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A new idea for exact solving of the complex interval linear systems

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Abstract. In this paper, the aim is to find a complex interval vector $[\mathbf{z}]$ such that satisfies the complex interval linear system $\mathbf{C}[\mathbf{z}] = [\mathbf{w}]$. For this, we present a new method by restricting the general solution set via applying some parameters. The numerical examples are given to show ability and reliability of the proposed method.

Keywords: Complex interval vector, Complex interval linear system, Crout decomposition method.

2010 Mathematics subject classification: 65G40, 65E05, 65F05.

1. Introduction

In some problems, for instance the electrical circuits [9, 10], we have a system of linear equations with uncertain complex parameters. Also, if we represent this uncertainty by intervals, then we obtain a problem that is called, "Complex interval linear system".

Unluckily, little researchers have presented the numerical and analytical methods for solving complex interval linear systems. Complex interval linear systems were studied in [8, 9, 10] and among others. In 2006, Djanybekov [6] have presented an outer estimation of solution set of a complex interval linear system by interval Householder method. In 2010, Hladik [8] have proposed a method for obtaining a very accurate

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approximation of the interval hull of the solution set of a complex interval linear system by a system of nonlinear inequalities. Also, in the same year, Popova et al. [11] have used the advanced technology of communication protocols for developing of new software, integrated between Mathematica and C-XSC, that solves complex-valued parametric linear systems. Recently, the author [7] have introduced an algorithm for presenting an inner estimation of the solution set of a complex interval linear system. Also, he showed that under some certain conditions, the obtained inner estimation is an algebraic solution.

In this paper, we focus on the complex interval linear systems that their coefficient matrix are complex crisp-valued and the right-hand-side columns are complex interval-valued. It can be easily investigated that, if we use the usual Crout decomposition method for solving a complex interval linear system, then we do not obtain an algebraic solution. In other words, Crout's solution vector does not satisfy all equations of system as algebraically. For this reason, in this paper, we try to eliminate this problem and obtain a solution vector such that satisfies all equations of a complex linear system as algebraically.

In proposed method, we first solve a complex interval linear system by a complex interval version of the classic Crout decomposition method. By doing this work, we obtain a general solution set for a complex interval linear system. In the next step, we restrict this general solution set by some parameters. Finally, we prove that the obtained complex interval vector is an algebraic solution, or in other words, it satisfies the complex interval linear system.

The structure of this paper is as follows. In Section 2 we present some basic definitions and concepts of complex interval theory. In Section 3, we represent a complex interval version of the classic Crout decomposition method. In Section 4, we propose a new method for obtaining the algebraic solution of a complex interval linear system. In Section 5, we give two numerical examples to show ability and reliability of our proposed method. Conclusion is drawn in Section 6.

2. Preliminaries

It is known that the real interval $[a]$ is showed as $[\underline{a}, \overline{a}]$ and also the set of all real intervals is denoted by $\mathbb{IR}$ [1, 3]. However, in this section, we focus on complex intervals and remind several basic concepts about them.

Definition 2.1. [4, 2] A complex interval $[z]$ is defined as

$$\begin{aligned} [z] &= [a] + i[b] \\ &= \{a + ib \in \mathbb{C} \mid \underline{a} \leqslant a \leqslant \overline{a},\ \underline{b} \leqslant b \leqslant \overline{b}\}, \end{aligned}$$

where $[a] = [\underline{a}, \overline{a}]$ and $[b] = [\underline{b}, \overline{b}]$ are two arbitrary real intervals, i.e. $[a], [b] \in \mathbb{IR}$.

Throughout the paper, we denote the set of all complex intervals by $\mathbb{IC}$. According to Definition 2.1 and similar to [6, 11], the complex interval $[z] = [\underline{a}, \overline{a}] + i[\underline{b}, \overline{b}]$ can be represented as

$$[z] = [\underline{z}, \overline{z}] = [\underline{a} + i\underline{b}, \overline{a} + i\overline{b}].$$

Therefore, we conclude

$$\underline{z} = \underline{a} + i\underline{b}, \qquad \overline{z} = \overline{a} + i\overline{b}.$$

Obviously, we can show $\mathbb{IR} \subset \mathbb{IC}$, because the real interval $[a]$ can be regarded as a complex interval $[a] = [a] + i[0, 0] \in \mathbb{IC}$.

To presentation of the complex interval arithmetic, it should be noted that most properties of the real interval arithmetic can be extended to the complex case as well. Let us consider the complex crisp number $c = a + ib$ and two complex intervals $[z_1] = [p_1] + i[q_1]$ and $[z_2] = [p_2] + i[q_2]$, where $[p_j] = [\underline{p_j}, \overline{p_j}]$ and $[q_j] = [\underline{q_j}, \overline{q_j}]$, $j = 1, 2$. Then, we will have

$$\begin{aligned} [z_1] + [z_2] &= ([p_1] + [p_2]) + i\ ([q_1] + [q_2]) \\ &= [\underline{p_1} + \underline{p_2}, \overline{p_1} + \overline{p_2}] + i\,[\underline{q_1} + \underline{q_2}, \overline{q_1} + \overline{q_2}], \end{aligned}$$

and

$$\begin{aligned} c \cdot [\widetilde{z}_1] &= (a + ib) \cdot ([p_1] + i\,[q_1]) \\ &= (a[p_1] - b[q_1]) + i\ (a[q_1] + b[p_1])\,. \end{aligned}$$

Definition 2.2. We define the center and width of the complex interval $[z] = [\underline{z}, \overline{z}]$, respectively as follows:

$$[z]^c = \frac{\underline{z} + \overline{z}}{2}, \qquad [z]^w = \overline{z} - \underline{z}.$$

Remark 2.3. Based on Definition 2.2, for the complex interval $[z] = [p] + i[q]$, it can be shown that

$$[z]^c = [p]^c + i[q]^c, \qquad [z]^w = [p]^w + i[q]^w.$$

In the next theorem, we present the center and width of a linear combination of the complex intervals.

Theorem 2.4. [7] *For the complex intervals $[z_j] = [p_j] + i[q_j]$, and the complex crisp numbers $c_j = a_j + ib_j$, $j = 1, 2, ..., n$ we have*

$$\left(\sum_{j=1}^{n} c_j\,[z_j]\right)^c = \left(\sum_{j=1}^{n} (a_j\ [p_j]^c - b_j\ [q_j]^c)\right) + i\left(\sum_{j=1}^{n} (a_j\ [q_j]^c + b_j\ [p_j]^c)\right).$$

$$\left(\sum_{j=1}^{n} c_j[z_j]\right)^w = \left(\sum_{j=1}^{n} \left(|a_j|\,[p_j]^w + |b_j|\,[q_j]^w\right)\right) + i\left(\sum_{j=1}^{n} \left(|a_j|\,[q_j]^w + |b_j|\,[p_j]^w\right)\right).$$

Obviously, the vector $[\mathbf{z}] = ([z_1], [z_2], \ldots, [z_n])^T$ where $[z_i]$, $i = 1, 2, \ldots, n$, are complex intervals, is called a complex interval vector. Now, we are going to define the center and width vectors of a complex interval vector.

Definition 2.5. We define the center and width vectors of the complex interval vector $[\mathbf{z}] = ([z_1], [z_2], \ldots, [z_n])^T$ respectively as follows:

$$[\mathbf{z}]^c = ([z_1]^c, [z_2]^c, \ldots, [z_n]^c)^T.$$

$$[\mathbf{z}]^w = ([z_1]^w, [z_2]^w, \ldots, [z_n]^w)^T.$$

In the continuation of these basic definitions, we state a generalized definition of the completely nonsingular matrices that was presented in [7].

Definition 2.6. Let $\mathbf{C} = (c_{kj})_{n\times n}$ be a complex crisp matrix, i.e. $c_{kj} = a_{kj} + i\,b_{kj}$ and also $\mathbf{A} = (a_{kj})_{n\times n}$ and $\mathbf{B} = (b_{kj})_{n\times n}$ be the real and imaginary parts of the matrix $\mathbf{C}$, respectively. We say that the matrix $\mathbf{C}$ is completely nonsingular, if all matrices $\mathbf{C}$, $|\mathbf{A}| + |\mathbf{B}|$ and $|\mathbf{A}| - |\mathbf{B}|$ are nonsingular, where $|\mathbf{A}| = (|a_{kj}|)_{n\times n}$ and $|\mathbf{B}| = (|b_{kj}|)_{n\times n}$ are two nonnegative real matrices.

Here, we define a complex interval linear system as follows.

Definition 2.7. The $n \times n$ linear system

$$\begin{cases} c_{11}\,[z_1] + c_{12}\,[z_2] + \cdots + c_{1n}\,[z_n] = [w_1], \\ c_{21}\,[z_1] + c_{22}\,[z_2] + \cdots + c_{2n}\,[z_n] = [w_2], \\ \quad\vdots \\ c_{n1}\,[z_1] + c_{n2}\,[z_2] + \cdots + c_{nn}\,[z_n] = [w_n], \end{cases} \tag{2.1}$$

where the coefficient matrix $\mathbf{C} = (c_{kj})_{n\times n}$, $c_{kj} = a_{kj} + ib_{kj}$, is an $n \times n$ complex crisp matrix and $[w_j] = [u_j] + i\,[v_j]$, $1 \leqslant j \leqslant n$ are complex intervals, is called a complex interval linear system.

We can denote the complex interval linear system (2.1) as

$$\mathbf{C}[\mathbf{z}] = [\mathbf{w}], \tag{2.2}$$

where

$$[\mathbf{z}] = ([z_1], [z_2], \ldots, [z_n])^T, \qquad [\mathbf{w}] = ([w_1], [w_2], \ldots, [w_n])^T,$$

are two complex interval vectors. Also, if we set $[z_j] = [p_j] + i\,[q_j]$ and $[w_j] = [u_j] + i\,[vj]$, $1 \leqslant j \leqslant n$, then we can write

$$\mathbf{C} = \mathbf{A} + i\,\mathbf{B}, \qquad [\mathbf{z}] = [\mathbf{p}] + i\,[\mathbf{q}],$$

and

$$[\mathbf{w}] = [\mathbf{u}] + i\,[\mathbf{v}].$$

In most papers, a solution of (2.1) is defined as a solution to a system $\mathbf{C}\mathbf{z}' = \mathbf{w}'$ for some $\mathbf{w}' \in [\mathbf{w}]$. Regarding to this note, the solution set of complex interval linear system (2.1) is defined as a set of all solutions, as follows.

Definition 2.8. The solution set of the complex interval linear system (2.1) is defined traditionally as

$$\Sigma = \left\{\mathbf{z}' \in \mathbb{C}^n \mid (\exists \mathbf{w}' \in [\mathbf{w}])(\mathbf{C}\,\mathbf{z}' = \mathbf{w}')\right\}.$$

Remark 2.9. Suppose that the coefficient matrix $\mathbf{C}$ is nonsingular, that means $det(\mathbf{C}) \neq 0{+}i0$. Therefore, if we consider $\mathbf{C}^{-1}\,[\mathbf{w}] = \left\{\mathbf{C}^{-1}\mathbf{w}' \mid \mathbf{w}' \in [\mathbf{w}]\right\}$, then obviously we have

$$\Sigma = \mathbf{C}^{-1}\,[\mathbf{w}].$$

In the following, we want to define a complex interval vector as an algebraic solution for the complex interval linear system (2.1).

Definition 2.10. A complex interval vector

$$[\mathbf{z}_A] = ([z_{1A}], [z_{2A}], \ldots, [z_{nA}])^T,$$

where $[z_{jA}] = [\underline{z_{jA}}, \overline{z_{jA}}]$, is an "algebraic solution" of the complex linear system (2.1) if it satisfies all equations of system (2.1), or in other words

$$\sum_{j=1}^{n} c_{kj}\,\left([\underline{z_{jA}}, \overline{z_{jA}}]\right) = [\underline{w_k}, \overline{w_k}], \qquad k = 1, 2, \ldots, n.$$

It should be noted that for the complex interval linear system (2.1), the algebraic solution may not exist. The following theorem, shows the relation between the algebraic solution $[\mathbf{z}_A]$ and the solution set Σ.

Theorem 2.11. [7] *If the complex interval linear system (2.1) has the algebraic solution $[\boldsymbol{z}_A]$, then we have*

$$[\boldsymbol{z}_A] \subseteq \Sigma.$$

In the next section, we want apply a complex interval version of the usual Crout decomposition method to obtain a general solution set of the complex interval linear system (2.1).

3. General solution set

The Crout complex interval decomposition method is obtained from its real version (see [5]), replacing the real numbers by the complex intervals and the real operations by the corresponding complex interval operations. Also, in this section, we show that the solution set Σ and the algebraic solution $[\mathbf{z}_A]$ (if it exists) are subsets of the obtained solution by the Crout complex interval decomposition method. For this reason, we call the obtained solution by the Crout complex interval decomposition method a "general solution set".

To this end, please consider the complex interval linear system (2.2). In a similar manner of the usual Crout decomposition, the coefficient matrix $\mathbf{C} = (c_{kj})_{n\times n}$ is decomposed into the product of the lower-triangular complex matrix $\mathbf{L} = (l_{kj})_{n\times n}$ and the upper-triangular complex matrix $\mathbf{U} = (u_{kj})_{n\times n}$, where the main diagonal of $\mathbf{U}$ consists of all $1 + 0i$s. In other words

$$\mathbf{C} = \mathbf{L}\mathbf{U}, \tag{3.1}$$

where

$$\mathbf{L} = \begin{pmatrix} l'_{11} + il''_{11} & 0 & \cdots & 0 \\ l'_{21} + il''_{21} & l'_{22} + il''_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l'_{n1} + il''_{n1} & l'_{n2} + il''_{n2} & \cdots & l'_{nn} + il''_{nn} \end{pmatrix},$$

and

$$\mathbf{U} = \begin{pmatrix} 1 & u'_{12} + iu''_{12} & \cdots & u'_{1n} + iu''_{1n} \\ 0 & 1 & \cdots & u'_{2n} + iu''_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

In the above decomposition, it should be noted that if all ones be on the main diagonal of $\mathbf{L}$, then the corresponding decomposition is called "*Doolittle decomposition*". From Eqs. (2.2) and (3.1), we conclude

$$\mathbf{L}\mathbf{U}[\mathbf{z}] = [\mathbf{w}]. \tag{3.2}$$

If we suppose $[\mathbf{y}] = \mathbf{U}[\mathbf{z}]$, we obtain $\mathbf{L}[\mathbf{y}] = [\mathbf{w}]$. Since $\mathbf{L}$ is a lower triangular matrix, we can obtain $[\mathbf{y}]$ by forward substitution, as follows

$$[y_1] = [w_1], \tag{3.3}$$

$$[y_k] = [w_k] - \sum_{j=1}^{k-1} l_{kj}[y_j], \qquad k = 2, 3, \ldots, n. \tag{3.4}$$

In the next step, since $\mathbf{U}$ is upper triangular, we can find the desired solution $[\mathbf{z}]$ from $\mathbf{U}[\mathbf{z}] = [\mathbf{y}]$ by backward substitution, as follows

$$[z_n] = \frac{1}{u_{nn}}[y_n], \tag{3.5}$$

$$[z_k] = \frac{1}{u_{kk}}\left([y_k] - \sum_{j=k+1}^{n} u_{kj}[z_j]\right), \qquad k = n-1, n-2, \ldots, 1. \tag{3.6}$$

We denote the obtained solution by the above process as

$$[\mathbf{z_{cr}}] = ([z_{1_{cr}}], [z_{2_{cr}}], \ldots, [z_{n_{cr}}])^T,$$

where $[z_{j_{cr}}] = [p_j] + i[q_j]$, $j = 1, 2, \ldots, n$. Also, it should be noted that the used arithmetic in the Eqs. (3.3)-(3.6) are the complex interval arithmetic defined in this paper.

In the following theorem, we show that the solution set Σ is a subset of the complex interval vector $[\mathbf{z_{cr}}]$.

Theorem 3.1. *If the coefficient matrix* $\mathbf{C}$ *of the system (2.1) be nonsingular, then we have*

$$\Sigma \subseteq [\boldsymbol{z_{cr}}].$$

Proof. Based on Remark 2.9, since the matrix $\mathbf{C}$ is nonsingular, we can write $\Sigma = \mathbf{C}^{-1}[\mathbf{w}]$. Now, suppose that $\mathbf{z}' \in \Sigma$. Then, there exists a $\mathbf{w}' \in [\mathbf{w}]$ such that $\mathbf{C}\,\mathbf{z}' = \mathbf{w}'$. According to the Crout decomposition, we obtain $\mathbf{LU}\,\mathbf{z}' = \mathbf{w}'$. If we set $\mathbf{y}' = \mathbf{U}\,\mathbf{z}'$, then we conclude that $\mathbf{L}\,\mathbf{y}' = \mathbf{w}'$. At first, for the lower triangular system $\mathbf{L}\,\mathbf{y}' = \mathbf{w}'$, by forward substitution we have

$$y'_1 = w'_1,$$

$$y'_k = w'_k - \sum_{j=1}^{k-1} l_{kj} y'_j, \qquad k = 2, 3, \ldots, n.$$

Since $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_n)^T \in [\mathbf{w}]$, then $\mathbf{y}' = (y'_1, y'_2, \ldots, y'_n)^T \in [\mathbf{y}]$, where $[\mathbf{y}]$ is defined in the Eqs. (3.2)-(3.4).

On the other hand, for the upper triangular system $\mathbf{y}' = \mathbf{U}\,\mathbf{z}'$, by backward substitution we have

$$z'_n = \frac{1}{u_{nn}} y'_n,$$

$$z'_k = \frac{1}{u_{kk}} \left(y'_k - \sum_{j=k+1}^{n} u_{kj} z'_j \right), \qquad k = n-1, n-2, \ldots, 1.$$

Now, since $\mathbf{y}' = (y'_1, y'_2, \ldots, y'_n)^T \in [\mathbf{y}]$, then $\mathbf{z}' = (z'_1, z'_2, \ldots, z'_n)^T \in [\mathbf{z_{cr}}]$, where $[\mathbf{z_{cr}}]$ is obtained by the Eqs. (3.5) and (3.6). Consequently $\Sigma \subseteq [\mathbf{z_{cr}}]$. □ □

Remark 3.2. From Theorems 3.3 and 2.11, we conclude that, if there exists an algebraic solution $[\mathbf{z}_A]$ for the system (2.1) and also the coefficient matrix $\mathbf{C}$ be nonsingular, then

$$[\mathbf{z}_A] \subseteq \Sigma \subseteq [\mathbf{z_{cr}}].$$

Therefore, in continuation of this paper, the complex interval vector $[\mathbf{z_{cr}}]$ is called as "General solution set".

In the following theorem, we show that the general solution set satisfies the center of the system (2.1).

Theorem 3.3. *If the coefficient matrix* $\mathbf{C}$ *be nonsingular, then*

$$\mathbf{C}\,[\mathbf{z}_{\boldsymbol{cr}}]^c = [\mathbf{w}]^c.$$

Proof. According to the process of obtaining of the general solution set $[\mathbf{z_{cr}}]$, we have

$$[y_1]^c = [w_1]^c,$$
$$[y_k]^c = [w_k]^c - \sum_{j=1}^{k-1} l_{kj}[y_j]^c, \qquad k = 2, 3, \ldots, n.$$

Since the center of a complex interval is a complex crisp number, the above equations can be written as its matrix form as follows

$$[\mathbf{y}]^c = \mathbf{L}^{-1}\,[\mathbf{w}]^c. \tag{3.7}$$

Similarly, we have

$$[z_{n_{cr}}]^c = \frac{1}{u_{nn}}[y_n]^c,$$
$$[z_{k_{cr}}]^c = \frac{1}{u_{kk}}\left([y_k]^c - \sum_{j=k+1}^{n} u_{kj}[z_{j_{cr}}]^c\right), \qquad k = n-1, n-2, \ldots, 1.$$

and therefore

$$[\mathbf{z_{cr}}]^c = \mathbf{U}^{-1}\,[\mathbf{y}]^c. \tag{3.8}$$

From Eqs. (3.7) and (3.8) we conclude

$$[\mathbf{z_{cr}}]^c = \mathbf{U}^{-1}\,\mathbf{L}^{-1}\,[\mathbf{w}]^c = \mathbf{C}^{-1}[\mathbf{w}]^c,$$

and consequently

$$\mathbf{C}[\mathbf{z_{cr}}]^c = [\mathbf{w}]^c. \qquad \square$$

$\square$

Remark 3.4. If $[w_k] = [\underline{u_k}, \overline{u_k}] + i\,[\underline{v_k}, \overline{v_k}]$, and $[z_{k_{cr}}] = [\underline{p_k}, \overline{p_k}] + i\,[\underline{q_k}, \overline{q_k}]$, for $k = 1, 2, \ldots, n$, from Theorem 3.3, we obtain

$$\sum_{j=1}^{n} a_{kj}\left(\underline{p_j} + \overline{p_j}\right) - \sum_{j=1}^{n} b_{kj}\left(\underline{q_j} + \overline{q_j}\right) = \underline{u_k} + \overline{u_k},$$
$$\sum_{j=1}^{n} a_{kj}\left(\underline{q_j} + \overline{q_j}\right) + \sum_{j=1}^{n} b_{kj}\left(\underline{p_j} + \overline{p_j}\right) = \underline{v_k} + \overline{v_k},$$

for $k = 1, 2, \ldots, n$.

In the following theorem, we show that under certain conditions, the center of the algebraic solution (if it exists), the solution set Σ and the general solution set $[\mathbf{z_{cr}}]$ are equal to each other.

Theorem 3.5. *Suppose that the complex interval linear system (2.1) has an algebraic solution and also the complex coefficient matrix* $\mathbf{C}$ *be nonsingular. Then, we have*

$$[\mathbf{z}_A]^c = \Sigma^c = [\mathbf{z}_{cr}]^c.$$

Proof. By Remark 2.9 and Theorem 3.3, it is obvious that $[\mathbf{z}_A]^c = \Sigma^c = \mathbf{C}^{-1}[\mathbf{w}]^c = [\mathbf{z_{cr}}]^c$. □

□

The following numerical example is presented to illustrate the above theorems and remarks. All numerical results are obtained using MATLAB software.

Example 3.6. Consider the following 4×4 complex interval linear system

$$\begin{cases} (1+i)[z_1] + (2-i)[z_2] + (i)[z_3] + (3+i)[z_4] = [-1,10] + i\,[7,17], \\ (-1+2i)[z_1] + (3-i)[z_2] + (1+2i)[z_3] + (-1-2i)[z_4] = [-8,7] + i\,[-3,11], \\ (i)[z_1] + (-3+i)[z_2] + (-1-2i)[z_3] + (2+i)[z_4] = [-6,7] + i\,[-6,6], \\ (-2+2i)[z_1] + (1-3i)[z_2] + (-1+i)[z_3] + (3-3i)[z_4] = [0,17] + i\,[-3,14], \end{cases}$$

with the unique algebraic solution

$$[\mathbf{z}_A] = \begin{pmatrix} [z_{1A}] \\ [z_{2A}] \\ [z_{3A}] \\ [z_{4A}] \end{pmatrix} = \begin{pmatrix} [1,2] + i\,[0,1] \\ [-1,0] + i\,[1,2] \\ [0,1] + i\,[-1,1] \\ [1,2] + i\,[1,2] \end{pmatrix}.$$

For the above system, by Remark 2.9, the solution set Σ can be obtained as follows

$$\Sigma = \mathbf{C}^{-1}[\mathbf{w}] = \begin{pmatrix} [-3.6740, 6.6740] + i\,[-4.7851, 5.7851] \\ [-7.5938, 6.5938] + i\,[-5.7062, 8.7062] \\ [-14.5243, 15.5243] + i\,[-15.0055, 15.0055] \\ [-4.0951, 7.0951] + i\,[-4.0887, 7.0887] \end{pmatrix}.$$

Also, by using the Crout complex interval decomposition method , we obtain

$$[\mathbf{z_{cr}}] = \begin{pmatrix} [z_{1_{cr}}] \\ [z_{2_{cr}}] \\ [z_{3_{cr}}] \\ [z_{4_{cr}}] \end{pmatrix} = \begin{pmatrix} [-756.0690, 759.0690] + i\,[-756.8810, 757.8810] \\ [-254.4258, 253.4258] + i\,[-252.5729, 255.5729] \\ [-176.2212, 177.2212] + i\,[-176.5332, 176.5332] \\ [-21.0535, 24.0535] + i\,[-21.0126, 24.0126] \end{pmatrix}.$$

From the above solutions, it is clear that

$$[\mathbf{z}_A] \subseteq \Sigma \subseteq [\mathbf{z_{cr}}],$$

$$[\mathbf{z}_A]^c = \Sigma^c = [\mathbf{z_{cr}}]^c = \begin{pmatrix} 1.5 + 0.5i \\ -0.5 + 1.5i \\ 0.5 \\ 1.5 + 1.5i \end{pmatrix},$$

and

$$\mathbf{C}\,[\mathbf{z}_G]^c = \mathbf{C}\,[\mathbf{z}_A]^c = \mathbf{C}\,\Sigma^c = [\mathbf{w}]^c = \begin{pmatrix} 4.5 + 12i \\ -0.5 + 4i \\ 0.5 \\ 8.5 + 5.5 \end{pmatrix}.$$

In the next section, we are going to extend the previous our work [**?**] on the Crout decomposition method and present a simple approach for obtaining the algebraic solution of a complex interval linear system.

4. Exact solving

In this section, we present an exact solving approach to obtain a complex interval vector such that it satisfies the complex linear system (2.1). In the proposed method, we firstly use the Crout complex interval decomposition method for obtaining the general solution set of the system (2.1). In the next step, we restrict the general solution set $[\mathbf{z_{cr}}] = ([z_{1_{cr}}], [z_{2_{cr}}], \ldots, [z_{n_{cr}}])^T$ by the limiting parameters. In Final, we obtain a complex interval vector as an algebraic solution. The main idea is based on Theorems 3.5. In the proposed method, we define

$$[\mathbf{z}_A] = \begin{pmatrix} [z_{1_A}] \\ [z_{2_A}] \\ \vdots \\ [z_{n_A}] \end{pmatrix} = \begin{pmatrix} [\underline{z_{1_{cr}}} + \theta_1\,,\ \overline{z_{1_{cr}}} - \theta_1] \\ [\underline{z_{2_{cr}}} + \theta_2\,,\ \overline{z_{2_{cr}}} - \theta_2] \\ \vdots \\ [\underline{z_{n_{cr}}} + \theta_n\,,\ \overline{z_{n_{cr}}} - \theta_n] \end{pmatrix}, \tag{4.1}$$

where θ_i, $i = 1, 2, \ldots, n$ are the complex crisp numbers such that satisy the following conditions

$$0 \leqslant Real(\theta_j) \leqslant \frac{1}{2} Real\left([z_{jD}]^{\Delta}\right), \quad i = 1, 2, \ldots, n, \tag{4.2}$$

$$0 \leqslant Imag(\theta_j) \leqslant \frac{1}{2} Imag\left([z_{jD}]^{\Delta}\right), \quad i = 1, 2, \ldots, n. \tag{4.3}$$

It should be noted that the above conditions guarantee that the Eq. (4.1) be a complex interval vector. When, we set $\theta_j = \alpha_j + i\beta_j$ and $[z_{j_{cr}}] = [p_j] + i[q_j]$ for $j = 1, 2, \ldots, n$, then $\underline{z_{j_{cr}}} = \underline{p_j} + i\underline{q_j}$, $\overline{z_{j_{cr}}} = \overline{p_j} + i\overline{q_j}$ and the Eqs. (4.1)-(4.3) can be rewritten as

$$[\mathbf{z}_A] = \begin{pmatrix} [\underline{p_1} + \alpha_1, \overline{p_1} - \alpha_1] + i\,[\underline{q_1} + \beta_1, \overline{q_1} - \beta_1] \\ [\underline{p_2} + \alpha_2, \overline{p_2} - \alpha_2] + i\,[\underline{q_2} + \beta_2, \overline{q_2} - \beta_2] \\ \vdots \\ [\underline{p_n} + \alpha_n, \overline{p_n} - \alpha_n] + i\,[\underline{q_n} + \beta_n, \overline{q_n} - \beta_n] \end{pmatrix}, \tag{4.4}$$

also, the conditions (4.2) and (4.3) can be replaced by

$$0 \leqslant \alpha_j \leqslant \frac{1}{2}[p_j]^{\Delta}, \quad j = 1, 2, \ldots, n, \tag{4.5}$$

$$0 \leqslant \beta_j \leqslant \frac{1}{2}[q_j]^{\Delta}, \quad j = 1, 2, \ldots, n. \tag{4.6}$$

Based on our method, we must determine the values of parameters α_j and β_j such that the complex interval vector (4.4) be an algebraic solution for the complex interval linear system (2.1). To this end, we assume that

$$\sum_{j=1}^{n} c_{kj} \left([\underline{p_j} + \alpha_j, \overline{p_j} - \alpha_j] + i\,[\underline{q_j} + \beta_j, \overline{q_j} - \beta_j] \right) = [w_k],$$

for $k = 1, 2, \ldots, n$. Supposing that $[w_k] = [\underline{u_k}, \overline{u_k}] + i\,[\underline{v_k}, \overline{v_k}]$ and $c_{kj} = a_{kj} + i\,b_{kj}$, we conclude

$$\begin{aligned}
\underline{u_k} &= \sum_{a_{kj} \geqslant 0} a_{kj} \left(\underline{p_j} + \alpha_j \right) + \sum_{a_{kj} < 0} a_{kj} \left(\overline{p_j} - \alpha_j \right) \\
&- \sum_{b_{kj} < 0} b_{kj} \left(\underline{q_j} + \beta_j \right) - \sum_{b_{kj} \geqslant 0} b_{kj} \left(\overline{q_j} - \beta_j \right), \\
\overline{u_k} &= \sum_{a_{kj} \geqslant 0} a_{kj} \left(\overline{p_j} - \alpha_j \right) + \sum_{a_{kj} < 0} a_{kj} \left(\underline{p_j} + \alpha_j \right) \\
&- \sum_{b_{kj} < 0} b_{kj} \left(\overline{q_j} - \beta_j \right) - \sum_{b_{kj} \geqslant 0} b_{kj} \left(\underline{q_j} + \beta_j \right), \\
\underline{v_k} &= \sum_{a_{kj} \geqslant 0} a_{kj} \left(\underline{q_j} + \beta_j \right) + \sum_{a_{kj} < 0} a_{kj} \left(\overline{q_j} - \beta_j \right) \\
&+ \sum_{b_{kj} \geqslant 0} b_{kj} \left(\underline{p_j} + \alpha_j \right) + \sum_{b_{kj} < 0} b_{kj} \left(\overline{p_j} - \alpha_j \right), \\
\overline{v_k} &= \sum_{a_{kj} \geqslant 0} a_{kj} \left(\overline{q_j} - \beta_j \right) + \sum_{a_{kj} < 0} a_{kj} \left(\underline{q_j} + \beta_j \right) \\
&+ \sum_{b_{kj} \geqslant 0} b_{kj} \left(\overline{p_j} - \alpha_j \right) + \sum_{b_{kj} < 0} b_{kj} \left(\underline{p_j} + \alpha_j \right).
\end{aligned}$$

From the above equations, we have

$$\begin{aligned}\overline{u_k}-\underline{u_k} &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{p_j}-\alpha_j\right)-\sum_{j=1}^{n}|a_{kj}|\left(\underline{p_j}+\alpha_j\right)\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{q_j}-\beta_j\right)-\sum_{j=1}^{n}|b_{kj}|\left(\underline{q_j}+\beta_j\right),\end{aligned}$$

$$\begin{aligned}\overline{v_k}-\underline{v_k} &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{q_j}-\beta_j\right)-\sum_{j=1}^{n}|a_{kj}|\left(\underline{q_j}+\beta_j\right)\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{p_j}-\alpha_j\right)-\sum_{j=1}^{n}|b_{kj}|\left(\underline{p_j}+\alpha_j\right).\end{aligned}$$

In other words

$$\begin{aligned}[u_k]^{\Delta} &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{p_j}-\underline{p_j}\right)-2\sum_{j=1}^{n}|a_{kj}|\alpha_j\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{q_j}-\underline{q_j}\right)-2\sum_{j=1}^{n}|b_{kj}|\beta_j,\end{aligned}$$

$$\begin{aligned}[v_k]^{\Delta} &= \sum_{j=1}^{n}|a_{kj}|\left(\overline{q_j}-\underline{q_j}\right)-2\sum_{j=1}^{n}|a_{kj}|\beta_j\\ &+ \sum_{j=1}^{n}|b_{kj}|\left(\overline{p_j}-\underline{p_j}\right)-2\sum_{j=1}^{n}|b_{kj}|\alpha_j,\end{aligned}$$

for $k=1,2,\dots,n$. Therefore, in the matrix form, we have

$$\begin{cases}[\mathbf{u}]^{\Delta}=|\mathbf{A}|\cdot[\mathbf{p}]^{\Delta}-2|\mathbf{A}|\cdot\alpha+|\mathbf{B}|\cdot[\mathbf{q}]^{\Delta}-2|\mathbf{B}|\cdot\beta,\\ [\mathbf{v}]^{\Delta}=|\mathbf{A}|\cdot[\mathbf{q}]^{\Delta}-2|\mathbf{A}|\cdot\beta+|\mathbf{B}|\cdot[\mathbf{p}]^{\Delta}-2|\mathbf{B}|\cdot\alpha,\end{cases}\tag{4.7}$$

where

$$|\mathbf{A}|=(|a_{kj}|)_{n\times n},\qquad |\mathbf{B}|=(|b_{kj}|)_{n\times n},$$
$$\alpha=(\alpha_1,\alpha_2,\dots,\alpha_n)^T,$$

and

$$\beta=(\beta_1,\beta_2,\dots,\beta_n)^T.$$

From Eq. (4.7) we conclude

$$\begin{cases}|\mathbf{A}|\,\alpha+|\mathbf{B}|\,\beta=\frac{1}{2}\left(|\mathbf{A}|\,[\mathbf{p}]^{\Delta}+|\mathbf{B}|\,[\mathbf{q}]^{\Delta}-[\mathbf{u}]^{\Delta}\right),\\ |\mathbf{B}|\,\alpha+|\mathbf{A}|\,\beta=\frac{1}{2}\left(|\mathbf{A}|\,[\mathbf{q}]^{\Delta}+|\mathbf{B}|\,[\mathbf{p}]^{\Delta}-[\mathbf{v}]^{\Delta}\right).\end{cases}\tag{4.8}$$

By solving the above $2n \times 2n$ real linear system (4.8) we estimate the values of parameters α_j and β_j. If the obtained values of α_j and β_j satisfy the conditions (4.5) and (4.6), then the Eq. (4.4) give an algebraic solution for the complex interval linear system (2.1). Otherwise, the complex interval linear system (2.1) does not have any algebraic solution, because the Eq. (4.4) does not construct a complex interval vector. On the other hand, the real linear system (4.8) has an unique solution if and only if its coefficient matrix be nonsingular.

Theorem 4.1. [7] *The coefficient matrix of the real linear system (4.8) is nonsingular if and only if the matrices* $|\mathbf{A}| + |\mathbf{B}|$ *and* $|\mathbf{A}| - |\mathbf{B}|$ *are both nonsingular.*

According to the above mentioned discussions, the proof of the following theorem is obvious.

Theorem 4.2. *If the complex interval linear system (2.1) has an algebraic solution, then its coefficient matrix is completely nonsingular.*

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5. Numerical examples

In continuation, we apply the presented method in the previous section to obtain the algebraic solution of a complex interval linear system. All numerical solutions are obtained via MATLAB software.

Example 5.1. Consider the complex interval linear system of Example 3.6. It can be easily verified that $det(\mathbf{C}) = 66.00 - 116.00\,i$, $det(|\mathbf{A}| + |\mathbf{B}|) = -16.00$ and $det(|\mathbf{A}| - |\mathbf{B}|) = 4.00$, where $\mathbf{C}$ is the coefficient matrix of the system and $\mathbf{A}$ and $\mathbf{B}$ are the real and imaginary parts of $\mathbf{C}$, respectively. Therefore, according to Definition 2.6, the matrix $\mathbf{C}$ is completely nonsingular. Now, by the proposed method, we must obtain the general solution set $[\mathbf{z_{cr}}]$ for the system of Example 3.6. According to the Crout complex interval decomposition method, we decompose the matrix $\mathbf{C}$ as follows:

$$\mathbf{C} = \mathbf{L}\,\mathbf{U},$$

where

$$\mathbf{L} = \begin{pmatrix} 1+1i & 0 & 0 & 0 \\ -1+2i & 0.5-3.5i & 0 & 0 \\ i & -4.5+0.5i & -1.56+1.08i & 0 \\ -2+2i & -1-7i & -4.64-0.48i & -13.8-2.7333i \end{pmatrix},$$

and

$$\mathbf{U} = \begin{pmatrix} 1 & 0.5-1.5i & 0.5+0.5i & 2-i \\ 0 & 1 & -0.32+0.76i & 1.92-0.56i \\ 0 & 0 & 1 & -5.4-0.8667i \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By solving the system $\mathbf{L}\,[\mathbf{y}] = [\mathbf{w}]$ by forward substitution and Eqs.(3.3) and (3.4) we have

$$[\mathbf{y}] = \begin{pmatrix} [y_1] \\ [y_2] \\ [y_3] \\ [y_4] \end{pmatrix} = \begin{pmatrix} [3.0000, 13.5000] + \; i\,[-1.5000, 9.0000] \\ [-4.2400, 10.3600] + \; i\,[-3.5000, 11.3400] \\ [-41.7213, 29.1213] - \; i\,[44.8187, 26.0187] \\ [-21.0535, 24.0535] - \; i\,[21.0126, 24.0126] \end{pmatrix}.$$

Now, by solving $\mathbf{U}\,[\mathbf{z}] = [\mathbf{y}]$ via backward substitution and Eqs. (3.5) and (3.6) we obtain the general solution set as follows:

$$[\mathbf{z_{cr}}] = \begin{pmatrix} [z_{1_{cr}}] \\ [z_{2_{cr}}] \\ [z_{3_{cr}}] \\ [z_{4_{cr}}] \end{pmatrix} = \begin{pmatrix} [-756.0690, 759.0690] + \; i\,[-756.8810, 757.8810] \\ [-254.4258, 253.4258] - \; i\,[252.5729, 255.5729] \\ [-176.2212, 177.2212] - \; i\,[176.5332, 176.5332] \\ [-21.0535, 24.0535] - \; i\,[21.0126, 24.0126] \end{pmatrix}.$$

In the next step, by solving the real linear system (4.8), we obtain the real and imaginary parts of the complex limiting factors $\theta_j = \alpha_j + i\,\beta_j$, $j = 1, 2, 3, 4$, respectively as follows:

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} 757.0690 \\ 253.4258 \\ 176.2212 \\ 22.0535 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} 756.8810 \\ 253.5729 \\ 175.5332 \\ 22.0126 \end{pmatrix}.$$

It can be easily investigated that the above obtained vectors α and β satisfy the conditions (4.5) and (4.6), respectively. Consequently, based on the proposed algorithm, by using the Eqs. (4.1) or (4.4) we obtain the unique algebraic solution of the complex interval linear system of Example 3.6 as follows

$$[\mathbf{z}_A] = \begin{pmatrix} [z_{1A}] \\ [z_{2A}] \\ [z_{3A}] \\ [z_{4A}] \end{pmatrix} = \begin{pmatrix} [1,2] + i\,[0,1] \\ [-1,0] + i\,[1,2] \\ [0,1] + i\,[-1,1] \\ [1,2] + i\,[1,2] \end{pmatrix}.$$

Example 5.2. Consider the following 5×5 complex interval linear system

$$\begin{cases} (1+i)[z_1] + (1-i)[z_2] + (2+i)[z_3] + (3-i)[z_4] + (2+i)[z_5] = [-6,17] + i[-5,13], \\ (-3-2i)[z_1] + (-1-i)[z_2] + (1+i)[z_3] + (-1+i)[z_4] + (3-i)[z_5] = [-5,16] + i[-9,10], \\ (2-i)[z_1] + (3+i)[z_2] + (2-3i)[z_3] + (4-i)[z_4] + (1+2i)[z_5] = [-13,16] + i[-4,25], \\ (2+2i)[z_1] + (3-2i)[z_2] + (2-i)[z_3] + (-2+i)[z_4] + (1-i)[z_5] = [-14,10] + i[-12,9], \\ (3-i)[z_1] + (1-3i)[z_2] + (2+2i)[z_3] + (1+i)[z_4] + (3-3i)[z_5] = [-10,18] + i[-7,21], \end{cases}$$

It can be easily computed that $det(\mathbf{C}) = 2304.00 - 690.00\,i$, $det(|\mathbf{A}| + |\mathbf{B}|) = 198$ and $det(|\mathbf{A}| - |\mathbf{B}|) = 42$. Therefore, we conclude that the matrix $\mathbf{C}$ is completely nonsingular. Now, based on our method, we decompose the matrix $\mathbf{C}$ into

$$\mathbf{C} = \mathbf{L}\,\mathbf{U},$$

where

$$\mathbf{L} = \begin{pmatrix} 1+i & 0 & 0 & 0 & 0 \\ -3-2i & 1-4i & 0 & 0 & 0 \\ 2-1i & 4+3i & 5.3529-6.5882i & 0 & 0 \\ 2+2i & 1 & -1.7941-4.6765i & -7.9743+4.7514i & 0 \\ 3-i & 2 & -1.5882+1.6471i & -1.0914+4.5029i & 0.5459-5.2061i \end{pmatrix},$$

and

$$\mathbf{U} = \begin{pmatrix} 1 & -i & 11.5-0.5i & 1-2i & 1.5-0.5i \\ 0 & 1 & -0.2059+1.6765i & 1.0588+1.2353i & 0.3824+2.0294i \\ 0 & 0 & 1 & 0.6343+0.0114i & 0.6629-0.0743i \\ 0 & 0 & 0 & 1 & 0.0571+0.2927i \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

By solving the system $\mathbf{L}\,[\mathbf{y}] = [\mathbf{w}]$ by forward substitution and also Eqs. (3.3) and (3.4), we obtain

$$[\mathbf{y}] = \begin{pmatrix} [y_1] \\ [y_2] \\ [y_3] \\ [y_4] \\ [y_5] \end{pmatrix} = \begin{pmatrix} [-5.5000, 15.0000] + i\,[-11.0000, 9.5000] \\ [-18.5000, 17.3529] + i\,[-12.6471, 23.5588] \\ [-27.2429, 29.5514] + i\,[-27.5998, 29.1884], \\ [-36.6842, 38.3916] + i\,[-35.9602, 39.0173] \\ [-82.4912, 82.4912] + i\,[-81.9026, 82.9026] \end{pmatrix}.$$

Now, by solving $\mathbf{U}\,[\mathbf{z}] = [\mathbf{y}]$ via backward substitution and Eqs. (3.5) and (3.6), we can find the general solution set $[\mathbf{z_{cr}}]$ as follows

$$[\mathbf{z_{cr}}] = \begin{pmatrix} [z_{1_{cr}}] \\ [z_{2_{cr}}] \\ [z_{3_{cr}}] \\ [z_{4_{cr}}] \\ [z_{5_{cr}}] \end{pmatrix} = \begin{pmatrix} [-1257.5676, 1254.5676] + \ i[-1255.0001, 1256.0001] \\ [-616.8486, 617.8486] + \ i[-616.2830, 619.2830] \\ -131.5498, 132.5498] + \ i[-132.4770, 131.4770] \\ [-65.3633, 67.3633] + \ i[-64.8350, 67.8350] \\ [-82.4912, 82.4912] + \ i[-81.9026, 82.9026] \end{pmatrix}.$$

In the next step, by solving the real linear system (4.8), we obtain the real parts α_j and the imaginary parts β_j of the complex limiting factors $\theta_j = \alpha_j + i\,\beta_j$, $j = 1, 2, \ldots, n$, as follows

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{pmatrix} = \begin{pmatrix} 1255.5676 \\ 0616.8486 \\ 0130.5498 \\ 0065.3633 \\ 0081.4912 \end{pmatrix},$$

and

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} = \begin{pmatrix} 1255.0001 \\ 617.2830 \\ 131.4770 \\ 65.8350 \\ 81.9026 \end{pmatrix}.$$

It can be easily investigated that the above obtained vectors α and β satisfy the conditions (4.5) and (4.6), respectively. Consequently, based on the proposed algorithm, by using the Eqs. (4.1) or (4.4), we obtain the unique algebraic solution as follows

$$[\mathbf{z}_A] = \begin{pmatrix} [z_{1A}] \\ [z_{2A}] \\ [z_{3A}] \\ [z_{4A}] \\ [z_{5A}] \end{pmatrix} = \begin{pmatrix} [-2,-1] + i\,[0,1] \\ [0,1] + i\,[1,2] \\ [-1,2] + i\,[-1,0] \\ [0,2] + i\,[1,2] \\ [-1,1] + i\,[0,1] \end{pmatrix}.$$

6. Conclusion

In this paper, we have introduced a new approach for exact solving a complex interval linear system. In the proposed method, we restricted the general solution set by some complex parameters. It is proved that the obtained solution vector is an algebraic solution for the system. For future work, we can extend our idea for other numerical or analytical methods.

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