified homotopy perturbation method [6], RBF collocation method [16], etc. Further computational methods for solving more general optimal control problems are also available at *e.g.* [8], [10].

In this paper, a novel SAM is proposed. We first derive the TPBVP from the PMP and then apply a novel SAM to solve it. This method could be applied to a large class of linear and nonlinear OCP's. The convergence of the proposed method is proved and a suboptimal control design algorithm with low computational complexity is presented. The simplicity and the efficiency of the proposed SAM is demonstrated through an illustrative example.

This paper is organized as follows. Section 2 describes the OCP and its associated extreme conditions. The novel SAM for solving the TPBVP is proposed in Section 3. The convergence of the proposed method is proved in section 4. In section 5, an efficient control design algorithm is presented. And finally, an illustrative example is given in section 6 to demonstrates the effectiveness of the new SAM.

2. Statement of the OCP and optimality conditions

Consider the following affine in control dynamical system

$$\begin{array}{l} \dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t), t \in [t_0, t_f] \\ x(t_0) = x^0. \end{array} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is denoting the state variable, $u(t) \in \mathbb{R}^m$ the control variable and x^0 the given initial state at t_0. Moreover, $f(t, x(t)) \in \mathbb{R}^n$ and $g(t, x(t)) \in \mathbb{R}^{n\times m}$ are two continuously differentiable functions in all arguments. Our aim is to minimize the objective functional

$$J[x, u] = \frac{1}{2}\int_{t_0}^{t_f} (Q(x(t)) + u^T(t)Ru(t))dt \tag{2.2}$$

subject to the dynamical system (2.1), for $Q(x(t))$ a positive semi-definite real function and $R \in \mathbb{R}^{m\times m}$ a positive definite matrix. Since the performance index (2.2) is convex, the following extreme necessary conditions are also sufficient for optimality:

$$\begin{array}{l} \dot{x} = f(t, x) + g(t, x)u^* \\ \dot{\lambda} = -H_x(x, u^*, \lambda) \\ u^* = \arg\min_u H(x, u, \lambda) \\ x(t_0) = x^0, \lambda(t_f) = 0. \end{array} \tag{2.3}$$

where $H(x, u, \lambda) = \frac{1}{2}[Q(x) + u^T Ru] + \lambda^T[f(t, x) + g(t, x)u]$ is referred to as the Hamiltonian. Equivalently, (2.3) can be written in the form of

([4, 18]):

$$\begin{aligned}
&\dot{x} = f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda] \\
&\dot{\lambda} = -\left(\tfrac{1}{2}\nabla Q(x) + (\tfrac{\partial f(t,x)}{\partial x})^T\lambda + \textstyle\sum_{i=1}^{n}\lambda_i[-R^{-1}g^T(t,x)\lambda]^T\tfrac{\partial g_i(t,x)}{\partial x}\right) \\
&x(t_0) = x^0, \lambda(t_f) = 0.
\end{aligned} \tag{2.4}$$

where $\lambda(t) \in \mathbb{R}^n$ is the co-state vector with the i^{th} component $\lambda_i(t)$, $i = 1, ..., n$ and $g(t,x) = [g_1(t,x), ..., g_n(t,x)]^T$ with $g_i(t,x) \in \mathbb{R}^m$, $i = 1, ..., n$. Also the optimal control law is obtained by

$$u^* = -R^{-1}g^T(t,x)\lambda. \tag{2.5}$$

For convenience, let us define the right hand sides of (2.4) as,

$$\begin{aligned}
&\Psi_1(t,x,\lambda) := f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda], \\
&\Psi_2(t,x,\lambda) := -\left(\tfrac{1}{2}\nabla Q(x) + (\tfrac{\partial f(t,x)}{\partial x})^T\lambda + \textstyle\sum_{i=1}^{n}\lambda_i[-R^{-1}g^T(t,x)\lambda]^T\tfrac{\partial g_i(t,x)}{\partial x}\right).
\end{aligned} \tag{2.6}$$

Thus the TPBVP in (2.4) changes to the operator form ([9, 21]) as follows:

$$\begin{aligned}
&\dot{X}(t) - \Psi(t, X(t)) = \mathcal{L}[X(t)] + \mathcal{N}[X(t)] = 0, \\
&x(t_0) = x^0, \quad \lambda(t_f) = 0,
\end{aligned} \tag{2.7}$$

where

$$X(t) = \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}, \quad \Psi(t, X(t)) = \begin{bmatrix} \Psi_1(t, X(t)) \\ \Psi_2(t, X(t)) \end{bmatrix},$$

and the linear and nonlinear operators $\mathcal{L}$ and $\mathcal{N}$ are defined as:

$$\begin{aligned}
&\mathcal{L}[X(t)] = \dot{X}(t) + p(t)X(t), \\
&\mathcal{N}[X(t)] = -p(t)X(t) - \Psi(t, X(t)) = \begin{bmatrix} \mathcal{N}_1[X(t)] \\ \mathcal{N}_2[X(t)] \end{bmatrix},
\end{aligned} \tag{2.8}$$

where $p(t)$ is a real $(2n)$ by $(2n)$ matrix as follows:

$$p(t) = \begin{bmatrix} p_1(t) & \mathrm{O} \\ \mathrm{O} & p_2(t) \end{bmatrix}, \; p_1(t), p_2(t) \in \mathbb{R}^{n\times n}.$$

Unfortunately, there is no analytical solution to this nonlinear TPBVP, in general. So, it is of high importance to calculate analytical approximation or numerical solutions for that. In recent decades, some new numerical and analytical approximation methods have been proposed for solving such a difficult problem in the context of ordinary differential equations. In the next section, we propose a new SAM, for this end.

3. A New SAM for Solving the TPBVP

In this section, we propose a new SAM to solve the TPBVP in (2.7), analytically. Construct a sequence of solutions for solving (2.7), as follows:

$$\mathcal{L}[X_{k+1}(t)] = -\mathcal{N}[X_k(t)], \tag{3.1}$$

with $x_{k+1}(t_0) = x^0$, $\lambda_{k+1}(t_f) = 0$ and $k \geq 0$. This linear ODE could be solved for $x_{k+1}(t)$ analytically (See *e.g.* [3, 17]).

In view of (2.8), solving (3.1) leads to:

$$X_{k+1}(t) = -\int_{t_0}^{t} \Phi(t,s)\mathcal{N}[X_k(s)]ds + \Phi(t,t_0)C, \tag{3.2}$$

where $\Phi(t,s) = e^{-\int_s^t p(\tau)d\tau}$ is the transfer matrix and $C \in \mathbb{R}^{2n}$ is constant. (3.2) can be equivalently written as:

$$\begin{array}{l} x_{k+1}(t) = -\int_{t_0}^{t} \Phi_1(t,s)\mathcal{N}_1[X_k(s)]ds + \Phi_1(t,t_0)C_1, \\ \lambda_{k+1}(t) = -\int_{t_0}^{t} \Phi_2(t,s)\mathcal{N}_2[X_k(s)]ds + \Phi_2(t,t_0)C_2. \end{array} \tag{3.3}$$

where the transfer matrix is decomposed as follows:

$$\Phi(t,s) = \begin{bmatrix} \Phi_1(t,s) & \mathrm{O} \\ \mathrm{O} & \Phi_2(t,s) \end{bmatrix}, \forall s \in [t_0,t] \text{ and } t \in [t_0,t_f],$$

and $\Phi_i(t,s) = e^{-\int_s^t p_i(\tau)d\tau}$, $i = 1,2$. Also $\Phi_i(t,s) \cdot \Phi_i(s,w) = \Phi_i(t,w)$, for all $t,s,w \in [t_0,t_f]$ and $i = 1,2$. Imposing the initial and final conditions, $x_{k+1}(t_0) = x^0$ and $\lambda_{k+1}(t_f) = 0$, for all $k \geq 0$, C_1 and C_2 can be readily calculated as:

$$\begin{array}{rcl} C_1 &=& x^0, \\ C_2 &=& \Phi_2^{-1}(t_f,t_0)\int_{t_0}^{t_f} \Phi_2(t_f,s)\mathcal{N}_2[X_k(s)]ds \\ &=& \int_{t_0}^{t_f} \Phi_2(t_0,t_f)\Phi_2(t_f,s)\mathcal{N}_2[X_k(s)]ds \\ &=& \int_{t_0}^{t_f} \Phi_2(t_0,s)\mathcal{N}_2[X_k(s)]ds. \end{array}$$

Therefore, the SAM formula becomes,

$$x_{k+1}(t) = -\int_{t_0}^{t} \Phi_1(t,s)\mathcal{N}_1[X_k(s)] + \Phi_1(t,t_0)x^0, \tag{3.4}$$

$$\lambda_{k+1}(t) = \int_{t}^{t_f} \Phi_2(t,s)\mathcal{N}_2[X_k(s)]ds. \tag{3.5}$$

Remark 3.1. The SAM formula (3.4)-(3.5) is directly dependent on the integration. In our optimal control problem, $\mathcal{N}_1$ and $\mathcal{N}_2$ are two nonlinear operators which might cause the right-hand-side integration of (3.4)-(3.5) to be very time consuming and complicated, even at the first few iterations of SAM. To overcome this undesirable case, the following correction, which omits the time consuming calculations from SAM can be applied:

$$\begin{aligned} x_{k+1}(t) &= -\int_{t_0}^{t} T_k(t,s)ds + \Phi_1(t,t_0)x^0, \\ \lambda_{k+1}(t) &= \int_t^{t_f} \widetilde{T}_k(t,s)ds. \end{aligned} \tag{3.6}$$

where $T_k(t,s)$ and $\widetilde{T}_k(t,s)$ are the k^{th} order of Taylor interpolating polynomial at $s = t_0$ of the integrands of (3.4) and (3.5), respectively.

4. Convergence Analysis

Now, we state and prove the convergence of the foregoing SAM sequence.

Definition 4.1. ([2]) Let V be a Banach space. For an operator $T : K \subseteq V \to V$, we say it is contractive with contractivity constant $\alpha \in [0,1)$, if

$$\|T(v) - T(w)\|_V \leq \alpha \|v - w\|_V, \quad \forall v, w \in K.$$

Theorem 4.2. *Assume that $\{x_k(t)\}$ and $\{\lambda_k(t)\}$ are two SAM sequences produced by (3.4)-(3.5). Furthermore, assume $N[v(t)]$ is continuous for any $v(t) \in \mathbb{R}^{2n}$, $t \in [t_0, t_f]$, and*

$$|N[v(t)] - N[w(t)]| \leq M_1 |v(t) - w(t)|, \quad \forall v, w \in C[t_0, t_f],$$

for some constant M_1. Then $\{x_k(t)\}$ and $\{\lambda_k(t)\}$ converge to the exact solutions of (2.7), for any initial continuous functions $x_0(t)$ and $\lambda_0(t)$, if the contractivity constant $M_1 M_2 (t_f - t_0) \in [0,1)$, where

$$M_2 = \sup \left\{ e^{-\int_s^t p(\tau)d\tau}, \ s \in [t_0, t], \ t \in [t_0, t_f] \right\}.$$

Proof. It is clear that the SAM sequences (3.4)-(3.5) are equivalent to (3.1) or (3.2). In the light of (3.2), define the operator T as:

$$T[v(t)] := -\int_{t_0}^{t} e^{-\int_s^t p(\tau)d\tau} \mathcal{N}[v(s)]ds + e^{-\int_{t_0}^t p(\tau)d\tau} C, \ C \in \mathbb{R}^{2n}. \tag{4.1}$$

Then for any continuous functions $v(t)$ and $w(t)$, we have:

$$\begin{aligned} |T[v(t)] - T[w(t)]| &= \left| \int_{t_0}^{t} e^{-\int_s^t p(\tau)d\tau} (\mathcal{N}[v(s)] - \mathcal{N}[w(s)]) ds \right| \\ &\leq M_2 \left| \int_{t_0}^{t} (\mathcal{N}[v(s)] - \mathcal{N}[w(s)]) \right| ds \\ &\leq M_1 M_2 \int_{t_0}^{t} |v(s) - w(s)| ds \\ &\leq M_1 M_2 (t - t_0) \|v - w\|_\infty \\ &\leq M_1 M_2 (t_f - t_0) \|v - w\|_\infty. \end{aligned}$$

Thus by Banach fixed-point theorem (page 133 of [2]), $\{x_k(t)\}$ and $\{\lambda_k(t)\}$ converge to some $\hat{x}(t)$ and $\hat{\lambda}(t)$. By taking limits from both sides of (3.1), we have:

$$\lim_{k\to\infty} \mathcal{L}[X_{k+1}] = -\lim_{k\to\infty} \mathcal{N}[X_k],$$

which the continuity of $\mathcal{N}$, gives

$$\mathcal{L}[\lim_{k\to\infty} X_{k+1}] = -\mathcal{N}[\lim_{k\to\infty} X_k],$$

or $\mathcal{L}[\hat{X}] = -\mathcal{N}[\hat{X}]$. Moreover, by (3.4)-(3.5), one can easily check that for all $k \geq 0$, $x_{k+1}(t_0) = x^0$ and $\lambda_{k+1}(t_f) = 0$. Hence, $\hat{x}(t_0) = x^0$ and $\hat{\lambda}(t_f) = 0$. That is, $\hat{x}(t)$ and $\hat{\lambda}(t)$ are the exact solutions of (2.7) which completes the proof. $\square$

Remark 4.3. The choice of $p(t)$ should be performed such that the condition $M_1M_2(t_f - t_0) \in [0, 1)$ in Theorem 4.1 holds. Some easy choices could be zero matrix, the linear parts at each equation of (2.4) or some linear term that we add to the both sides of equations in (2.4).

Theorem 4.4. *Under the assumptions of Theorem 4.2, the sequences $\{u_k(t)\}$ and $\{J_k\}$ defined by*

$$u_k(t) = -R^{-1}g^T(t, x_k(t))\lambda_k(t), \tag{4.2}$$

$$J_k = \frac{1}{2}\int_{t_0}^{t_f} (Q(x_k(t)) + u_k^T(t)Ru_k(t))dt, \tag{4.3}$$

converge to the optimal control law and optimal objective value, respectively.

Proof. Theorem 4.2 states that $\{x_k(t)\}$ and $\{\lambda_k(t)\}$ converge to the optimal state and costate vectors, say $\hat{x}(t)$ and $\hat{\lambda}(t)$, respectively. Taking limits from (4.2), the continuity assumption of $g(t, x)$ gives

$$\begin{aligned} \hat{u}(t) &:= \lim_{k\to\infty} u_k(t) = -R^{-1}g^T(t, \lim_{k\to\infty} x_k(t)) \lim_{k\to\infty} \lambda_k(t) \\ &= -R^{-1}g^T(t, \hat{x}(t))\hat{\lambda}(t), \end{aligned}$$

which is the optimal control law, since $\hat{x}(t)$ and $\hat{\lambda}(t)$ are the optimal state and costate vectors. Also by the continuity assumption of $Q(x(t))$, taking limits from (4.3) yields:

$$\begin{aligned} \hat{J} &:= \lim_{k\to\infty} J_k = \frac{1}{2}\lim_{k\to\infty}\int_{t_0}^{t_f} (Q(x_k(t)) + u_k^T(t)Ru_k(t))dt \\ &= \frac{1}{2}\int_{t_0}^{t_f} (Q(\lim_{k\to\infty} x_k(t)) + \lim_{k\to\infty} u_k^T(t)R \lim_{k\to\infty} u_k(t))dt \\ &= \frac{1}{2}\int_{t_0}^{t_f} (Q(\hat{x}(t)) + \hat{u}^T(t)R\hat{u}(t))dt. \end{aligned}$$

Therefore $\hat{J}$ is the optimal objective value. □

5. Suboptimal control design algorithm

The solution guidelines for TPBVP (2.7) has been discussed in previous section. In this section, we give a more reliable way for finding the desired optimal control and the optimal state and then we present an algorithm for this end.

From Theorem 4.4, we conclude that for large number of iterations, N, suboptimal control law is derived by

$$u^*(t) \cong u_N(t) = -R^{-1}g^T(t,x)\lambda_N(t), \tag{5.1}$$

and the approximate suboptimal state is $x^*(t) \cong x_N(t)$. Applying this pair of suboptimal control and state to the objective functional (2.2), results in the suboptimal objective value of the problem, i.e.

$$J^* \cong J_N = \frac{1}{2}\int_{t_0}^{t_f} (Q(x_N(t)) + u_N^T(t)Ru_N(t))dt. \tag{5.2}$$

For the accuracy analysis, we consider the following criterion. The suboptimal control (5.1) has the desirable accuracy, if for given $\epsilon > 0$, the following condition holds,

$$\left|\frac{J_N - J_{N-1}}{J_N}\right| < \epsilon. \tag{5.3}$$

If the tolerance limit ϵ is sufficiently small, according to Theorem 4.4, the suboptimal value is very close to the optimal value J^*. Now, we present an algorithm of the proposed method with low computational complexity, in order to maintain the accuracy of solutions.

Algorithm:
Step 1. Let $N = 1$, $x_0(t) = x^0$, $\lambda_0(t) = 0$ and $\epsilon > 0$ be any given sufficiently small tolerance.
Step 2. Update state and costate functions implementing the SAM (3.4)-(3.5) or (3.6), to find $x_N(t)$ and $\lambda_N(t)$.
Step 3. Determine the suboptimal control $u_N(t)$ and the suboptimal objective value J_N by (5.1)-(5.2).
Step 4. If criterion (5.3) holds, go to Step 5, otherwise let $N = N + 1$ and go to Step 2.
Step 5. Stop the algorithm. $u_N(t)$ is the desirable suboptimal control law.

6. Illustrative example

The following example is given to illustrate the simplicity and efficiency of the proposed method. The codes are developed using computation softwares MAPLE 15 and MATLAB, and the calculations are implemented on a machine with Intel Core 2 Due Processor 2.53 Ghz and 4 GB RAM.

Example 6.1. Consider the nonlinear system described by

$$\begin{aligned} &\dot{x}_1 = x_2 + x_1 x_2 \\ &\dot{x}_2 = -x_1 + x_2 + x_2^2 + u \\ &x_1(0) = -0.8, \quad x_2(0) = 0 \end{aligned} \tag{6.1}$$

and the cost functional

$$J = \frac{1}{2}\int_0^{\frac{1}{2}} (x_1^2 + x_2^2 + u^2)dt. \tag{6.2}$$

The extreme conditions are

$$\begin{aligned} &\dot{x}_1 = x_2 + x_1 x_2 \\ &\dot{x}_2 = -x_1 + x_2 + x_2^2 - \lambda_2 \\ &\dot{\lambda}_1 = -(x_1 + \lambda_1 x_2 - \lambda_2) \\ &\dot{\lambda}_2 = -(x_2 + \lambda_1(1 + x_1) + \lambda_2(1 + 2x_2)) \\ &x_1(0) = -0.8,\ x_2(0) = 0,\ \lambda_1(\tfrac{1}{2}) = 0,\ \lambda_2(\tfrac{1}{2}) = 0, \end{aligned}$$

and the optimal control is $u = -\lambda_2$. In view of (2.8), the linear and the nonlinear operators of the above TPBVP can be defined in several ways as follows:

$$\mathcal{L}[X] = \dot{X}(t) + p(t)X(t),$$

where $X = [x_1, x_2, \lambda_1, \lambda_2]^T$ and

(a)

$$p(t) = \mathrm{O}_{4\times 4},\ \mathcal{N}[\mathrm{X}] = \begin{bmatrix} -x_2 - x_1 x_2 \\ x_1 - x_2 - x_2^2 + \lambda_2 \\ x_1 + \lambda_1 x_2 - \lambda_2 \\ x_2 + \lambda_1(1 + x_1) + \lambda_2(1 + 2x_2) \end{bmatrix},\ \alpha = 0.4016$$

(b)

$$p(t) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix},\ \mathcal{N}[X] = \begin{bmatrix} -x_1 x_2 \\ -x_2^2 + \lambda_2 \\ x_1 + \lambda_1 x_2 \\ x_2 + \lambda_1 x_1 + 2\lambda_2 x_2 \end{bmatrix},\ \alpha = 0.6554$$

where $\alpha = M_1 M_2 (t_f - t_0)$ is the contractivity constant used in Theorem 4.2. In case (a), we first set the $p(t)$ as a zero matrix and other terms as nonlinear terms. This implies a contraction constant as $\alpha < 1$. In case

TABLE 1. Simulation results of SAM in case (a) and (b), based on the relative errors of objective value, Example 1.

N (Itr.)	Case (a)	Case (b)
5	4.82861×10^{-3}	9.30044×10^{-3}
10	9.64918×10^{-5}	1.36117×10^{-4}
15	1.58638×10^{-6}	1.24144×10^{-6}
20	4.89518×10^{-8}	1.08149×10^{-8}
25	2.23602×10^{-10}	7.22779×10^{-11}

(b), $p(t)$ is linear terms of each equation in extreme conditions. i.e. the extreme conditions are written as:

$$\dot{x}_1 = x_2 + x_1 x_2 \tag{6.3}$$

$$\dot{x}_2 - x_2 = -x_1 + x_2^2 - \lambda_2 \tag{6.4}$$

$$\dot{\lambda}_1 = -\left(x_1 + \lambda_1 x_2 - \lambda_2\right) \tag{6.5}$$

$$\dot{\lambda}_2 + \lambda_2 = -\left(x_2 + \lambda_1 x_1 + 2\lambda_2 x_2\right) \tag{6.6}$$

The left hand sides of the above equation are the linear parts and the right hand sides are nonlinear terms. Of course, other choices are available as discussed in Remark 4.3. It is important to note that the convergence of SAM is guaranteed whenever $\alpha \in [0,1)$, which is true in our case (a) and (b).

Implementing the algorithm described in Section 5, one can obtain the suboptimal solution for given $\epsilon = 5 \times 10^{-6}$, after $N = 15$ iterations. Table 1 shows the relative error of optimal objective values for several iterations. It is seen that SAM (a) and (b) reach the tolerance limit after 15 iterations. For $N = 15$, the suboptimal control and objective value can be found using SAM (a) as:

$$\begin{aligned} u^*(t) \cong u_{15}(t) &= 8.2005 \times 10^{-7}\, t^{16} + 0.23575 \times 10^{-5}\, t^{15} - 0.18623 \times 10^{-4}\, t^{14} \\ &-0.12099 \times 10^{-4}\, t^{13} - 0.10112 \times 10^{-3}\, t^{12} - 0.16282 \times 10^{-3}\, t^{11} + 0.59612 \times 10^{-3}\, t^{10} \\ &+0.21462 \times 10^{-3}\, t^{9} + 0.21739 \times 10^{-2}\, t^{8} + 0.27712 \times 10^{-2}\, t^{7} - 0.022487\, t^{6} \\ &-0.78086 \times 10^{-2}\, t^{5} - 0.036403\, t^{4} - 0.035923\, t^{3} + 0.49182\, t^{2} + 0.049226\, t - 0.14024, \end{aligned}$$

$$J^* \cong J_{15} = 0.175683.$$

To illustrate the efficiency of the proposed method, we compare the results of SAM (a), with two recent methods, VIM [18] and HPM [4]. The number of iterations and the CPU time of these methods are summarized in Table 2, for different tolerance limits (5.3).

According to Table 2, the proposed SAM is faster than two other methods. For instance, the proposed SAM reaches the tolerance limit $\epsilon = 5 \times 10^{-12}$, in less than one second while the modified VIM and HPM do

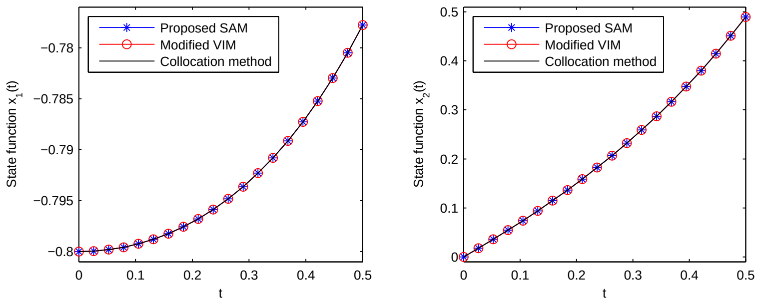

FIGURE 1. The suboptimal states of Example 1.

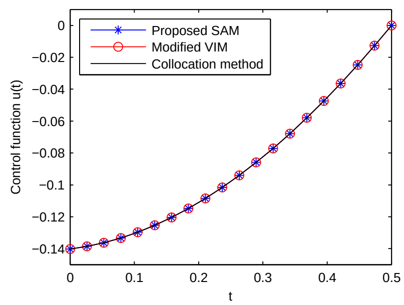

FIGURE 2. The suboptimal control of Example 1.

TABLE 2. Comparison of the proposed SAM, Modified VIM [18] and HPM [4], Example 1.

Tolerance	Proposed SAM		Modified VIM [18]		HPM [4]	
limit (ϵ)	N	CPU Time*	N	CPU Time	N	CPU Time
5×10^{-6}	15	0.266	10	0.546	11	8.752
5×10^{-9}	22	0.406	17	2.761	-	-
5×10^{-12}	29	0.811	23	15.71	-	-

* CPU time (sec.)

not. In fact, HPM could not reach the tolerance limit less than 5×10^{-9}, because of the complicated calculations and the CPU time of the modified VIM grows rapidly for a large number of iterations.
Also, the values of objective functional for these methods are presented in Table 3. According to this table, the sub-optimal objective value is

TABLE 3. Comparison of objective values of the proposed SAM (a)-(b), Modified VIM [18] and HPM [4] for $m = 10$ iterations, Example 1.

Method	J_m
Proposed SAM (a)	0.1756936822
Proposed SAM (b)	0.1756786212
Modified VIM	0.1756827686
HPM	0.1756827686

approximately $J^* = 0.1757$. Figures 1 and 2 show the suboptimal solutions after $N = 15$ algorithm iterations, compared to the Modified VIM and collocation method [13].

7. Conclusions

In this paper, a novel analytical approximate method called SAM has been proposed for solving a broad class of optimal control problems. This method can solve the TPBVP obtained from PMP recursively. The proposed SAM does not need any complex computations in comparison with other recent methods. The convergence of the proposed SAM is proved and an illustrative example demonstrated the effectiveness and good results in low CPU time.

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