Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **8**(2)(2019), 120-136

Controlled $*$-G-Frames and their $*$-G-Multipliers in Hilbert C^*-Modules

Zahra Ahmadi Moosavi [1] and Akbar Nazari [2]
[1] Department of Pure Mathematics, Faculty of Mathematics and Computer Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran.
[2] Department of Pure Mathematics, Faculty of Mathematics and Computer Shahid Bahonar University of Kerman, 76169-14111, Kerman, Iran.

Abstract. In this paper we introduce controlled $*$-g-frame and $*$-g-multipliers in Hilbert C^*-modules and investigate their properties . We demonstrate that any controlled $*$-g-frame is equivalent to a $*$-g-frame and define multipliers for (C, C')-controlled $*$-g-frames.

Keywords: $*$-g-frame, $*$-g-multiplier, controlled $*$-g-frame, controlled $*$-g-Bessel sequence, (C, C')-controlled $*$-g-frame, (C, C')-controlled $*$-g-multiplier operator.

2000 Mathematics subject classification: 42C15; Secondary 46L08

1. Introduction

Frank and Larson [14] generalized the definition of frames in Hilbert spaces to Hilbert C^*-modules and then Khosravi and Khosravi [17] proposed a definition of g-frames in Hilbert C^*-modules. We note that due to the complexity of the C^*-algebras involved in the Hilbert C^*-modules and fact that some useful techniques available in Hilbert spaces are either absent or unknown in Hilbert C^*-modules, the generalizations of

[1]Corresponding author: zmoosavi@uk.ac.ir
Received: 02 July 2019
Revised: 6 August 2019
Accepted: 15 August 2019

frame theory from Hilbert spaces to Hilbert C^*-modules are not trivial. The properties of frames and g-frames in Hilbert C^*-modules were further studied in [2, 16].

Controlled frames improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces [4]; they have been also used earlier as a tool for spherical wavelets [5]. Gabor multipliers [10, 13], Gabor filters [19] and other applications of frames led Peter Balazs to introduce Bessel and frame multipliers for abstract Hilbert spaces H_1 and H_2. A. Rahimi and A. Freydooni [21] defined the concept of controlled g-frames and showed that any controlled g-frame is equivalent to a g-frame. In this paper we generalize the concept of controlled frames and Bessel sequences defined [3, 4, 21, 22, 23], to $*$-g-frames and $*$-g-Bessel sequences in Hilbert C^*-modules and extend the concepts of multipliers from g-frames to $*$-g-Bessel sequences and $*$-g-frames. Moreover we show that a C^2-controlled $*$-g-frame is equivalent to a $*$-g-frame. Finally, we define the multiplier for C^2-controlled $*$-g-frames in Hilbert C^*-modules.

2. Preliminaries

In the following we briefly recall some definitions and basic properties of Hilbert C^*-modules.

Throughout this paper J is a finite or countably index set and $\mathcal{A}$ is a unital C^*-algebra with identity $1_{\mathcal{A}}$, and $|a|^2 = a^*a$ for any a $\in \mathcal{A}$. The spectrum $sp(a)$ of $a \in \mathcal{A}$ is the set $\{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - a \quad \text{is not invertible}\}$. An element a of $\mathcal{A}$ is positive if a is Hermitian and $\sigma(a) \subseteq R^+$. We write $a \geq 0$ to mean that a is positive, and denote by $\mathcal{A}^+$ the set of positive elements of $\mathcal{A}$.

Definition 2.1. [18] Let H be a left $\mathcal{A}$-module such that the linear structures of $\mathcal{A}$ and H are compatible, H is called a pre-Hilbert $\mathcal{A}$-module if H is equipped with an $\mathcal{A}$-valued inner product,
$\langle \cdot, \cdot \rangle : H \times H \longrightarrow \mathcal{A}$ such that:

(1) $\langle f, f \rangle \geq 0$ for all $f \in H$ and $\langle f, f \rangle = 0$ if and only if $f = 0$;
(2) $\langle f, g \rangle = \langle g, f \rangle^*$ for all $f, g \in H$;
(3) $\langle af + g, h \rangle = a\langle f, h \rangle + \langle g, h \rangle$ for all $a \in \mathcal{A}$ and $f, g, h \in H$.

For every $f \in H$, we define $\|f\|^2 = \|\langle f, f \rangle\|$ and $|f|^2 = \langle f, f \rangle$. If H is complete with respect to the norm, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-module over $\mathcal{A}$.

From now on, we assume that H and K are finitely or countably generated Hilbert $\mathcal{A}$-modules and $\{H_j\}_{j \in J}$ is a sequence of closed Hilbert submodules of H, For each $j \in J$, $\mathrm{End}^*_{\mathcal{A}}(H, H_j)$ is the collection of all adjointable $\mathcal{A}$-linear maps from H to H_j. Let $gl(H)$ be the set of all bounded operators with a bounded inverse and $gl^+(H)$ be the set of positive operators in $gl(H)$.

We also write

$$\bigoplus_{j\in J} H_j = \{g = \{g_j\}_{j\in J} : g_j \in H_j \text{ and } \sum_{j\in J}\langle g_j, g_j\rangle \quad \text{is norm convegent in } \mathcal{A}\}.$$

For any $f = \{f_j\}_{j\in J}$ and $g = \{g_j\}_{j\in J}$, if the $\mathcal{A}$-valued inner product is defined by $\langle f, g\rangle = \sum_{j\in J}\langle f_j, g_j\rangle$ and the norm is defined by
$\|f\|^2 = \|\langle f, f\rangle\|$, then $\bigoplus_{j\in J} H_j$ is a Hilbert $\mathcal{A}$-module (see [18]).

A bounded operator $T : H \longrightarrow H$ is called positive, if $\langle Tf, f\rangle \geq 0$ for all $f \in H$. The nonzero element a is called strictly nonzero if zero does not belong to $\sigma(a)$, and a is said to be strictly positive if it is strictly nonzero and positive. The relation " $\leq$ " given by:

$$a \leq b \quad \text{if and only if} \quad b - a \quad \text{is positive;}$$

define a partial ordering on $\mathcal{A}$. Some elementary facts about " $\leq$ " are given in the following statements for $a, b, c \in \mathcal{A}$;

(1) $a \leq \|a\|$;
(2) $0 \leq a \leq b$ implies $\|a\| \leq \|b\|$, $ab \geq 0$, $a + b \geq 0$, and $a^t \leq b^t$ for $t \in (0, 1)$;
(3) if $a \leq b$, then $cac^* \leq cbc^*$. Moreover, if c commutes with a and b, then $ca \leq cb$ for $c \geq 0$;
(4) If a and b are positive invertible elements and $a \leq b$, then $0 \leq b^{-1} \leq a^{-1}$.

2.1. **Some equivalencies of $*$-g-frames in Hilbert C^*-modules.** In this section, we will study equivalencies of $*$-g-frames in Hilbert C^*-modules from several aspects.

Definition 2.2. A sequence $\Lambda = \{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\}$ is called a generalized $*$-frame, or simply, a $*$-g-frame, for H with respect to $\{H_j : j \in J\}$ if there exist two strictly nonzero elements A and B in $\mathcal{A}$ such that

$$A\langle f, f\rangle A^* \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq B\langle f, f\rangle B^*, \qquad (\forall f \in H). \tag{2.1}$$

The elements A and B are called the lower and upper $*$-g-frame bounds, respectively. If $\lambda = A = B$ then the $*$-g-frame $\{\Lambda_j\}_{j\in J}$ is said to be a λ-tight $*$-g-frame. In the special case $A = B = 1_{\mathcal{A}}$, it is called a Parseval $*$-g-frame or normalized $*$-g-frame.
If $\{\Lambda_j\}_{j\in J}$ possesses an upper $*$-g-frame bound, but not necessarily a lower $*$-g-frame bound, we called it a $*$-g-Bessel sequence for H with $*$-g-Bessel bound B.

The bounded linear operator T_Λ defined by:

$$T_\Lambda : \bigoplus_{j\in J} H_j \to H, \qquad T_\Lambda(\{g_j\}_{j\in J}) = \sum_{j\in J} \Lambda_j^* g_j, \tag{2.2}$$

is called the pre-∗-g-frame operator of $\{\Lambda_j\}_{j\in J}$. Also, the linear operator S_Λ defined by:

$$S_\Lambda : H \to H, \qquad S_\Lambda(f) = \sum_{j\in J} \Lambda_j^* \Lambda_j f,$$

is called ∗-g-frame operator of $\{\Lambda_j\}_{j\in J}$.

We mentioned that the set of all of g-frames in Hilbert $\mathcal{A}$-modules can be considered as a subset of the family of ∗-g-frames. To illustrate this, let $\{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\}$ be a g-frame for the Hilbert $\mathcal{A}$-module H with respect to $\{H_j : j \in J\}$with real bounds A and B. Note that for $f \in H$,

$$(\sqrt{A})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{A})1_{\mathcal{A}} \le \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle \le (\sqrt{B})1_{\mathcal{A}}\langle f, f\rangle(\sqrt{B})1_{\mathcal{A}}.$$

Therefore, every g-frame for H with real bounds A and B is a ∗-g-frame for H with $\mathcal{A}$-valued ∗-g-frame bounds $(\sqrt{A})1_{\mathcal{A}}$ and $(\sqrt{B})1_{\mathcal{A}}$.

Example 2.3 ([1])**.** Let $\mathcal{A} = \ell^\infty$ and let $H = C_0$, the Hilbert $\mathcal{A}$-module of the set of all null sequences equipped with the $\mathcal{A}$-inner product

$$\langle (x_i)_{i\in N}, (y_i)_{i\in N}\rangle = (x_i\overline{y_i})_{i\in N}.$$

The action of each sequence $(a_i)_{i\in N} \in \mathcal{A}$ on a sequence $(x_i)_{i\in N} \in H$ is implemented as $(a_i)_{i\in N}(x_i)_{i\in N} = (a_i x_i)_{i\in N}$. Let $j \in J = N$ and $(1 + \frac{1}{i})_{i\in N} \in \ell^\infty$. Define $\Lambda_j \in End^*_{\mathcal{A}}(H)$ by

$$\Lambda_j (x_i)_{i\in N} = (\delta_{ij} a_j x_j)_{i\in N}, \qquad \forall (x_i)_{i\in N} \in H.$$

We observe that

$$\sum_{j\in N} \langle \Lambda_j x, \Lambda_j x\rangle = ((1 + \frac{1}{i})^2 x_i\overline{x_i})_{i\in N} = (1 + \frac{1}{i})_{i\in N}\langle x, x\rangle(1 + \frac{1}{i})_{i\in N},$$

for all $x = (x_i)_{i\in N} \in H$.

Thus$\{\Lambda_j\}_{j\in J}$ is a ∗-g-frame with bounds $(1 + \frac{1}{i})_{i\in N}$.

Lemma 2.4 ([2])**.** *Let $T \in End^*_{\mathcal{A}}(H)$ and $T = T^*$. Then the following assertions are true.*

(1) *If T is injective and has a closed range, then T^*T is an invertible, self-adjoint operator satisfying,*

$$\|(T^*T)^{-1}\|^{-1} \le T^*T \le \|T\|^2; \tag{2.3}$$

(2) *If T is surjective, then T^*T is an invertible, self-adjoint operator satisfying,*

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2. \tag{2.4}$$

Theorem 2.5. *Let $\mathcal{A}$ be a unital C^*-algebra, $T \in End^*_{\mathcal{A}}(H)$ and $T = T^*$. Then the following are equivalent:*

(1) *T is surjective;*
(2) *T^* is bounded with respect to norm, i.e $\exists\ m \in \mathcal{A}^+$ such that $\|T^*x\| \geq \|m\|\|x\|$;*
(3) *T^* is bounded with respect to inner product i.e $\exists\ m' \in \mathcal{A}^+$ such that $\langle T^*x, T^*x\rangle \geq (m')\langle x, x\rangle(m')^*$.*

Proof. (1)$\Longrightarrow$(3) Let T be surjective, by Lemma 2.4, T^*T is an invertible and positive operator and

$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Write,

$$\|(TT^*)^{-1}\|^{-1}1_{\mathcal{A}} = m'(m')^*.$$

Then by Lemma 4.1 [18], $TT^* - m'(m')^* \geq 0$. This is equivalent to

$$\langle (TT^* - m'(m')^*)x, x\rangle \geq 0. \tag{2.5}$$

for all $x \in H$, i.e $\langle T^*x, T^*x\rangle \geq (m')\langle x, x\rangle(m')^*$ for all $x \in H$. The implication (3)$\Longrightarrow$ (2) is trivial.

(2)$\Longrightarrow$(1) Suppose that T^* is bounded below with respect to the norm then T^* is clearly injective. Since $T = T^*$ therefore T is injective, and Ker$T = \{0\}$. We now show Img$T$ is closed. Let $\{u_n\} \subseteq H$ be a sequence in Img$T$ such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$.

Then we can find $\{v_n\} \subseteq H$ such that $T(v_n) = u_n$. By (2), we have $\|(v_n - v_m)\|\|m\| \leq \|T(v_n - v_m)\|$. Since $T(v_n)$ is a Cauchy sequence, $\|T(v_n - v_m)\| \longrightarrow 0$ as $m, n \longrightarrow \infty$. Therefore the sequence $\{v_n\}$ is a Cauchy sequence in H and hence there exists $v \in H$ such that $v_n \longrightarrow v$ as $n \longrightarrow \infty$ implies that $u_n = T(v_n) \longrightarrow Tv = u$. It concludes that Img$T$ is closed. By Theorem 3.2 of [18], ImgT^* is closed and

$$H = \mathrm{Ker}T^* \bigoplus \mathrm{Img} = \mathrm{Img}T.$$

□

Lemma 2.6 ([20])**.** *For self-adjoint $f \in C(X)$, the following are equivalent:*

(1) *$f \geq 0$;*
(2) *For all $t \geq \|f\|$, we have $\|f - t\| \leq t$;*
(3) *For at least one $t \geq \|f\|$, we have $\|f - t\| \leq t$.*

It is immediate from Lemma 2.6 that $\mathcal{A}^+$ is closed in $\mathcal{A}$.

Proposition 2.7 ([18])**.** *Let $T \in End^*_{\mathcal{A}}(H, H_j)$, then for all $x \in H$ we have:*

$$\langle Tx, Tx\rangle \leq \|T\|^2 \langle x, x\rangle. \tag{2.6}$$

Theorem 2.8. *Let $\{\Lambda_j\}_{j\in J} \in End^*_{\mathcal{A}}(H, H_j)$, and $\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle$ converge in norm $\mathcal{A}$. Then $\{\Lambda_j\}_{j\in J}$ is a $*$-g-frame for H with respect to $\{H_j\}_{j\in J}$ if and only if*

$$\| A^{-1} \|^{-2} \| \langle f, f\rangle\| \leq \| \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\| \leq \| B \|^2 \| \langle f, f\rangle\| \tag{2.7}$$

for all $f \in H$ and strictly nonzero elements $A, B \in \mathcal{A}$.

Proof. By the definition of $*$-g-frame we have $\langle f, f\rangle \leq A^{-1}\langle Sf, f\rangle (A^*)^{-1}$ and $\langle Sf, f\rangle \leq B\langle f, f\rangle B^*$. Hence

$$\| A^{-1} \|^{-2} \| \langle f, f\rangle\| \leq \| \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\| \leq \| B \|^2 \| \langle f, f\rangle\|, \forall f \in H. \tag{2.8}$$

For the converse, assume that (2.7) holds. For any $f \in H$, we define $Tf := \sum_{j\in J} \Lambda_j^* \Lambda_j f$ then

$$\begin{aligned}
\|Tf\|^4 &= \|\langle Tf, Tf\rangle\|^2 = \|\langle Tf, \sum_{j\in J}\Lambda_j^*\Lambda_j f\rangle\|^2 \\
&= \| \sum_{J\in j}\langle \Lambda_j Tf, \Lambda_j f\rangle\|^2 \\
&\leq \| \sum_{j\in J}\langle \Lambda_j Tf, \Lambda_j Tf\rangle\| \| \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\| \\
&\leq \|B\|^2\|Tf\|^2\|B\|^2\|f\|^2.
\end{aligned}$$

Hence $\|Tf\|^2 \leq \|B\|^4\|f\|^2$.
It is easy to check that $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in H$, so T is bounded and $T = T^*$. From $\langle Tf, f\rangle = \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \geq 0$ for all $f \in H$, it follows that $T \geq 0$. Now $\langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} f\rangle \leq \|T^{\frac{1}{2}}\|^2\langle f, f\rangle$. On the other hand we have, $\|(T^{\frac{1}{2}})^*(T^{\frac{1}{2}})\|\langle f, f\rangle = \|T\|\langle f, f\rangle$, therefore we get $\langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} f\rangle \leq \|T\|\langle f, f\rangle \leq \|B\|^2 1_{\mathcal{A}}\langle f, f\rangle$. Therefore

$$\langle Tf, f\rangle = \langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} f\rangle \leq (\|B\|1_{\mathcal{A}})\langle f, f\rangle(\|B\|1_{\mathcal{A}})^*. \tag{2.9}$$

However $\|\langle Tf, f\rangle\| = \|\langle T^{\frac{1}{2}} f, T^{\frac{1}{2}}\rangle\| = \|T^{\frac{1}{2}} f\|^2$ and by inequality (2.7), $\|A^{-1}\|^{-2}\|\langle f, f\rangle\| \leq \|T^{\frac{1}{2}}\|^2$. We conclude that

$$\|A^{-1}\|^{-1}\|f\| \leq \|T^{\frac{1}{2}} f\|.$$

So by Theorem 2.5, we obtain lower bound for $\{\Lambda_j\}_{j\in J}$. This shows that $\{\Lambda_j\}_{j\in J}$ is $*$-g-frame for H with respect to $\{H_j\}_{j\in J}$. □

2.2. **Multipliers of $*$-g-Bessel sequences.** In the following, the concept of mutipliers for g-Bessel sequences will be extended to $*$-g-Bessel sequences and some of their properties will be shown.

Proposition 2.9. *Let*

$$\Lambda = \{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\}$$

and

$$\Theta = \{\Theta_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\}$$

be $*$-g-Bessel sequences with bounds B_Λ, B_Θ and $m = \{m_j\}_{j\in J} \in \ell^\infty(R)$ then the operator

$$M_{m,\Lambda,\Theta} : H \longrightarrow H, \qquad M_{m,\Lambda,\Theta} f := \sum_{j\in J} m_j \Lambda_j^* \Theta_j f, \tag{2.10}$$

for all $f \in H$ is a well-defined bounded operator.

Proof. Let Λ and Θ be $*$-g-Bessel sequences for H with bounds B_Λ, B_Θ, respectively. For any $f, g \in H$ and finite subset $I \subseteq J$,

$$\begin{aligned}\|\sum_{j\in I} m_j \Lambda_j^* \Theta_j f\|^2 &= \sup_{g\in H, \|g\|=1} \|\langle \sum_{j\in I} m_j \Lambda_j^* \Theta_j f, g\rangle\|^2 \\ &= \sup_{g\in H, \|g\|=1} \|\sum_{j\in I} \langle m_j \Theta_j f, \Lambda_j g\rangle\|^2 \\ &\le \sup_{g\in H, \|g\|=1} \|\sum_{j\in I} \langle m_j \Theta_j f, m_j \Theta_j f\rangle\| \|\sum_{j\in I} \langle \Lambda_j g, \Lambda_j g\rangle\|,\end{aligned}$$

since

$$\begin{aligned}\sum_{j\in I} \langle m_j \Theta_j f, m_j \Theta_j f\rangle &= \sum_{j\in I} \mid m_j \mid^2 \langle \Theta_j f, \Theta_j f\rangle \\ &\le \|m\|_\infty^2 \sum_{j\in I} \langle \Theta_j f, \Theta_j f\rangle \le \|m\|_\infty^2 B_\Theta \langle f, f\rangle B_\Theta^*.\end{aligned}$$

Hence

$$\begin{aligned}\|\sum_{j\in I} m_j \Lambda_j^* \Theta_j f\|^2 &\le \sup_{g\in H, \|g\|=1} \|m\|_\infty^2 \|B_\Theta\|^2 \|f\|^2 \|B_\Lambda\|^2 \|g\|^2 \\ &= \|m\|_\infty^2 \|B_\Theta\|^2 \|f\|^2 \|B_\Lambda\|^2.\end{aligned}$$

This shows that $M_{m,\Lambda,\Theta}$ is well-defined and

$$\|M_{m,\Lambda,\Theta}\| \le \|m\|_\infty \|B_\Lambda\| \|B_\Theta\|.$$ □

Now, the map M in the above proposition is called a $*$-g-multiplier of Λ, Θ and m.

Lemma 2.10. *Let*

$$\Lambda = \{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\}$$

and

$$\theta = \{\Theta_j \in End^*_A(H, H_j) : j \in J\}$$

be $*$*-g-Bessel sequences with respect to* $\{H_j : j \in J\}$ *with bounds* B_Λ, B_Θ *respectively. Let* $m = \{m_j\}_{j\in J} \in \ell^\infty(R)$ *then the operator,* $M = M_{m,\Lambda,\Theta} : H \longrightarrow H$ *defined by* $\langle Mf, g\rangle = \sum_{j\in J} m_j\langle\Theta_j f, \Lambda_j g\rangle$*, is well-defined and* $(M_{m,\Lambda,\Theta})^* = M_{\overline{m},\Theta,\Lambda}$*.*

Proof. By Proposition 2.9, M is well-defined. We claim that

$$(M_{m,\Lambda,\Theta})^* = M_{\overline{m},\Theta,\Lambda}.$$

Let $f, g \in H$, then

$$\begin{aligned}
\langle f, (M_{m,\Lambda,\Theta})^* g\rangle &= \langle (M_{m,\Lambda,\Theta} f), g\rangle \\
&= \sum_{j\in J} m_j \langle\Theta_j f, \Lambda_j g\rangle \\
&= \sum_{j\in J} \langle\Theta_j f, \overline{m_j}\Lambda_j g\rangle \\
&= \sum_{j\in J} \langle f, \Theta_j^*\overline{m_j}\Lambda_j g\rangle \\
&= \sum_{j\in J} \langle f, \overline{m_j}\Theta_j^*\Lambda_j g\rangle \\
&= \langle f, M_{\overline{m},\Theta,\Lambda}\rangle.
\end{aligned}$$

□

3. Controlled ∗-g-frames

Weighted and controlled frames have been introduced recently to improve the numerical efficiency of iterative algorithms for inverting the frame operator. In [4], it was shown that controlled frames are equivalent to standard frames. In this section, the concepts of controlled g-frames and controlled g-Bessel sequences will be extended to controlled ∗-g-frames and we will show that controlled ∗-g-frames are equivalent to ∗-g-frames.

Definition 3.1. [21] Let $C, C' \in gl^+(H)$. The family

$$\Lambda = \{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\},$$

will be called a (C, C')-controlled g-frame for H with respect to $\{H_j\}_{j\in J}$, if $\Lambda = \{\Lambda_j\}_{j\in J}$ is a g-Bessel sequence and there exist constants $A > 0$ and $B < \infty$ such that

$$A\|f\|^2 \le \sum_{j\in J}\langle\Lambda_j Cf, \Lambda_j C' f\rangle \le B\|f\|^2, \quad \forall f \in H. \tag{3.1}$$

A and B will be called (C, C')-controlled g-frame bounds. If $C' = I$, (or, $C = C'$), we call $\Lambda = \{\Lambda_j\}_{j \in J}$ a C-controlled g-frame. (respectively, C^2- controlled g-frame) for H with bounds A, B. If the second part of the above inequality holds, it will be called (C, C')-controlled g-Bessel sequence with bound B.

Definition 3.2. Let $C, C' \in gl^+(H)$. The family

$$\Lambda = \{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\}$$

will be called a (C, C')-controlled $*$-g-frame for H with respect to $\{H_j\}_{j \in J}$, if $\Lambda = \{\Lambda_j\}_{j \in J}$ is a $*$-g-Bessel sequence and

$$A\langle f, f\rangle A^* \leq \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C' f\rangle \leq B\langle f, f\rangle B^* \tag{3.2}$$

for all $f \in H$ and strictly nonzero elements $A, B \in \mathcal{A}$.

A and B will be called (C, C')-controlled $*$-g-frame bounds. If $C' = I$, (or, $C = C'$), we call $\Lambda = \{\Lambda_j\}_{j \in J}$ a C-controlled $*$-g-frame. (respectively, C^2- controlled $*$-g-frame) for H with bounds A, B. If the second part of the above inequality holds, it will be called (C, C')-controlled $*$-g-Bessel sequence with bound B.

The proof of the following lemmas is straightforward.

Lemma 3.3. *Let $C \in gl^+(H)$. The $*$-g-Bessel sequence and*

$$\Lambda = \{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\},$$

is a C^2-controlled $*$-g-Bessel sequence (or, C^2-controlled $*$-g-frame) if and only if

$$\sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C f\rangle \leq B\langle f, f\rangle B^*, \qquad \forall f \in H \tag{3.3}$$

(or $A\langle f, f\rangle A^* \leq \sum_{j \in J} \langle \Lambda_j C f, \Lambda_j C f\rangle \leq B\langle f, f\rangle B^*, \qquad \forall f \in H$).

Example 3.4. Let $\mathcal{A} = \ell^\infty$ and let $H = C_0$, the Hilbert $\mathcal{A}$-module of the set of all null sequences equipped with the $\mathcal{A}$-inner product

$$\langle (x_i)_{i \in N}, (y_i)_{i \in N}\rangle = (x_i \overline{y_i})_{i \in N}.$$

The action of each sequence $(a_i)_{i \in N} \in \mathcal{A}$ on a sequence $(x_i)_{i \in N} \in H$ is implemented as $(a_i)_{i \in N}(x_i)_{i \in N} = (a_i x_i)_{i \in N}$. Let $j \in J = N$ and $(1 + \frac{1}{i})_{i \in N} \in \ell^\infty$. Define $\Lambda_j \in End^*_{\mathcal{A}}(H)$ by

$$\Lambda_j (x_i)_{i \in N} = (\delta_{ij} a_j x_j)_{i \in N}, \qquad \forall (x_i)_{i \in N} \in H.$$

Now define $Cx = 2x$ and $C'x = \frac{1}{2}x$. Then for any $x \in H$, we can estimate

$$\sum_{j \in N} \langle \Lambda_j C x, \Lambda_j C' x\rangle = ((1 + \frac{1}{i})^2 x_i \overline{x_i})_{i \in N} = (1 + \frac{1}{i})_{i \in N} \langle x, x\rangle (1 + \frac{1}{i})_{i \in N},$$

for all $x = (x_i)_{i\in N} \in H$. This shows that $\Lambda = \{\Lambda_j \in End^*_{\mathcal{A}}(H) : j \in N\}$ is a (C, C')-controlled tight $*$-g-frame for H .

Suppose that $\{\Lambda_j \in End^*_{\mathcal{A}}(H, H_j) : j \in J\}$ be a (C, C')-controlled $*$-g-frame for the Hilbert C^*-module H with respect $\{H_j\}_{j\in J}$. The bounded linear operator $T_{(C,C')} : \bigoplus_{j\in J} H_j \to H$ defined by:

$$T_{(C,C')}(\{g_j\}_{j\in J}) = \sum_{j\in J} (CC')^{\frac{1}{2}\Lambda^*_j g_j, \forall \{g_j\}_{j\in J} \in \bigoplus_{j\in J} H_j} (3.4)$$

is called the synthesis operator for the (C, C')-controlled $*$-g-frame $\{\Lambda_j\}_{j\in J}$.
The adjoint operator $T^*_{(C,C')} : H \to \bigoplus_{j\in J} H_j$ given by

$$T^*_{(C,C')}(f) = \{\Lambda_j (C'C)^{\frac{1}{2} f\}_{j\in J}} (3.5)$$

is called the analysis operator for the (C, C')-controlled $*$-g-frame $\{\Lambda_j\}_{j\in J}$. When C and C' commute with each other, and also commute with the operator $\Lambda^*_j \Lambda_j$ for each j, then the (C, C')-controlled $*$-g-frame operator $S_{(C,C')} : H \to H$ is defined as:

$$S_{(C,C')} f = T_{(C,C')} T^*_{(C,C')} f = \sum_{j\in j} C' \Lambda^*_j \Lambda_j C f. \qquad (3.6)$$

For the above result one is referred to Hua and Huang [15].
From now on we assume that C and C' commute with each other, and commute with the operator $\Lambda^*_j \Lambda_j$ for all j.

Proposition 3.5. *Let $\{\Lambda_j : j \in J\}$ be a (C, C')-controlled $*$-g-frame for the Hilbert C^*-module H with respect to $\{H_j\}_{j\in J}$. Then the (C, C')-controlled $*$-g-frame operator $S_{(C,C')}$ is positive, self adjoint and invertible.*

Proof. The frame operator $S_{(C,C')}$ for the (C, C')-controlled $*$-g-frame is $S_{(C,C')} f = \sum_{j\in j} C' \Lambda^*_j \Lambda_j C f$. As $\{\Lambda_j : j \in J\}$ is a (C, C')-controlled $*$-g-frame, from the identity,

$$\textstyle\sum_{j\in J} \langle \Lambda_j C f, \Lambda_j C' f\rangle = \langle \sum_{j\in J} C' \Lambda^*_j \Lambda_j C f\rangle = \langle S_{(C,C')} f, f\rangle,$$

we clearly see that $S_{(C,C')}$ is a positive operator. It is clearly boundded and linear.

$$\begin{aligned}\langle S_{(C,C')}f,g\rangle &= \langle\sum_{j\in J} C'\Lambda_j^*\Lambda_j Cf,g\rangle\\ &= \sum_{j\in J}\langle C'\Lambda_j^*\Lambda_j Cf,g\rangle\\ &= \sum_{j\in J}\langle f, C\Lambda_j^*\Lambda_j C'g\rangle\\ &= \sum_{j\in J}\langle f, S_{(C',C)}g\rangle.\end{aligned}$$

Hence $S^*_{(C,C')} = S_{(C',C)}$ is positive and hence self adjoint. Also as C and C' commute with each other and commute with $\Lambda_j^*\Lambda_j$, we have $S_{(C,C')} = S_{(C',C)}$. From the controlled $*$-g-frame identity we have

$$A\langle f,f\rangle A^* \le \langle S_{(C,C')}f,f\rangle \le B\langle f,f\rangle B^*.$$

So

$$A\ Id_H\ A^* \le \langle S_{(C,C')}f,f\rangle \le B\ Id_H\ B^*,$$

where Id_H is the identity operator in H. Thus the controlled $*$-g-frame operator $S_{(C,C')}$ is invertible.

□

Theorem 3.6. *Let $\{\Lambda_j\}_{j\in J} \in End^*_{\mathcal{A}}(H,H_j)$, and $\sum_{j\in J}\langle\Lambda_j Cf, \Lambda_j C'f\rangle$ converge in norm $\mathcal{A}$. Then $\{\Lambda_j\}_{j\in J}$ is a (C,C')-controlled $*$-g-frame for H with respect to $\{H_j\}_{j\in J}$ if and only if*

$$\| A^{-1} \|^{-2} \| \langle f,f\rangle\| \le \| \sum_{j\in J}\langle\Lambda_j Cf, \Lambda_j C'f\rangle\| \le \| B \|^2 \| \langle f,f\rangle\| \qquad (3.7)$$

for all $f \in H$ and strictly nonzero elements $A, B \in \mathcal{A}$.

Proof. By the definition of (C,C')-controlled$*$-g-frame we conclude that

$$\langle f,f\rangle \le A^{-1}\langle S_{(C,C')}f,f\rangle (A^*)^{-1} \text{ and } \langle S_{(C,C')}f,f\rangle \le B\langle f,f\rangle B^*.$$

Hence

$$\| A^{-1} \|^{-2} \| \langle f,f\rangle\| \le \| \sum_{j\in J}\langle\Lambda_j Cf, \Lambda_j C'f\rangle\| \le \| B \|^2 \| \langle f,f\rangle\| \qquad (3.8)$$

for all $f \in H$. Conversely, suppose that

$$\| A^{-1} \|^{-2} \| \langle f,f\rangle\| \le \| \sum_{j\in J}\langle\Lambda_j Cf, \Lambda_j C'f\rangle\| \le \| B \|^2 \| \langle f,f\rangle\|, \qquad (3.9)$$

From Proposition 3.5, the (C, C')-controlled $*$-g-frame operator is positive, self adjoint and invertible. Hence

$$\langle (S_{(C,C')})^{\frac{1}{2}} f, (S_{(C,C')})^{\frac{1}{2}} f\rangle = \langle S_{(C,C')} f, f\rangle = \sum_{j\in J}\langle \Lambda_j C f, \Lambda_j C' f\rangle. \quad (3.10)$$

Using inequality (3.10) in inequality (3.9), we get

$$\| A^{-1} \| \| f \| \leq \| (S_{(C,C')})^{\frac{1}{2}} \| \leq \|B\| \|f\|, \quad (3.11)$$

According to Theorem 2.5 and inequality (3.11), $\{\Lambda_j : j \in J\}$ is a (C, C')-controlled $*$-g-frame for H with respect to $\{H_j\}_{j\in J}$.

□

The following theorem shows that any $*$-g-frame is a C^2-controlled $*$-g-frame and vice versa.

Theorem 3.7. *Let $C \in gl^+(H)$. The family $\{\Lambda_j\}_{j\in J} \in End^*_{\mathcal{A}}(H, H_j)$, is a $*$-g-frame if and only if $\Lambda = \{\Lambda_j\}_{j\in J}$ is a C^2-controlled $*$-g-frame.*

Proof. Let Λ is a C^2-controlled $*$-g-frame with bounds A, B. Then

$$\sum_{j\in J}\langle \Lambda_j C f, \Lambda_j C f\rangle \leq B\langle f, f\rangle B^*, \qquad \forall f \in H.$$

For $f \in H$ we have

$$\begin{aligned} A\langle f, f\rangle A^* &= A\langle CC^{-1} f, CC^{-1} f\rangle A^* \leq A\|C\|^2\langle C^{-1} f, C^{-1} f\rangle A^* \\ &\leq \|C\|^2 \sum_{j\in J}\langle \Lambda_j CC^{-1} f, \Lambda_j CC^{-1} f\rangle = \|C\|^2 \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle. \end{aligned}$$

Hence

$$A\|C\|^{-1}\langle f, f\rangle A^*\|C\|^{-1} \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle.$$

On the other hand for every $f \in H$,

$$\begin{aligned} \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle &= \sum_{j\in J}\langle \Lambda_j CC^{-1} f, \Lambda_j CC^{-1} f\rangle \leq B\langle C^{-1} f, C^{-1} f\rangle B^* \\ &\leq B\|C^{-1}\|^2\langle f, f\rangle B^* = B\|C^{-1}\|\langle f, f\rangle B^*\|C^{-1}\|. \end{aligned}$$

These inequalities yield that Λ is a $*$-g-frame with bounds $A\|C\|^{-1}, B\|C\|^{-1}$. For the converse, assume that Λ is a $*$-g-frame with bounds A', B'. Then for all $f \in H$,

$$A'\langle f, f\rangle (A')^* \leq \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq B'\langle f, f\rangle (B')^*.$$

So for all $f \in H$,

$$\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle \leq B'\|C\|^2\langle f,f\rangle(B')^*.$$

For lower bound, since Λ is a $*$-g-frame for any $f \in H$,

$$A'\langle f,f\rangle(A')^* = A'\langle C^{-1}Cf, C^{-1}Cf\rangle(A')^* \leq \|C^{-1}\|^2\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j Cf\rangle.$$

Therefore Λ is a C^2-controlled $*$-g-frame with bounds $A'\|C^{-1}\|^{-1}, B'\|C\|$. □

Proposition 3.8. *Assume that $\{\Lambda_j : j \in J\}$ is a $*$-g-frame for the Hilbert C^*-module H with respect to $\{H_j\}_{j\in J}$. Let S_Λ be the $*$-g-frame operator with the $*$-g-frame $\{\Lambda_j : j \in J\}$. Let $C, C' \in gl^+(H)$. Then $\{\Lambda_j : j \in J\}$ is a (C, C')-controlled $*$-g-frame.*

Proof. $\{\Lambda_j : j \in J\}$ is a $*$-g-frame for the Hilbert C^*-module H with respect to $\{H_j\}_{j\in J}$ with bounds A and B . By inequality (2.7), we have:

$$\| A^{-1} \|^{-2}\| \langle f,f\rangle\| \leq\| \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\| \leq\| B \|^2\| \langle f,f\rangle\|, \tag{3.12}$$

Again we have

$$\| \textstyle\sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle \| = \| \langle S_{(C,C')}f, f\rangle \| ,$$

and

$$\| \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle \| = \| C \|\| C' \|\| \langle S_\Lambda f, f\rangle \| . \tag{3.13}$$

From (3.12) and (3.13), we have

$$\begin{aligned}\| A^{-1} \|^{-2}\| C \|\| C' \|\| f \|^2 &\leq \sum_{j\in J}\langle \Lambda_j Cf, \Lambda_j C'f\rangle \| \\ &\leq\| B \|^2\| C \|\| C' \|\| f \|^2,\end{aligned}$$

for all $f \in H$. So $\{\Lambda_j : j \in J\}$ is a (C, C')-controlled $*$-g-frame with bounds $\| A^{-1} \|^{-1}\| C \|\| C' \|, \| B \|\| C \|\| C' \|$. □

Theorem 3.9. *Suppose that $C, C' \in gl^+(H)$, $\{\Lambda_j : j \in J\} \subset End^*(H, H_j)$ and C, C' commute with each other and commute with $\Lambda_j^*\Lambda_j$ for all $j \in J$. If the operator $T : \bigoplus_{j\in J} H_j \to H$ given by*

$$T_{(C,C')}(\{g_j\}_{j\in J}) = \sum_{j\in J}(CC')^{\frac{1}{2}\Lambda_j^* g_j, \forall\{g_j\}_{j\in J}\in\bigoplus_{j\in J} H_j}(3.14)$$

is well defined and bounded operator with $\| T_{(C,C')} \|\leq\| B \|$, then the sequence $\{\Lambda_j : j \in J\}$ is a (C, C')-controlled $*$-g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ with bound $\| B \|$.

Proof. Let $\{\Lambda_j : j \in J\}$ be a (C, C')-controlled $*$-g-Bessel sequence for H with respect to $\{H_j\}_{j\in J}$ with bound B. As a result of Theorem 3.6,

$$\| \sum_{j\in J} \langle \Lambda_j C f, \Lambda_j C' f\rangle \| \leq \| B \|^2 \| \langle f, f\rangle \|. \tag{3.15}$$

For any sequence $\{g_j\}_{j\in J} \in \bigoplus_{j\in J} H_j$,

$$\begin{aligned}
\| T_{(C,C')}(\{g_j\}_{j\in J}) \|^2 &= sup_{f\in H, \|f\|=1} \| \langle T_{(C,C')}(\{g_j\}_{j\in J}), f\rangle \|^2 \\
&= sup_{f\in H, \|f\|=1} \| \langle \sum_{j\in J} (CC')^{\frac{1}{2}\Lambda_j^* g_j, f\rangle \|^2} \\
&= sup_{f\in H, \|f\|=1} \| \sum_{j\in J} \langle (CC')^{\frac{1}{2}\Lambda_j^* g_j, f\rangle \|^2} \\
&= sup_{f\in H, \|f\|=1} \| \sum_{j\in J} \langle \langle g_j, \Lambda_j (CC')^{\frac{1}{2} f\rangle \|^2} \\
&\leq sup_{f\in H, \|f\|=1} \| \sum_{j\in J} \langle g_j, g_j\rangle \| \\
&\quad \| \sum_{j\in J} \langle \Lambda_j (CC')^{\frac{1}{2} f, \Lambda_j (CC')^{\frac{1}{2} f\rangle \|}} \\
&= sup_{f\in H, \|f\|=1} \| \sum_{j\in J} \langle g_j, g_j\rangle \| \\
&\quad \| \sum_{j\in J} \langle \Lambda_j C f, \Lambda_j C' f\rangle \| \\
&\leq sup_{f\in H, \|f\|=1} \| \sum_{j\in J} \langle g_j, g_j\rangle \| \| B \|^2 \| f \|^2 \\
&= \| B \|^2 \| \{g_j\} \|^2
\end{aligned}$$

Therefore, the sum $\sum_{j\in J} (CC')^{\frac{1}{2}\Lambda_j^* g_j}$ is convergent and we have

$$\| T_{(C,C')}(\{g_j\}_{j\in J}) \|^2 \leq \| B \|^2 \| \{g_j\} \|^2.$$

So

$$\| T_{(C,C')} \|^2 \leq \| B \|^2.$$

Hence the operator $T_{(C,C')}$ is well defined, bounded and

$$\| T_{(C,C')} \| \leq \| B \|.$$

□

4. Multipliers of Controlled $*$-g-frames

In this section, we define the multiplier of a controlled $*$-g-frame for C-controlled $*$-g-frames in Hilbert C^*- modules. The definition of general case (C, C')-controlled $*$-g-frames is similar.

Lemma 4.1. *Let $C, C' \in gl^+(H)$ and*

$$\Lambda = \{\Lambda_j \in End_A^*(H, H_j) : j \in J\} \;, \Theta = \{\Theta_j \in End_A^*(H, H_j) : j \in J\}$$

be C'^2 and C^2-controlled $*$-g-Bessel sequences for H, respectively. Let $m = \{m_j\}_{j\in J} \in \ell^\infty(R)$. The operator

$$M_{m,C,\Theta,\Lambda,C'} : H \longrightarrow H,$$

defined by

$$M_{m,C,\Theta,\Lambda,C'} f := \sum_{j\in J} m_j C\Theta_j^* \Lambda_j C' f,$$

is a well-defined bounded operator.

Proof. Let Λ, Θ be C'^2 and C^2-controlled $*$-g-Bessel sequences with bounds B, B', respectively. For any $f, g \in H$ and finite subset $I \subseteq J$,

$$\begin{aligned}\left\|\sum_{j\in I} m_j C\Theta_j^*\Lambda_j C' f\right\|^2 &= \sup_{g\in H, \|g\|=1} \|\sum_{j\in I}\langle m_j\Lambda_j C' f, \Theta_j C g\rangle\|^2 \\ &\leq \sup_{g\in H, \|g\|=1} \|\sum_{j\in I}\langle m_j\Lambda_j C' f, m_j\Lambda_j C' f\rangle\| \|\sum_{j\in I}\langle \Theta_j C g, \Theta_j C g\rangle\|,\end{aligned}$$

since

$$\begin{aligned}\sum_{j\in I}\langle m_j\Lambda_j C' f, m_j\Lambda_j C' f\rangle &= \sum_{j\in I} \mid m_j \mid^2 \langle \Lambda_j C' f, \Lambda_j C' f\rangle \\ &\leq \|m\|_\infty^2 \sum_{j\in I}\langle \Lambda_j C' f, \Lambda_j C' f\rangle \leq \|m\|_\infty^2 B\langle f, f\rangle B^*.\end{aligned}$$

Hence

$$\begin{aligned}\|\sum_{j\in I} m_j C\Theta_j^*\Lambda_j C' f\|^2 &\leq \sup_{g\in H, \|g\|=1} \|m\|_\infty^2 \|B\|^2 \|f\|^2 \|B'\|^2 \|g\|^2 \\ &\leq \|m\|_\infty^2 \|B\|^2 \|f\|^2 \|B'\|^2.\end{aligned}$$

This shows that $M_{m,C,\Theta,\Lambda,C'}$ is well-defined and

$$\|M_{m,C,\Theta,\Lambda,C'}\| \leq \|m\|_\infty \|B\| \|B'\|.$$ □

Definition 4.2. Let $C, C' \in gl^+(H)$ and

$$\Lambda = \{\Lambda_j \in End_A^*(H, H_j) : j \in J\}$$

and

$$\Theta = \{\Theta_j \in End_A^*(H, H_j) : j \in J\}$$

be C'^2 and C^2-controlled $*$-g-Bessel sequences for H, respectively. Let $m = \{m_j\}_{j\in J} \in \ell^\infty(R)$. The operator

$$M_{m,C,\Theta,\Lambda,C'} : H \longrightarrow H,$$

defined by

$$M_{m,C,\Theta,\Lambda,C'} f := \sum_{j\in J} m_j C \Theta_j^* \Lambda_j C' f,$$

is called the (C, C')-controlled multiplier operator with symbol m.

5. Conclusions

In this article, the concept of multipliers from g-frames to $*$-g-Bessel sequences and $*$-g-frames is extended. Controlled frames and controlled Bessel sequences are extended to controlled $*$-g-frames and controlled $*$-g-Bessel sequences. At the end of this paper, the concept of a multiplier for C^2- controlled and C'^2-controlled $*$-g-Bessel sequences is defined.

References

[1] A. Alijani , Generalized Frames with C^*-Valued Bounds and their Operator Duals, *Published by Faculty of Sciences and Mathematics. University of Nis. Sebria. (2015), 1469-1479.*

[2] A. Alijani , M. A. Dehghan, G-Frames And Their Duals For Hilbert C^*-Modules, *Bull. Iran. Math. Soc.* **38 (3)** *(2012), 567-580.*

[3] P. Balazs, D. Bayer, A. Rahimi, Multipliers for continuos frames in Hilbert spaces, *J. Phys. A:Math. Theor.* **45** *(2012), 1-24.*

[4] P. Balazs, J. P. Antoine, A. Grybos, Weighted and Controlled Frames, *Int. J. Wavelets Multiresolut. Inf. Process* . **8 (1)** *(2010), 109-132.*

[5] I. Bogdanova, P. Vandergheynst, J. P. Antoine, L. Jacques, M. Morvidone, Stereographic wavelet frames on the sphere, *Applied Comput. Harmon. Anal.* **19** *(2005), 223-252.*

[6] H. Bolcskei, F. Hlawatsch, H.G Feichtinger, Frame-theoretic analysis of over sampled filter banks, *IEEE Trans Signal Process.* **46(12)** *(1998), 3256-3268.*

[7] E. J Candes, D. L Donoho , New tight frames of curvelets and optimal representations of objects with piecewise C^2 singularities, *Commun. Pure Appl. Math.* **57(2)** *(2004), 219-266.*

[8] O. Christensen, An Introduction to Frames and Riesz bases, Birkhauser, Boston (2003).

[9] I. Daubechiesuffin, A. Grossmann, A. Meyer, Painless nonorthogonal expansions, *J. Math. Phys.* **72(5)** *(1986), 1271-1283.*

[10] M. Dorfler, Gabor analysis for a class of singels called music, Phd. Thesis, University of Vienna, (2003) .

[11] R. Duffin, A. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc.* **72** *(1952), 341-366.*
[12] M. H. Faroughi and M. Rahmani, Bochner p-g-frames, *J. Inequal. Appl.* **5** *(2012), 1-16.*
[13] H. G. Feichtinger and K. Nowak, A first survey of Gabor multipliers, *Advances in Gabor Analysis, Appl. Numer. Harmon. Anal, Birkhauser, Boston, MA. (2003) , 99-128.*
[14] M. Frank and D. R. Larson, Frames in Hilbert C^*-modules and C^*-algebras, *J Operator Theory.* **48** *(2002), 273-314.*
[15] D. Hua and Y. Huang, Controlled K-g-frames, *Results in Math.* **72(3)** *(2017),1227-1238.*
[16] W. Jing , Frames in Hilbert C^*-modules , Phd. University of Central Florida. Orlando. FL. USA. (2006).
[17] A. Khosravi and B. Khosravi, Fusion frames and g-frames in Hilbert C^*-modules, *Int J Wavelets Multiresolut Inf Process .* **6** *(2008), 433-446.*
[18] E. C. Lance, Hilbert C^*-modules, A Toolkit for operator Algebraists, London Mathematical Society Lecture Note Series, 210, Combridge University Press, Cambridge, 1995.
[19] G. Mats and F. Hlawatsch, Linear Time-Frequency Filters, One-line Algorithms and Applications, eds. A. Papandredreou-Suppappola, Boca Raton (FL):CRC Press,Ch. **6** (2002) 205-271.
[20] G. J. Murphy, C^*-algebra and Operator Theory, Academic Press. 1990.
[21] A. Rahimi and A. Freydooni, Controlled G-Frames and Their G-Multipliers in Hilbert spaces, *An. St. Ovidius.* **21** *(2013), 223-236.*
[22] A. Rahimi, Multipliers of Generalized frames in Hilbert spaces, *Bull. Iranian Math. Soc.* **37(1)** *(2011), 63-88.*
[23] A. Rahimi, P. Balazs, Multipliers of P-Bessel sequences in Banach spaces, *Integr. Equ. Oper. Theory,* **68***(2) (2010), 193-205.*
[24] W. Sun, g-frame and g-Riesz bases, *J. Math. Anal. Appl.* **322** *(2006), 437-452.*
[25] N. E. Wegge-Olsen, K-Theory and C^*-Algebra, A Friendly Approch, Oxford University Press, Oxford, England, 1993.