Controlled $\ast$-G-frames and their $\ast$-G-multipliers in Hilbert $C^*$-modules

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Abstract. In this paper we introduce controlled $\ast$-g-frame and $\ast$-g-multipliers in Hilbert $C^*$-modules and investigate their properties. We demonstrate that any controlled $\ast$-g-frame is equivalent to a $\ast$-g-frame and define multipliers for $(C, C')$-controlled $\ast$-g-frames.

Keywords: $\ast$-g-frame, $\ast$-g-multiplier, controlled $\ast$-g-frame, controlled $\ast$-g-Bessel sequence, $(C, C')$-controlled $\ast$-g-frame, $(C, C')$-controlled $\ast$-g-multiplier operator.

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1. Introduction

Frank and Larson [14] generalized the definition of frames in Hilbert spaces to Hilbert $C^*$-modules and then Khosravi and Khosravi [17] proposed a definition of g-frames in Hilbert $C^*$-modules. We note that due to the complexity of the $C^*$-algebras involved in the Hilbert $C^*$-modules and fact that some useful techniques available in Hilbert spaces are either absent or unknown in Hilbert $C^*$-modules, the generalizations of

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frame theory from Hilbert spaces to Hilbert $C^*$-modules are not trivial. The properties of frames and g-frames in Hilbert $C^*$-modules were further studied in [2, 16].

Controlled frames improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [4]; they have been also used earlier as a tool for spherical wavelets [5]. Gabor multipliers [10, 13], Gabor filters [19] and other applications of frames led Peter Balazs to introduce Bessel and frame multipliers for abstract Hilbert spaces $H_1$ and $H_2$. A. Rahimi and A. Freydooni [21] defined the concept of controlled g-frames and showed that any controlled g-frame is equivalent to a g-frame. In this paper we generalize the concept of controlled frames and Bessel sequences defined [3, 4, 21, 22, 23], to $\ast$-g-frames and $\ast$-g-Bessel sequences in Hilbert $C^*$-modules and extend the concepts of multipliers from g-frames to $\ast$-g-Bessel sequences and $\ast$-g-frames. Moreover we show that a $C^2$-controlled $\ast$-g-frame is equivalent to a $\ast$-g-frame. Finally, we define the multiplier for $C^2$-controlled $\ast$-g-frames in Hilbert $C^*$-modules.

2. Preliminaries

In the following we briefly recall some definitions and basic properties of Hilbert $C^*$-modules.

Throughout this paper $J$ is a finite or countably index set and $\mathcal{A}$ is a unital $C^*$-algebra with identity $1\mathcal{A}$, and $|a|^2 = a^*a$ for any $a \in \mathcal{A}$. The spectrum $sp(a)$ of $a \in \mathcal{A}$ is the set $\{\lambda \in \mathbb{C} : \lambda 1\mathcal{A} - a \text{ is not invertible}\}$. An element $a$ of $\mathcal{A}$ is positive if $a$ is Hermitian and $\sigma(a) \subseteq R^+$. We write $a \geq 0$ to mean that $a$ is positive, and denote by $\mathcal{A}^+$ the set of positive elements of $\mathcal{A}$.

**Definition 2.1.** [18] Let $H$ be a left $\mathcal{A}$-module such that the linear structures of $\mathcal{A}$ and $H$ are compatible, $H$ is called a pre-Hilbert $\mathcal{A}$-module if $H$ is equipped with an $\mathcal{A}$-valued inner product, $(\cdot, \cdot) : H \times H \rightarrow \mathcal{A}$ such that:

1. $(f, f) \geq 0$ for all $f \in H$ and $(f, f) = 0$ if and only if $f = 0$;
2. $(f, g) = (g, f)^*$ for all $f, g \in H$;
3. $(af + g, h) = a(f, h) + (g, h)$ for all $a \in \mathcal{A}$ and $f, g, h \in H$.

For every $f \in H$, we define $\|f\|^2 = \|(f, f)\|$ and $|f|^2 = (f, f)$. If $H$ is complete with respect to the norm, it is called a Hilbert $\mathcal{A}$-module or a Hilbert $C^*$-module over $\mathcal{A}$.

From now on, we assume that $H$ and $K$ are finitely or countably generated Hilbert $\mathcal{A}$-modules and $\{H_j\}_{j \in J}$ is a sequence of closed Hilbert submodules of $H$. For each $j \in J$, $\text{End}_\mathcal{A}(H, H_j)$ is the collection of all
adjointable $\mathcal{A}$-linear maps from $H$ to $H_j$. Let $gl(H)$ be the set of all bounded operators with a bounded inverse and $gl^+(H)$ be the set of positive operators in $gl(H)$.

We also write

$$
\bigoplus_{j \in J} H_j = \{ g = \{ g_j \}_{j \in J} : g_j \in H_j \text{ and } \sum_{j \in J} \langle g_j, g_j \rangle \text{ is norm convergent in } \mathcal{A} \}.
$$

For any $f = \{ f_j \}_{j \in J}$ and $g = \{ g_j \}_{j \in J}$, if the $\mathcal{A}$-valued inner product is defined by $\langle f, g \rangle = \sum_{j \in J} \langle f_j, g_j \rangle$ and the norm is defined by $\| f \|^2 = \| \langle f, f \rangle \|$, then $\bigoplus_{j \in J} H_j$ is a Hilbert $\mathcal{A}$-module (see [18]).

A bounded operator $T : H \to H$ is called positive, if $\langle Tf, f \rangle \geq 0$ for all $f \in H$. The nonzero element $a$ is called strictly nonzero if zero does not belong to $\sigma(a)$, and $a$ is said to be strictly positive if it is strictly nonzero and positive. The relation “$\leq$” given by:

$$
a \leq b \text{ if and only if } b - a \text{ is positive;}
$$

define a partial ordering on $\mathcal{A}$. Some elementary facts about “$\leq$” are given in the following statements for $a, b, c \in \mathcal{A}$;

1. $a \leq \| a \|$
2. $0 \leq a \leq b$ implies $\| a \| \leq \| b \|$, $ab \geq 0$, $a + b \geq 0$, and $a^t \leq b^t$ for $t \in (0, 1)$;
3. if $a \leq b$, then $cac^* \leq cbc^*$. Moreover, if $c$ commutes with $a$ and $b$, then $ca \leq cb$ for $c \geq 0$;
4. If $a$ and $b$ are positive invertible elements and $a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$.

2.1. Some equivalencies of $^*$-g-frames in Hilbert $C^*$-modules.

In this section, we will study equivalencies of $^*$-g-frames in Hilbert $C^*$-modules from several aspects.

Definition 2.2. A sequence $\Lambda = \{ \Lambda_j \in \operatorname{End}^*_\mathcal{A}(H, H_j) : j \in J \}$ is called a generalized $^*$-frame, or simply, a $^*$-g-frame, for $H$ with respect to $\{ H_j : j \in J \}$ if there exist two strictly nonzero elements $A$ and $B$ in $\mathcal{A}$ such that

$$
A \langle f, f \rangle A^* \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq B \langle f, f \rangle B^*, \quad (\forall f \in H).
$$

The elements $A$ and $B$ are called the lower and upper $^*$-g-frame bounds, respectively. If $\lambda = A = B$ then the $^*$-g-frame $\{ \Lambda_j \}_{j \in J}$ is said to be a $\lambda$-tight $^*$-g-frame. In the special case $A = B = 1_\mathcal{A}$, it is called a Parseval $^*$-g-frame or normalized $^*$-g-frame.

If $\{ \Lambda_j \}_{j \in J}$ possesses an upper $^*$-g-frame bound, but not necessarily a lower $^*$-g-frame bound, we called it a $^*$-g-Bessel sequence for $H$ with $^*$-g-Bessel bound $B$. 

The bounded linear operator \( T_\Lambda \) defined by:

\[
T_\Lambda : \bigoplus_{j \in J} H_j \to H, \quad T_\Lambda(\{g_j\}_{j \in J}) = \sum_{j \in J} \Lambda_j^* g_j, \tag{2.2}
\]
is called the pre-\( \ast \)-g-frame operator of \( \{\Lambda_j\}_{j \in J} \). Also, the linear operator \( S_\Lambda \) defined by:

\[
S_\Lambda : H \to H, \quad S_\Lambda(f) = \sum_{j \in J} \Lambda_j^* \Lambda_j f,
\]
is called \( \ast \)-g-frame operator of \( \{\Lambda_j\}_{j \in J} \).

We mentioned that the set of all \( \ast \)-frames in Hilbert \( \mathcal{A} \)-modules can be considered as a subset of the family of \( \ast \)-g-frames. To illustrate this, let \( \{\Lambda_j \in \text{End}_\mathcal{A}(H, H_j) : j \in J\} \) be a \( \ast \)-g-frame for the Hilbert \( \mathcal{A} \)-module \( H \) with respect to \( \{H_j : j \in J\} \) with real bounds \( A \) and \( B \). Note that for \( f \in H \),

\[
(\sqrt{A})_A \langle f, f \rangle (\sqrt{A})_A \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq (\sqrt{B})_A \langle f, f \rangle (\sqrt{B})_A.
\]

Therefore, every \( \ast \)-g-frame for \( H \) with real bounds \( A \) and \( B \) is a \( \ast \)-g-frame for \( H \) with \( \mathcal{A} \)-valued \( \ast \)-g-frame bounds \((\sqrt{A})_A\) and \((\sqrt{B})_A\).

**Example 2.3 ([1])**. Let \( \mathcal{A} = \ell^\infty \) and let \( H = C_0 \), the Hilbert \( \mathcal{A} \)-module of the set of all null sequences equipped with the \( \mathcal{A} \)-inner product

\[
\langle (x_i)_{i \in N}, (y_i)_{i \in N} \rangle = (x_i \overline{y_i})_{i \in N}.
\]

The action of each sequence \((a_i)_{i \in N} \in \mathcal{A}\) on a sequence \((x_i)_{i \in N} \in H\) is implemented as \((a_i)_{i \in N}(x_i)_{i \in N} = (a_i x_i)_{i \in N}\). Let \( j \in J = N \) and \((1 + \frac{1}{\ell})_{i \in N} \in \ell^\infty \). Define \( \Lambda_j \in \text{End}_\mathcal{A}(H) \) by

\[
\Lambda_j(x_i)_{i \in N} = (\delta_{ij} a_j x_j)_{i \in N}, \quad \forall (x_i)_{i \in N} \in H.
\]

We observe that

\[
\sum_{j \in N} \langle \Lambda_j x, \Lambda_j x \rangle = ((1 + \frac{1}{\ell})^2 x_i \overline{x_i})_{i \in N} = (1 + \frac{1}{\ell})_{i \in N} \langle x, x \rangle (1 + \frac{1}{\ell})_{i \in N},
\]

for all \( x = (x_i)_{i \in N} \in H \).

Thus \( \{\Lambda_j\}_{j \in J} \) is a \( \ast \)-g-frame with bounds \((1 + \frac{1}{\ell})_{i \in N}\).

**Lemma 2.4 ([2])**. Let \( T \in \text{End}_\mathcal{A}(H) \) and \( T = T^* \). Then the following assertions are true.

1. If \( T \) is injective and has a closed range, then \( T^* T \) is an invertible, self-adjoint operator satisfying,

\[
\| (T^* T)^{-1} \|^{-1} \leq T^* T \leq \| T \|^2; \tag{2.3}
\]
(2) If $T$ is surjective, then $T^*T$ is an invertible, self-adjoint operator satisfying,
\[
\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.
\] (2.4)

**Theorem 2.5.** Let $\mathcal{A}$ be a unital $C^*$-algebra, $T \in \text{End}_\mathcal{A}^*(H)$ and $T = T^*$. Then the following are equivalent:

1. $T$ is surjective;
2. $T^*$ is bounded with respect to norm, i.e. $\exists \ m \in \mathcal{A}^+$ such that $\|T^*x\| \geq \|m\|\|x\|$;
3. $T^*$ is bounded with respect to inner product i.e. $\exists \ m' \in \mathcal{A}^+$ such that $\langle T^*x, T^*x \rangle \geq \langle m'(x, x)(m')^* \rangle$.

**Proof.** $(1) \implies (3)$ Let $T$ be surjective, by Lemma 2.4, $T^*T$ is an invertible and positive operator and
\[
\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.
\] Write,
\[
\|(TT^*)^{-1}\|^{-1}1_{\mathcal{A}} = m'(m'^*)^*.
\]
Then by Lemma 4.1 [18], $TT^* - m'(m'^*)^* \geq 0$. This is equivalent to
\[
\langle (TT^* - m'(m'^*)^*)x, x \rangle \geq 0.
\] (2.5) for all $x \in H$, i.e. $\langle T^*x, T^*x \rangle \geq \langle m'(x, x)(m'^*)^* \rangle$ for all $x \in H$.

The implication $(3) \implies (2)$ is trivial.

$(2) \implies (1)$ Suppose that $T^*$ is bounded below with respect to the norm then $T^*$ is clearly injective. Since $T = T^*$ therefore $T$ is injective, and $\text{Ker}T = \{0\}$. We now show $\text{Img}T$ is closed. Let $\{v_n\} \subseteq H$ be a sequence in $\text{Img}T$ such that $v_n \to u$ as $n \to \infty$.

Then we can find $\{v_n\} \subseteq H$ such that $T(v_n) = u_n$. By (2), we have $\|(v_n - v_m)\| \leq \|T(v_n - v_m)\|$. Since $T(v_n)$ is a Cauchy sequence, $\|T(v_n - v_m)\| \to 0$ as $m, n \to \infty$. Therefore the sequence $\{v_n\}$ is a Cauchy sequence in $H$ and hence there exists $v \in H$ such that $v_n \to v$ as $n \to \infty$ implies that $u_n = T(v_n) \to Tv = u$. It concludes that $\text{Img}T$ is closed. By Theorem 3.2 of [18], $\text{Img}T^*$ is closed and
\[
\text{Im} = \text{Ker}T^* \oplus \text{Img} = \text{Img}T.
\]

\[\square\]

**Lemma 2.6** ([20]). For self-adjoint $f \in C(X)$, the following are equivalent:

1. $f \geq 0$;
2. For all $t \geq \|f\|$, we have $\|f - t\| \leq t$;
3. For at least one $t \geq \|f\|$, we have $\|f - t\| \leq t$. 

It is immediate from Lemma 2.6 that \( A^+ \) is closed in \( A \).

**Proposition 2.7** ([18]). Let \( T \in \text{End}_A(H, H_j) \), then for all \( x \in H \) we have:

\[
\langle Tx, Tx \rangle \leq \|T\|^2 \|x, x\|.  \tag{2.6}
\]

**Theorem 2.8.** Let \( \{\Lambda_j\}_{j \in J} \in \text{End}_A(H, H_j) \), and \( \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \) converge in norm \( A \). Then \( \{\Lambda_j\}_{j \in J} \) is a \( * \)-g-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \) if and only if

\[
\| A^{-1} \|^{-2} \| \langle f, f \rangle \| \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \| \leq B \| \langle f, f \rangle \|  \tag{2.7}
\]

for all \( f \in H \) and strictly nonzero elements \( A, B \in A \).

**Proof.** By the definition of \( * \)-g-frame we have \( \langle f, f \rangle \leq A^{-1} \langle S f, f \rangle (A^*)^{-1} \) and \( \langle S f, f \rangle \leq B \langle f, f \rangle B^* \). Hence

\[
\| A^{-1} \|^{-2} \| \langle f, f \rangle \| \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \| \leq B \| \langle f, f \rangle \|, \forall f \in H.  \tag{2.8}
\]

For the converse, assume that (2.7) holds. For any \( f \in H \), we define \( T f := \sum_{j \in J} \Lambda_j^* \Lambda_j f \) then

\[
\|T f\|^4 = \|\langle T f, T f \rangle\|^2 = \|\langle T f, \sum_{j \in J} \Lambda_j^* \Lambda_j f \rangle\|^2 \\
= \|\sum_{j \in J} \langle \Lambda_j T f, \Lambda_j f \rangle\|^2 \\
\leq \|\sum_{j \in J} \langle \Lambda_j f, \Lambda_j T f \rangle\|\| \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle\| \\
\leq \|B\|^2 \|T f\|^2 \|B\|^2 \|f\|^2 .
\]

Hence \( \|T f\|^2 \leq \|B\|^4 \|f\|^2 \).

It is easy to check that \( \langle T f, g \rangle = \langle f, T g \rangle \) for all \( f, g \in H \), so \( T \) is bounded and \( T = T^* \). From \( \langle T f, f \rangle = \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \geq 0 \) for all \( f \in H \), it follows that \( T \geq 0 \). Now \( \langle T^1 f, T^1 f \rangle \leq \|T^1 f\|^2 \langle f, f \rangle \). On the other hand we have, \( \|\langle T^1 f, T^1 f \rangle\| \langle f, f \rangle = \|T\| \|f, f \rangle \), therefore we get

\[
\langle T^1 f, T^1 f \rangle \leq \|T\| \|f, f \rangle \leq \|B\|^2 \|1_A\| \langle f, f \rangle .
\]

Therefore

\[
\langle T f, f \rangle = \langle T^1 f, T^1 f \rangle \leq (\|B\| \|1_A\| \langle f, f \rangle) (\|B\| \|1_A\|^*) . \tag{2.9}
\]

However \( \|\langle T f, f \rangle\| = \|\langle T^1 f, T^1 f \rangle\| = \|T^1 f\|^2 \) and by inequality (2.7), \( \|A^{-1}\|^{-2} \|\langle f, f \rangle\| \leq \|T^1 f\|^2 \). We conclude that

\[
\|A^{-1}\|^{-1} \|f\| \leq \|T^1 f\| .
\]

So by Theorem 2.5, we obtain lower bound for \( \{\Lambda_j\}_{j \in J} \). This shows that \( \{\Lambda_j\}_{j \in J} \) is \( * \)-g-frame for \( H \) with respect to \( \{H_j\}_{j \in J} \).
2.2. Multipliers of $\ast$-$g$-Bessel sequences. In the following, the concept of multipliers for $g$-Bessel sequences will be extended to $\ast$-$g$-Bessel sequences and some of their properties will be shown.

**Proposition 2.9.** Let

$$\Lambda = \{\Lambda_j \in \text{End}_A^\ast(H, H_j) : j \in J\}$$

and

$$\Theta = \{\Theta_j \in \text{End}_A^\ast(H, H_j) : j \in J\}$$

be $\ast$-$g$-Bessel sequences with bounds $B_{\Lambda}, B_{\Theta}$ and $m = \{m_j\}_{j \in J} \in \ell^\infty(\mathbb{R})$ then the operator

$$M_{m, \Lambda, \Theta} : H \rightarrow H, \quad M_{m, \Lambda, \Theta} f := \sum_{j \in J} m_j \Lambda_j^* \Theta_j f,$$

for all $f \in H$ is a well-defined bounded operator.

**Proof.** Let $\Lambda$ and $\Theta$ be $\ast$-$g$-Bessel sequences for $H$ with bounds $B_{\Lambda}, B_{\Theta}$, respectively. For any $f, g \in H$ and finite subset $I \subseteq J$,

$$\left\| \sum_{j \in I} m_j \Lambda_j^* \Theta_j f \right\|^2 = \sup_{g \in H, \|g\| = 1} \left\| \sum_{j \in I} m_j \Lambda_j^* \Theta_j f, g \right\|^2$$

$$= \sup_{g \in H, \|g\| = 1} \left\| \sum_{j \in I} m_j \Theta_j f, \Lambda_j g \right\|^2$$

$$\leq \sup_{g \in H, \|g\| = 1} \left\| \sum_{j \in I} m_j \Theta_j f, m_j \Theta_j f \right\| \sum_{j \in I} \left\| \Lambda_j g, \Lambda_j g \right\|,$$

since

$$\sum_{j \in I} \langle m_j \Theta_j f, m_j \Theta_j f \rangle = \sum_{j \in I} |m_j|^2 \langle \Theta_j f, \Theta_j f \rangle$$

$$\leq \|m\|_\infty^2 \sum_{j \in I} \langle \Theta_j f, \Theta_j f \rangle \leq \|m\|_\infty^2 B_{\Theta}(f, f) B_{\ast}^\Theta.$$

Hence

$$\left\| \sum_{j \in I} m_j \Lambda_j^* \Theta_j f \right\|^2 \leq \sup_{g \in H, \|g\| = 1} \|m\|_\infty^2 B_{\Theta} \|f\|^2 \|f\|^2 \|B_{\Lambda}\|^2 \|g\|^2$$

$$= \|m\|_\infty^2 \|B_{\Theta}\|^2 \|f\|^2 \|B_{\Lambda}\|^2.$$

This shows that $M_{m, \Lambda, \Theta}$ is well-defined and

$$\|M_{m, \Lambda, \Theta}\| \leq \|m\|_\infty \|B_{\Lambda}\| \|B_{\Theta}\|.$$  \hfill \(\square\)

Now, the map $M$ in the above proposition is called a $\ast$-$g$-multiplier of $\Lambda, \Theta$ and $m$.

**Lemma 2.10.** Let

$$\Lambda = \{\Lambda_j \in \text{End}_A^\ast(H, H_j) : j \in J\}$$
and
\[ \theta = \{ \Theta_j \in \text{End}_A^*(H, H_j) : j \in J \} \]
be \( \ast \)-\( g \)-Bessel sequences with respect to \( \{ H_j : j \in J \} \) with bounds \( B_\Lambda, B_\Theta \) respectively. Let \( m = \{ m_j \}_{j \in J} \in \ell^\infty(R) \) then the operator,
\[ M = M_{m, \Lambda, \Theta} : H \rightarrow H \]
defined by \( \langle Mf, g \rangle = \sum_{j \in J} m_j \langle \Theta_j f, \Lambda_j g \rangle \), is well-defined and \( (M_{m, \Lambda, \Theta})^* = M_{\overline{m}, \Theta, \Lambda} \).

**Proof.** By Proposition 2.9, \( M \) is well-defined. We claim that
\[ (M_{m, \Lambda, \Theta})^* = M_{\overline{m}, \Theta, \Lambda}. \]

Let \( f, g \in H \), then
\[
\langle f, (M_{m, \Lambda, \Theta})^* g \rangle = \langle (M_{m, \Lambda, \Theta} f), g \rangle \\
= \sum_{j \in J} m_j \langle \Theta_j f, \Lambda_j g \rangle \\
= \sum_{j \in J} \langle \Theta_j f, \overline{m_j} \Lambda_j g \rangle \\
= \sum_{j \in J} \langle f, \Theta_j^* \overline{m_j} \Lambda_j g \rangle \\
= \sum_{j \in J} \langle f, m_j \Theta_j^* \Lambda_j g \rangle \\
= \langle f, M_{\overline{m}, \Theta, \Lambda} \rangle. \quad \Box
\]

3. **Controlled \( \ast \)-\( g \)-frames**

Weighted and controlled frames have been introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator. In [4], it was shown that controlled frames are equivalent to standard frames. In this section, the concepts of controlled \( g \)-frames and controlled \( g \)-Bessel sequences will be extended to controlled \( \ast \)-\( g \)-frames and we will show that controlled \( \ast \)-\( g \)-frames are equivalent to \( \ast \)-frames.

**Definition 3.1.** [21] Let \( C, C' \in gl^+(H) \). The family
\[ \Lambda = \{ \Lambda_j \in \text{End}_A^*(H, H_j) : j \in J \}, \]
will be called a \( (C, C') \)-controlled \( g \)-frame for \( H \) with respect to \( \{ H_j \}_{j \in J} \), if \( \Lambda = \{ \Lambda_j \}_{j \in J} \) is a \( g \)-Bessel sequence and there exist constants \( A > 0 \) and \( B < \infty \) such that
\[
A\|f\|^2 \leq \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \leq B\|f\|^2, \quad \forall f \in H. \quad (3.1)
\]
Let $C$ be implemented as 

\[ \{ \text{the set of all null sequences equipped with the} \ A \text{module of} \ C, \ \text{part of the above inequality holds, it will be called} \ (C, C') \text{-controlled g-Bessel sequence with bound} \ B. \]

**Definition 3.2.** Let $C, C' \in gl^+(H)$. The family 

\[ \Lambda = \{ \Lambda_j \in \text{End}_A^*(H, H_j) : j \in J \} \]

will be called a $(C, C')$-controlled $*$-g-frame for $H$ with respect to \{H_j\}_{j \in J},$ if $\Lambda = \{ \Lambda_j \}_{j \in J}$ is a $*$-g-Bessel sequence and 

\[ A(f, f)A^* \geq \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \leq B(f, f)B^* \quad (3.2) \]

for all $f \in H$ and strictly nonzero elements $A, B \in \mathcal{A}$.

$A$ and $B$ will be called $(C, C')$-controlled $*$-g-frame bounds. If $C' = I$, (or, $C = C'$), we call $\Lambda = \{ \Lambda_j \}_{j \in J}$ a $C$-controlled $*$-g-frame. (respectively, $C^2$-controlled $*$-g-frame) for $H$ with bounds $A, B$. If the second part of the above inequality holds, it will be called $(C, C')$-controlled $*$-g-Bessel sequence with bound $B$.

The proof of the following lemmas is straightforward.

**Lemma 3.3.** Let $C \in gl^+(H)$. The $*$-g-Bessel sequence and 

\[ \Lambda = \{ \Lambda_j \in \text{End}_A^*(H, H_j) : j \in J \}, \]

is a $C^2$-controlled $*$-g-Bessel sequence (or, $C^2$-controlled $*$-g-frame) if and only if 

\[ \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \leq B(f, f)B^*, \quad \forall f \in H \quad (3.3) \]

(or $A(f, f)A^* \geq \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \leq B(f, f)B^*, \quad \forall f \in H$).

**Example 3.4.** Let $\mathcal{A} = \ell^\infty$ and let $H = C_0$, the Hilbert $\mathcal{A}$-module of the set of all null sequences equipped with the $\mathcal{A}$-inner product 

\[ \langle (x_i)_{i \in N}, (y_i)_{i \in N} \rangle = \langle x_i \overline{y_i} \rangle_{i \in N}. \]

The action of each sequence $(a_{i})_{i \in N} \in \mathcal{A}$ on a sequence $(x_i)_{i \in N} \in H$ is implemented as $(a_{i})_{i \in N}(x_i)_{i \in N} = (a_{i}x_i)_{i \in N}$. Let $j \in J = N$ and $(1 + \frac{1}{i})_{i \in N} \in \ell^\infty$. Define $\Lambda_j \in \text{End}_A^*(H)$ by 

\[ \Lambda_j (x_i)_{i \in N} = (\delta_{ij}a_j x_j)_{i \in N}, \quad \forall (x_i)_{i \in N} \in H. \]

Now define $Cx = 2x$ and $C'x = \frac{1}{2}x$. Then for any $x \in H$, we can estimate 

\[ \sum_{j \in N} \langle \Lambda_j C x, \Lambda_j C' x \rangle = ((1 + \frac{1}{i})^2 x_i \overline{x_i})_{i \in N} = (1 + \frac{1}{i})_{i \in N}h(x, x)(1 + \frac{1}{i})_{i \in N}, \]
for all \( x = (x_i)_{i \in N} \in H \). This shows that \( \Lambda = \{ \Lambda_j \in \text{End}_A^*(H) : j \in N \} \) is a \((C, C')\)-controlled tight \(*\)-g-frame for \( H \).

Suppose that \( \{ \Lambda_j \in \text{End}_A^*(H, H_j) : j \in J \} \) be a \((C, C')\)-controlled \(*\)-g-frame for the Hilbert \( C^*\)-module \( H \) with respect \( \{ H_j \}_{j \in J} \). The bounded linear operator \( T_{(C, C')} : \bigoplus_{j \in J} H_j \to H \) defined by:

\[
T_{(C, C')} (\{ g_j \}_{j \in J}) = \sum_{j \in J} (CC')^{1/2} \Lambda_j^* g_j \Lambda_j C, \quad \forall \{ g_j \}_{j \in J} \in \bigoplus_{j \in J} H_j
\]

is called the synthesis operator for the \((C, C')\)-controlled \(*\)-g-frame \( \{ \Lambda_j \}_{j \in J} \).

The adjoint operator \( T_{(C, C')}^* : H \to \bigoplus_{j \in J} H_j \) given by

\[
T_{(C, C')}^* (f) = \{ \Lambda_j (C'C)^{1/2} f \}_{j \in J}
\]

is called the analysis operator for the \((C, C')\)-controlled \(*\)-g-frame \( \{ \Lambda_j \}_{j \in J} \).

When \( C \) and \( C' \) commute with each other, and also commute with the operator \( \Lambda_j^* \Lambda_j \) for each \( j \), then the \((C, C')\)-controlled \(*\)-g-frame operator \( S_{(C, C')} : H \to H \) is defined as:

\[
S_{(C, C')} f = T_{(C, C')} T_{(C, C')}^* f = \sum_{j \in J} C'^* \Lambda_j^* \Lambda_j C f.
\]

For the above result one is referred to Hua and Huang [15]. From now on we assume that \( C \) and \( C' \) commute with each other, and commute with the operator \( \Lambda_j^* \Lambda_j \) for all \( j \).

**Proposition 3.5.** Let \( \{ \Lambda_j : j \in J \} \) be a \((C, C')\)-controlled \(*\)-g-frame for the Hilbert \( C^*\)-module \( H \) with respect \( \{ H_j \}_{j \in J} \). Then the \((C, C')\)-controlled \(*\)-g-frame operator \( S_{(C, C')} \) is positive, self adjoint and invertible.

**Proof.** The frame operator \( S_{(C, C')} \) for the \((C, C')\)-controlled \(*\)-g-frame is \( S_{(C, C')} f = \sum_{j \in J} C'^* \Lambda_j^* \Lambda_j C f \). As \( \{ \Lambda_j : j \in J \} \) is a \((C, C')\)-controlled \(*\)-g-frame, from the identity,

\[
\sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle = \langle \sum_{j \in J} C'^* \Lambda_j^* \Lambda_j C f \rangle = \langle S_{(C, C')} f, f \rangle,
\]
we clearly see that $S_{(C,C')} = \sum_{j \in J} C' \Lambda_j^* \Lambda_j C f, g$ is a positive operator. It is clearly bounded and linear.

$$\langle S_{(C,C')} f, g \rangle = \sum_{j \in J} \langle C' \Lambda_j^* \Lambda_j C f, g \rangle = \sum_{j \in J} \langle f, C \Lambda_j^* \Lambda_j C' g \rangle = \sum_{j \in J} \langle f, S_{(C',C)} g \rangle.$$  

Hence $S_{(C,C')}^* = S_{(C',C)}$ is positive and hence self-adjoint. Also as $C$ and $C'$ commute with each other and commute with $\Lambda_j^* \Lambda_j$, we have $S_{(C,C')} = S_{(C',C)}$. From the controlled $*$-g-frame identity we have

$$A(f,f)A^* \leq \langle S_{(C,C')} f, f \rangle \leq B \langle f, f \rangle B^*.$$  

So

$$A \text{Id}_H A^* \leq \langle S_{(C,C')} f, f \rangle \leq B \text{Id}_H B^*,$$

where $\text{Id}_H$ is the identity operator in $H$. Thus the controlled $*$-g-frame operator $S_{(C,C')}$ is invertible.

**Theorem 3.6.** Let $\{\Lambda_j\}_{j \in J} \in \text{End}_A^*(H, H_j)$, and $\sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle$ converge in norm $A$. Then $\{\Lambda_j\}_{j \in J}$ is a $(C, C')$-controlled $*$-g-frame for $H$ with respect to $\{H_j\}_{j \in J}$ if and only if

$$\text{for all } f \in H \text{ and strictly nonzero elements } A, B \in A.$$

**Proof.** By the definition of $(C, C')$-controlled $*$-g-frame we conclude that

$$(f,f) \leq A^{-1} \langle S_{(C,C')} f, f \rangle (A^*)^{-1} \text{ and } \langle S_{(C,C')} f, f \rangle \leq B \langle f, f \rangle B^*.$$  

Hence

$$\text{for all } f \in H.$$  

Conversely, suppose that

$$\text{for all } f \in H. \text{ Conversely, suppose that}$$  

$$\| A^{-1} \|^{-2} \| \langle f, f \rangle \| \leq \| \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \| \leq B \| \langle f, f \rangle \| \text{ (3.7)}$$

for all $f \in H$. Conversely, suppose that

$$\| A^{-1} \|^{-2} \| \langle f, f \rangle \| \leq \| \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \| \leq B \| \langle f, f \rangle \|, \text{ (3.9)}$$
From Proposition 3.5, the \((C, C')\)-controlled \(\ast\)-g-frame operator is positive, self adjoint and invertible. Hence
\[
\langle \frac{1}{2}((S_{(C, C')})^2)_{f, (S_{(C, C')})^2}^{1/2}) \rangle_{f, f} = \langle \sum_{j \in J} (\Lambda_j f, \Lambda_j f) \rangle_{f, f}.
\]

Using inequality (3.10) in inequality (3.9), we get
\[
\| A^{-1} \| \| f \| \leq \| (S_{(C, C')})^2 \| \leq \| B \| \| f \|.
\]

According to Theorem 2.5 and inequality (3.11), \(\{ \Lambda_j : j \in J \}\) is a \((C, C')\)-controlled \(\ast\)-g-frame for \(H\) with respect to \(\{ H_j \}_{j \in J}\).

The following theorem shows that any \(\ast\)-g-frame is a \(C^2\)-controlled \(\ast\)-g-frame and vice versa.

**Theorem 3.7.** Let \(C \in gl^+(H)\). The family \(\{ \Lambda_j \}_{j \in J} \in \text{End}_A(H, H_j)\), is a \(\ast\)-g-frame if and only if \(\Lambda = \{ \Lambda_j \}_{j \in J}\) is a \(C^2\)-controlled \(\ast\)-g-frame.

**Proof.** Let \(\Lambda\) be a \(C^2\)-controlled \(\ast\)-g-frame with bounds \(A, B\). Then
\[
\sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C f \rangle \leq B' \langle f, f \rangle B'^* \quad \forall f \in H.
\]

For \(f \in H\) we have
\[
A' \langle f, f \rangle A'^* = A(CC^{-1} f, CC^{-1} f) A'^* \leq A\| C \|^2 \langle C^{-1} f, C^{-1} f \rangle A'^*
\]
\[
\leq \| C \|^2 \sum_{j \in J} \langle \Lambda_j C^{-1} f, \Lambda_j C^{-1} f \rangle = \| C \|^2 \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle.
\]

Hence
\[
A' \| C \|^{-1} \langle f, f \rangle A'^* \| C \|^{-1} \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle.
\]

On the other hand for every \(f \in H\),
\[
\sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle = \sum_{j \in J} \langle \Lambda_j C^{-1} f, \Lambda_j C^{-1} f \rangle \leq B \langle C^{-1} f, C^{-1} f \rangle B'^*
\]
\[
\leq B \| C^{-1} \|^2 \langle f, f \rangle B'^* = B \| C^{-1} \| \langle f, f \rangle B'^* \| C^{-1} \|.
\]

These inequalities yield that \(\Lambda\) is a \(\ast\)-g-frame with bounds \(A \| C \|^{-1}, B \| C \|^{-1}\).

For the converse, assume that \(\Lambda\) is a \(\ast\)-g-frame with bounds \(A', B'\). Then for all \(f \in H\),
\[
A' \langle f, f \rangle (A')^* \leq \sum_{j \in J} \langle \Lambda_j f, \Lambda_j f \rangle \leq B' \langle f, f \rangle (B')^*.
\]
So for all \( f \in H \),
\[
\sum_{j \in J} \langle A_j C f, A_j C f \rangle \leq B'\|C\|^2 \langle f, f \rangle (B')^*.
\]

For lower bound, since \( \Lambda \) is a \(-g\)-frame for any \( f \in H \),
\[
A'\langle f, f \rangle (A')^* = A'(C^{-1}Cf, C^{-1}Cf)(A')^* \leq \|C^{-1}\|^2 \sum_{j \in J} \langle A_j C f, A_j C f \rangle.
\]

Therefore \( \Lambda \) is a \( C^2 \)-controlled \(-g\)-frame with bounds \( A'\|C^{-1}\|^{-1}, B'\|C\| \).

**Proposition 3.8.** Assume that \( \{\Lambda_j : j \in J\} \) is a \(-g\)-frame for the Hilbert \( C^*\)-module \( H \) with respect to \( \{H_j\}_{j \in J} \). Let \( S_\Lambda \) be the \(-g\)-frame operator with the \(-g\)-frame \( \{\Lambda_j : j \in J\} \). Let \( C, C' \in gl^+(H) \). Then \( \{\Lambda_j : j \in J\} \) is a \((C, C')\)-controlled \(-g\)-frame.

**Proof.** \( \{\Lambda_j : j \in J\} \) is a \(-g\)-frame for the Hilbert \( C^*\)-module \( H \) with respect to \( \{H_j\}_{j \in J} \) with bounds \( A \) and \( B \). By inequality (2.7), we have:
\[
\| A^{-1} \|^2 \| \langle f, f \rangle \| \leq \sum_{j \in J} \|\langle A_j f, A_j f \rangle\| \leq \| B \|^2 \| \langle f, f \rangle \|, \quad (3.12)
\]

Again we have
\[
\| \sum_{j \in J} \langle A_j C f, A_j C' f \rangle \| = \| \langle S_{(C, C')} f, f \rangle \|,
\]
and
\[
\| \sum_{j \in J} \langle A_j C f, A_j C' f \rangle \| = \| C \| \| C' \| \| \langle S_\Lambda f, f \rangle \|. \quad (3.13)
\]

From (3.12) and (3.13), we have
\[
\| A^{-1} \|^2 \| C \| \| C' \| \| f \|^2 \leq \sum_{j \in J} \|\langle A_j C f, A_j C' f \rangle\| \leq \| B \|^2 \| C \| \| C' \| \| f \|^2,
\]
for all \( f \in H \). So \( \{\Lambda_j : j \in J\} \) is a \((C, C')\)-controlled \(-g\)-frame with bounds \( \| A^{-1} \|^{-1}\| C \| \| C' \|, \| B \| \| C \| \| C' \| \).

**Theorem 3.9.** Suppose that \( C, C' \in gl^+(H) \), \( \{\Lambda_j : j \in J\} \subset End^+(H, H_j) \) and \( C, C' \) commute with each other and commute with \( \Lambda_j^* \Lambda_j \) for all \( j \in J \).

If the operator \( T : \bigoplus_{j \in J} H_j \to H \) given by
\[
T_{(C, C')} = \sum_{j \in J} \frac{1}{2^{\Lambda_j^* g_j} \Lambda_j^* g_j} (g_j)_{j \in J} \in \bigoplus_{j \in J} H_j \quad (3.14)
\]

is well defined and bounded operator with \( \| T_{(C, C')} \| \leq \| B \| \), then the sequence \( \{\Lambda_j : j \in J\} \) is a \((C, C')\)-controlled \(-g\)-Bessel sequence for \( H \) with respect to \( \{H_j\}_{j \in J} \) with bound \( \| B \| \).
Proof. Let \( \{ \Lambda_j : j \in J \} \) be a \((C, C')\)-controlled \(*\)-\(g\)-Bessel sequence for \( H \) with respect to \( \{ H_j \}_{j \in J} \) with bound \( B \). As a result of Theorem 3.6,

\[
\| \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \| \leq B \| f \|^2 \| \langle f, f \rangle \|. \tag{3.15}
\]

For any sequence \( \{ g_j \}_{j \in J} \in \bigoplus_{j \in J} H_j \),

\[
\| T_{(C, C')} (\{ g_j \}_{j \in J}) \|^2 = \sup_{f \in H, \| f \| = 1} \| \langle T_{(C, C')} (\{ g_j \}_{j \in J}), f \rangle \|^2
\]

\[
= \sup_{f \in H, \| f \| = 1} \| \sum_{j \in J} \langle CC' \frac{1}{2} \Lambda_j g_j, f \rangle \|^2
\]

\[
= \sup_{f \in H, \| f \| = 1} \sum_{j \in J} \| \langle CC' \frac{1}{2} \Lambda_j g_j, f \rangle \|^2
\]

\[
= \sup_{f \in H, \| f \| = 1} \sum_{j \in J} \| \langle g_j, \Lambda_j (CC')^{\frac{1}{2}} f \rangle \|^2
\]

\[
\leq \sup_{f \in H, \| f \| = 1} \| \sum_{j \in J} \langle g_j, g_j \rangle \|
\]

\[
\| \sum_{j \in J} \langle \Lambda_j (CC')^{\frac{1}{2}} f, \Lambda_j (CC')^{\frac{1}{2}} f \rangle \|
\]

\[
= \sup_{f \in H, \| f \| = 1} \| \sum_{j \in J} \langle g_j, g_j \rangle \|
\]

\[
\| \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f \rangle \|
\]

\[
\leq \sup_{f \in H, \| f \| = 1} \| \sum_{j \in J} \langle g_j, g_j \rangle \| B \| f \|^2
\]

\[
= \| B \|^2 \| \{ g_j \} \|^2
\]

Therefore, the sum \( \sum_{j \in J} (CC')^{\frac{1}{2}} \Lambda_j g_j \) is convergent and we have

\[
\| T_{(C, C')} (\{ g_j \}_{j \in J}) \|^2 \leq \| B \|^2 \| \{ g_j \} \|^2.
\]

So

\[
\| T_{(C, C')} \|^2 \leq \| B \|^2.
\]

Hence the operator \( T_{(C, C')} \) is well defined, bounded and

\[
\| T_{(C, C')} \| \leq \| B \|.
\]
4. Multipliers of Controlled $\ast$-g-frames

In this section, we define the multiplier of a controlled $\ast$-g-frame for $C$-controlled $\ast$-g-frames in Hilbert $C^*$-modules. The definition of general case $(C, C')$-controlled $\ast$-g-frames is similar.

Lemma 4.1. Let $C, C' \in gl^+(H)$ and

\[ \Lambda = \{ \Lambda_j \in End^*_A(H, H_j) : j \in J \}, \Theta = \{ \Theta_j \in End^*_A(H, H_j) : j \in J \} \]

be $C^2$ and $C^2$-controlled $\ast$-g-Bessel sequences for $H$, respectively. Let $m = \{ m_j \}_{j \in J} \in \ell^\infty(R)$. The operator

\[ m, C, \Theta, \Lambda, C' : H \to H, \]

defined by

\[ M_{m, C, \Theta, \Lambda, C'} f := \sum_{j \in J} m_j C^* \Lambda_j C' f, \]

is a well-defined bounded operator.

Proof. Let $\Lambda, \Theta$ be $C^2$ and $C^2$-controlled $\ast$-g-Bessel sequences with bounds $B, B'$, respectively. For any $f, g \in H$ and finite subset $I \subseteq J$,

\[ \sum_{j \in I} m_j \langle \langle \Lambda_j C' f, \Lambda_j C' f \rangle, \Theta_j C' f \rangle \leq \| m \|_\infty \sum_{j \in I} \langle \langle \Lambda_j C' f, \Lambda_j C' f \rangle, \Theta_j C' f \rangle \leq \| m \|_\infty B \langle \langle f, f \rangle, B' \rangle. \]

Hence

\[ \| \sum_{j \in I} m_j C^* \Lambda_j C' f \| \leq \sup_{g \in H, \| g \| = 1} \| m \|_\infty \| B \| \| f \| \| B' \| \| g \|^2 \]

This shows that $M_{m, C, \Theta, \Lambda, C'}$ is well-defined and

\[ \| M_{m, C, \Theta, \Lambda, C'} \| \leq \| m \|_\infty \| B \| \| B' \|. \]

Definition 4.2. Let $C, C' \in gl^+(H)$ and
\[ \Lambda = \{ \Lambda_j \in \text{End}^*_A(H, H_j) : j \in J \} \]

and

\[ \Theta = \{ \Theta_j \in \text{End}^*_A(H, H_j) : j \in J \} \]

be \( C^{\alpha^2} \) and \( C^2 \)-controlled \( \ast \)-g-Bessel sequences for \( H \), respectively. Let \( m = \{ m_j \}_{j \in J} \in \ell^\infty(R) \). The operator

\[ M_{m, C, \Theta, \Lambda, C'} : H \rightarrow H, \]

defined by

\[ M_{m, C, \Theta, \Lambda, C'} f := \sum_{j \in J} m_j C \Theta_j \ast \Lambda_j C' f, \]

is called the \((C, C')\)-controlled multiplier operator with symbol \( m \).

5. CONCLUSIONS

In this article, the concept of multipliers from g-frames to \( \ast \)-g-Bessel sequences and \( \ast \)-g-frames is extended. Controlled frames and controlled Bessel sequences are extended to controlled \( \ast \)-g-frames and controlled \( \ast \)-g-Bessel sequences. At the end of this paper, the concept of a multiplier for \( C^2 \)-controlled and \( C^{\alpha^2} \)-controlled \( \ast \)-g-Bessel sequences is defined.

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