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Category of H-groups

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Abstract. This paper develops a basic theory of H-groups. We introduce a special quotient of H-groups and extend some algebraic constructions of topological groups to the category of H-groups and H-maps and then present a functor from this category to the category of quasitopological groups.

Keywords: H-group, sub-H-group, quotient of H-group.

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1. Introduction and Motivation

An H-group is a homotopy associative H-space with a given homotopy inverse. There are two main classes of motivating examples of H-groups. The first is the class of topological groups and the second is the class of loop spaces. Topological groups have been studied from a variety of viewpoint. Specially, there is an enriched developed basic theory for topological groups similar to abstract group theory. However, it seems that there is no such a basic theory for H-groups. One can find only

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the concept of sub-H-group of an H-group in [10] and some elementary properties in [6, 11, 12]. One of the main objects in this paper is to develop a basic theory for H-groups similar to abstract group theory.

After giving main definitions and notations in Section 2, we introduce in Section 3 cosets of a sub-H-group, a normal sub-H-group and a quotient of an H-group in order to provide preliminaries to begin a basic theory for H-groups similar to elementary group theory. We develop the theory in Section 4 by introducing the kernel of an H-homomorphisms in order to give H-isomorphism theorems.

In Section 5, we give a topology to a quotient of an H-group which makes it a quasitopological group in the sense of [1], that is, a group with a topology such that inversion and all translations are continuous. We also study the path component space of an H-group and give some conditions for significance of semilocally 0-connectedness.

Loop spaces have the main role in homotopy groups especially in topological homotopy groups [7]. Finally, we give some examples in topological homotopy groups. The topological n-th homotopy group of a pointed space (X, x) is the familiar n-th homotopy group $\pi_n(X, x)$ by endowing a topology on it as a quotient of the n-loop space $\Omega^n(X, x)$ equipped with the compact-open topology, denoted by $\pi_n^{top}(X, x)$ [7]. More precisely, among reproving some of the known results, we give some new results for discreteness and indiscreteness of $\pi_n^{top}(X, x)$, for $n \geq 1$. Also, we find out a family of spaces by using n-Hawaiian like spaces introduced in [8] such that their topological fundamental groups are indiscrete topological groups.

2. Notations and preliminaries

We recall from [6, 11] that an H-space (P, μ, c) consists of a pointed topological space (P, p_0) together with a continuous pointed map $\mu : (P \times P, (p_0, p_0)) \longrightarrow (P, p_0)$ and the constant map $c : (P, p_0) \longrightarrow (P, p_0)$ for which $\mu \circ (1_P, c) \simeq \mu \circ (c, 1_P) \simeq 1_P \text{ rel}\{p_0\}$, where $(1_P, c) : P \longrightarrow P \times P$ defined by $(1_P, c)(p) = (p, p_0)$ and $(c, 1_P) : P \longrightarrow P \times P$ defined by $(c, 1_P)(p) = (p_0, p)$. The continuous multiplication μ is said to be *homotopy associative* if $\mu \circ (\mu \times 1_P) \simeq \mu \circ (1_P \times \mu) \text{ rel}\{(p_0, p_0, p_0)\}$, where $(\mu \times 1_P) : P \times P \times P \longrightarrow P \times P$ defined by $(\mu \times 1_P)(x, y, z) = (\mu(x, y), z)$ and $(1_P \times \mu) : P \times P \times P \longrightarrow P \times P$ defined by $(1_P \times \mu)(x, y, z) = (x, \mu(y, z))$.

An H-group (P, μ, η, c) consists of an H-space (P, μ, c) together with a continuous pointed map $\eta : (P, p_0) \longrightarrow (P, p_0)$ for which $\mu \circ (\eta, 1_P) \simeq \mu \circ (1_P, \eta) \simeq c \text{ rel}\{p_0\}$, where $(1_P, \eta) : P \longrightarrow P \times P$ defined by $(1_P, \eta)(p) = (p, \eta(p))$ and $(\eta, 1_P) : P \longrightarrow P \times P$ defined by $(\eta, 1_P)(p) = (\eta(p), p)$. The maps μ, η and c are called *multiplication*, *homotopy inverse* and *homotopy identity*, respectively. As two important examples, one can show that any topological group with group multiplication, group inverse and group identity and also every loop space with path concatenation, path inverse and path identity are H-groups. Moreover, P is called an Abelian H-group if $\mu \circ T \simeq \mu$, where $T : P \times P \longrightarrow P \times P$ defined by $T(x, y) = (y, x)$.

The following notions and results are needed in the sequel.

Definition 2.1. ([6, XIX, 3 and 6]). Let (P, μ, c) and (P', μ', c') be two H-spaces. A continuous map $\varphi : P \longrightarrow P'$ is called an *H-homomorphism* whenever $\varphi \circ \mu \simeq \mu' \circ (\varphi \times \varphi)$. Moreover, if (P, μ, η, c) and (P', μ', η', c') be two H-groups, then a continuous map $\varphi : P \longrightarrow P'$ is called an *H-homomorphism* whenever $\varphi \circ \mu \simeq \mu' \circ (\varphi \times \varphi)$ and $\varphi \circ \eta \simeq \eta' \circ \varphi$. Also, φ is called an *H-isomorphism* if there exists an H-homomorphism $\psi : P' \longrightarrow P$ such that $\varphi \circ \psi \simeq 1_{P'}$ and $\psi \circ \varphi \simeq 1_P$; in this event, the H-structures are called H-isomorphic.

Remark 2.2. It is straightforward to check that H-morphisms of H-groups preserve homotopy associativity, that means

$$\varphi \circ \mu \circ (\mu \times 1_P) \simeq \mu' \circ (\mu' \times 1_P) \circ (\varphi \times \varphi, \varphi).$$

Example 2.3. ([6, XIX, 3]). Let $x, y \in X$, and let α be any path in X from x to y. The map $\alpha^+ : \Omega(X, x) \longrightarrow \Omega(X, y)$ by setting $\alpha^+(\beta) = \alpha^{-1} * (\beta * \alpha)$ is an H-isomorphism by $(\alpha^{-1})^+ : \Omega(X, y) \longrightarrow \Omega(X, x)$ as inverse, where $*$ means the concatenation of paths. Also, for each continuous map $f : (X, x) \longrightarrow (Y, y)$, $\Omega f : \Omega(X, x) \longrightarrow \Omega(Y, y)$ by $(\Omega f)(\alpha) = f \circ \alpha$ is an H-homomorphism and if f is a homotopy equivalence, Ωf is an H-isomorphism.

Proposition 2.4. *(*[6, XIX, Theorem 7.2]*). If (P, μ, η, c) is an H-group with the based point p_0, then $\pi_0(P)$, the set of all path components of P, is a group with the multiplication $[g_1][g_2] = [\mu(g_1, g_2)]$, for all $[g_1]$, $[g_2] \in \pi_0(P)$, and with the equivalence class of p_0 as the identity. Also, for any H-homomorphism $\varphi : P \longrightarrow P'$, $\pi_0(\varphi) : \pi_0(P) \longrightarrow \pi_0(P')$ is a group homomorphism.*

Definition 2.5. ([10, Definition 3.1]). A pointed subspace P' of an H-group (P, μ, η, c) with the same based point p_0 is called a sub-H-group of P if P' is itself an H-group such that the inclusion map $i : P' \hookrightarrow P$ is an homomorphism in the sense of Spanier [11] i.e. $i : P' \hookrightarrow P$ is an H-homomorphism of H-spaces in the sense of Dugundji [6].

Example 2.6. Given a pointed space (Y, y_0) with (Y', y_0) as a pointed subspace. Then the loop space $\Omega(Y', y_0)$ is a sub-H-group of the loop space $\Omega(Y, y_0)$.

Theorem 2.7. *(*[10, Proposition 3.8]*). If P' is a sub-H-group of an H-group (P, μ, η, c), then*
(i) There exists a continuous multiplication $\mu' : P' \times P' \longrightarrow P'$ such that $i \circ \mu' \simeq \mu \circ (i \times i)$, where $i : P' \hookrightarrow P$ is the inclusion map;
(ii) For the constant map $c' : P' \longrightarrow P'$ we have $i \circ c' = c \circ i$;
(iii) There exists a continuous map $\eta' : P' \longrightarrow P'$ such that $i \circ \eta' \simeq \eta \circ i$.

Remark 2.8. Note that at the proof of the above theorem [10, Proposition 3.8] it is proved that the continuous multiplication μ', the homotopy identity c' and the homotopy inverse for μ', η', of P' as a sub-H-group of P satisfy (i), (ii) and (iii), respectively. Hence we can assume in the definition of sub-H-group (Definition 2.4) that the inclusion map $i : P' \hookrightarrow P$ is an H-homomorphism of H-groups.

Let $hTop_*$ be the category of pointed topological spaces with homotopy class of pointed maps as morphism. A morphism $f : (X, x_0) \longrightarrow (Y, y_0)$ is called *monic* if and only if it is a left-cancellative morphism, that is, for any morphisms $g_1, g_2 : (Z, z_0) \longrightarrow (X, x_0)$ the homotopy $f \circ g_1 \simeq f \circ g_2$ implies that $g_1 \simeq g_2$. Also a morphism $f' : (X, x_0) \longrightarrow (Y, y_0)$ is called *epic* if and only if it is a right-cancellative morphism, that is, for any morphisms $h_1, h_2 : (Y, y_0) \longrightarrow (Z', z'_0)$ if $h_1 \circ f' \simeq h_2 \circ f'$, then $h_1 \simeq h_2$.

Theorem 2.9. *(*[10, Proposition 3.9]*). Let P' be a pointed subspace of an H-group (P, μ, η, c). Suppose that the statements (i), (ii) and (iii) given in Theorem 2.6 are satisfied and the inclusion map $i : P' \longrightarrow P$ is monic. Then P' is a sub-H-group of P.*

We introduce the following notations which are used throughout the paper to simplify most of the proofs.
(i) Let X be a topological space, Y be any subset of X, and x be any element of X. Then we say that x *pathly belongs to* Y, denoted by $x \tilde{\in} Y$ if and only if there exists a path $\alpha : I \longrightarrow X$ such that $\alpha(0) = x$ and $\alpha(1) \in Y$.
(ii) For any two points a and b of a topological space X, we say that a and b are *pathly equal*, denoted by $a \tilde{=} b$ if and only if there exists a path $\alpha : I \longrightarrow X$ such that $\alpha(0) = a$ and $\alpha(1) = b$. Clearly $\tilde{=}$ is an equivalent relation on X and if $a \tilde{=} b$ and $b \tilde{\in} Y$, then $a \tilde{\in} Y$.
(iii) Let (P, μ, η, c) be an H-group, then we use the notation $g_1 g_2$ instead of the μ-multiplication $\mu(g_1, g_2)$ and the notation g^{-1} instead of the η-inversion $\eta(g)$. Note that if (P', μ', η', c') is a sub-H-group of (P, μ, η, c), then in order to avoid ambiguity we use only the notations $g_1 g_2$ and g^{-1} instead of the μ-multiplication and the η-inversion, respectively, but not instead of the μ'-multiplication and the η'-inversion.

(*iv*) If α is a path in an H-group (P,μ,η,c) and $g\in P$, then by $g\alpha$ we mean the path $g\alpha : I\longrightarrow P$ given by $g\alpha(t)=\mu(g,\alpha(t))=g\alpha(t)$.

The following lemmas will be used frequently throughout the paper.

Lemma 2.10. *Let (P,μ,η,c) be an H-group with the pase point p_0. Then the following statements hold.*
(i) For any $g\in P$, $gp_0\tilde{=}g\tilde{=}p_0g$.
(ii) For any $g\in P$, $g^{-1}g\tilde{=}p_0\tilde{=}gg^{-1}$.
(iii) For any $g_1,g_2,g_3\in P$, $g_1(g_2g_3)\tilde{=}(g_1g_2)g_3$.
(iv) For any $g_1,g_1',g_2,g_2'\in P$, if $g_1\tilde{=}g_1'$ and $g_2\tilde{=}g_2'$, then $g_1g_2\tilde{=}g_1'g_2'$.
(v) For any $g_1,g_2,g_3\in P$, if $g_1g_3\tilde{=}g_2g_3$, then $g_1\tilde{=}g_2$. Also if $g_1g_2\tilde{=}g_1g_3$, then $g_2\tilde{=}g_3$.
(vi) For any $g\in P$, $(g^{-1})^{-1}\tilde{=}g$.
(vii) For any $g_1,g_2\in P$, $(g_1g_2)^{-1}\tilde{=}g_2^{-1}g_1^{-1}$.
(viii) $p_0^{-1}\tilde{=}p_0$.
(ix) For any $g_1,g_2\in P$, if $g_1\tilde{=}g_2$, then $g_1^{-1}\tilde{=}g_2^{-1}$.

Proof. (*i*) Since $\mu\circ(1_P,c)\simeq 1_P$ rel$\{p_0\}$, there exists a continuous map $F:P\times I\longrightarrow P$ such that $F(g,0)=\mu\circ(1_P,c)(g)=\mu(g,p_0)=gp_0$ and $F(g,1)=1_P(g)=g$, for all $g\in P$. Define $\lambda_g : I\longrightarrow P$ by $\lambda_g(t)=F(g,t)$. Then λ_g is a path in P from gp_0 to g and hence by Notation 2.9 $gp_0\tilde{=}g$. Also, since $1_P\simeq\mu\circ(c,1_P)$ rel$\{p_0\}$, by a similar method we have $g\tilde{=}p_0g$.
(*ii*) Similar to (*i*) by using homotopies $\mu\circ(\eta,1_P)\simeq c\simeq\mu\circ(1_P,\eta)$ rel$\{p_0\}$.
(*iii*) Similar to (*i*) by using homotopies $\mu\circ(\mu\times 1_P)\simeq\mu\circ(1_P\times\mu)$ rel$\{p_0\}$.
(*iv*) Since $g_1\tilde{=}g_1'$ and $g_2\tilde{=}g_2'$, there are two paths $\lambda_1 : I\longrightarrow P$, $\lambda_2\longrightarrow P$ such that $\lambda_1(0)=g_1$, $\lambda_1(1)=g_1'$, $\lambda_2(0)=g_2$, $\lambda_2(1)=g_2'$. Put $\gamma=\mu\circ(\lambda_1\times\lambda_2) : I\longrightarrow P$ defined by $\gamma(t)=\mu(\lambda_1(t),\lambda_2(t))$. Clearly γ is a path in P from $\mu(g_1,g_1')$ to $\mu(g_2,g_2')$ and hence $g_1g_1'\tilde{=}g_2g_2'$.
(*v*) By (*iv*) since $g_1g_3\tilde{=}g_2g_3$ and $g_3^{-1}\tilde{=}g_3^{-1}$, we have $(g_1g_3)g_3^{-1}\tilde{=}(g_2g_3)g_3^{-1}$. Using (*iii*), (*ii*), (*i*) we have $g_1\tilde{=}g_2$.
(*vi*) Using (*ii*) we have $(g^{-1})^{-1}g^{-1}\tilde{=}p_0\tilde{=}gg^{-1}$. Now (*v*) implies $g_1g_1'\tilde{=}g_2g_2'$.
(*vii*) By (*ii*) $(g_1g_2)^{-1}(g_1g_2)\tilde{=}p_0$. Using (*iv*) we have $(g_1g_2)^{-1}(g_1g_2)g_2^{-1}\tilde{=}p_0g_2^{-1}$ and so by (*iii*), (*ii*), (*i*) we have $(g_1g_2)^{-1}g_1\tilde{=}g_2^{-1}$. By a similar method we have $(g_1g_2)^{-1}\tilde{=}g_2^{-1}g_1^{-1}$.
(*viii*) By (*i*) $p_0^{-1}p_0\tilde{=}p_0\tilde{=}p_0p_0$. Hence by (*v*) $p_0^{-1}\tilde{=}p_0$.
(*ix*) By (*ii*) $g_1^{-1}g_1\tilde{=}p_0\tilde{=}g_2^{-1}g_2$. Since $g_1\tilde{=}g_2$, then by (*v*) $g_1^{-1}\tilde{=}g_2^{-1}$. □

Lemma 2.11. *Let (P',μ',η',c') be a sub-H-group of (P,μ,η,c), then the following statements hold.*
(i) For any $g_1,g_2\in P'$, $g_1g_2=\mu(g_1,g_2)\tilde{=}\mu'(g_1,g_2)$.
(ii) For any $g\in P'$, $g^{-1}=\eta(g)\tilde{=}\eta'(g)$.
(iii) If $g_1,g_2\tilde{\in}P'$, then $g_1g_2\tilde{\in}P'$ and $g_1^{-1}\tilde{\in}P'$.

Proof. Since (P',μ',η',c') is a sub-H-group of (P,μ,η,c), by Remark 2.7 the inclusion $i : P' \hookrightarrow P$ is an H-homomorphism of H-groups i.e $i\circ\mu' \simeq \mu\circ(i\times i)$ and $\eta\circ i \simeq i\circ\eta'$. Then there are continuous maps $H : P'\times P'\times I \longrightarrow P$ and $L : P'\times I \longrightarrow P$ such that $H(g_1,g_2,0) = i\circ\mu(g_1,g_2) = \mu(g_1,g_2)$, $H(g_1,g_2,1) = \mu'\circ(i\times i)(g_1,g_2) = \mu'(g_1,g_2)$ and $L(g,0) = \eta\circ i(g) = \eta(g)$, $L(g,1) = i\circ\eta'(g) = \eta'(g)$, for all $g_1,g_2,g \in P'$. Define $\alpha_{g_1,g_2} : I \longrightarrow P$ by $\alpha_{g_1,g_2}(t) = H(g_1,g_2,t)$ and $\beta_g : I \longrightarrow P$ by $\beta_g(t) = L(g,t)$. Then α_{g_1,g_2} is a path in P from $\mu(g_1,g_2)$ to $\mu'(g_1,g_2)$ and β_g is a path in P from $\eta(g)$ to $\eta'(g)$ and hence by Notation 2.9 we have $\mu(g_1,g_2)\widetilde{=}\mu'(g_1,g_2)$ and $\eta(g)\widetilde{=}\eta'(g)$. Hence (i) and (ii) hold.
(iii) Suppose $g_1,g_2\widetilde{\in}P'$, then there are two paths α,β in P such that $\alpha(0) = g_1$, $\alpha(1) = g_1'$, $\beta(0) = g_2$ and $\beta_(1) = g_2'$, for some $g_1',g_2' \in P'$. Then $g_1\widetilde{=}g_1'$ and $g_2\widetilde{=}g_2'$, and hence by Lemma 2.10 (iv) we have $g_1g_2\widetilde{=}g_1'g_2'$. Now by (i) we have $g_1'g_2' = \mu(g_1',g_2')\widetilde{=}\mu'(g_1,g_2) \in P'$ which implies that $g_1g_2\widetilde{\in}P'$. Also, since $g_1\widetilde{=}g_1'$, by Lemma 2.10 (ix) $g_1^{-1}\widetilde{=}{g_1'}^{-1}$ and by (ii) ${g_1'}^{-1}\widetilde{=}\eta'(g_1) \in P'$ which implies $g_1^{-1}\widetilde{\in}P'$. □

3. Quotient H-groups

In this section, we assume that (P,μ,η,c) is an H-group and (P',μ',η',c') is a sub-H-group of P and use Notation 2.9 extensively.

Definition 3.1. (i) Let P' be a sub-H-group of P and $g \in P$. Then we define the right coset of P' with representative g as follows:

$$P'g = \{g' \in P | g'g^{-1}\widetilde{\in}P'\}.$$

Similarly, the left coset of P' with representative g is defined as follows:

$$gP' = \{g' \in P | g^{-1}g'\widetilde{\in}P'\}.$$

Note that by Lemma 2.10 it is easy to see that

$$P'g = \{g' | g'\widetilde{=}p'g \text{ for some } p' \in P'\} \supseteq \{p'g\ | x \in P'\}$$

and

$$gP' = \{g' | g'\widetilde{=}gp' \text{ for some } p' \in P'\} \supseteq \{gp' | x \in P'\}.$$

(ii) Motivating by the above equalities, we define gA, Ag and AB for any $g \in P$ and any non-empty subsets A, B of P as follows:
$gA = \{g' \mid g'\widetilde{=}ga \text{ for some } a \in A\}$, $Ag = \{g' \mid g'\widetilde{=}ag \text{ for some } a \in A\}$ and $AB = \{g' \mid g'\widetilde{=}ab \text{ for some } a \in A,\ b \in B\}$.

Remark 3.2. By Lemma 2.10, one can easily show that $gA = \{g\}A$, $Ag = A\{g\}$ and $A(BC) = (AB)C$ for any $g \in P$ and any non-empty subsets A, B, C of P.

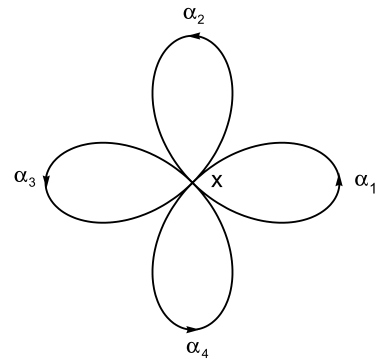

FIGURE 1.

Lemma 3.3. *For every sub-H-group* (P', μ', η', c') *of an H-group* (P, μ, η, c) *the following statements hold.*
(i) For each $g \in P$, $g \in gP'$.
(ii) For each $g_1, g_2 \in P$, $g_2^{-1} g_1 \widetilde{\in} P'$ *if and only if* $g_1 P' = g_2 P'$.
(iii) For each $g_1, g_2 \in P$ *if* $g_1 \cong g_2$, *then* $g_1 P' = g_2 P'$.

Proof. (i) By Lemma 2.10 (i) $g \cong g p_0$. Since $p_0 \in P'$, by definition we have $g \in gP'$.
(ii) If $g_1 P' = g_2 P'$, then by (i) $g_1 \in g_2 P'$. Thus by Lemma 2.11 (iii) $g_2^{-1} g_1 \widetilde{\in} P'$. Conversely, let $g' \in g_1 P'$, then $g' \cong g_1 p'$ for some $p \widetilde{\in} P'$. By Lemma 2.10

$$g_1^{-1} g' \cong g_1^{-1} (g_1 p') \cong (g_1^{-1} g_1) p' \cong p_0 p' \cong p'$$

and so $g_2^{-1} g' \widetilde{\in} P'$ which implies by the definition that $g' \in g_2 P'$. Hence $g_1 P' \subseteq g_2 P'$. Similarly $g_2 P' \subseteq g_1 P'$.
(iii) If $g_1 \cong g_2$, then by Lemma 2.10 $g_1^{-1} g_1 \cong g_1^{-1} g_2$ and so $g_1^{-1} g_2 \cong p_0 \in P'$ which implies by (ii) that $g_1 P' = g_2 P'$. □

Example 3.4. Consider loops $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in $\mathbb{R}^2$ as in Figure (1). If $X = \bigcup_{i=1}^4 Im(\alpha_i)$ and $Y = Im(\alpha_1) \bigcup Im(\alpha_3)$, then ΩY is a sub-H-group of ΩX and $\alpha_1, \alpha_3 \in \Omega Y$. Hence $\alpha_1 \Omega Y = \alpha_3 \Omega Y$, but α_1 is not homotopic to α_3. This shows that the converse of Lemma 3.3 (iii) does not hold.

The following proposition is a consequence of Lemma 2.10.

Proposition 3.5. *For any sub-H-group* P' *of* P, *the relation* $\overset{P'}{\sim}$ *on* P *defined by*

$$g_1 \overset{P'}{\sim} g_2 \Leftrightarrow g_1^{-1} g_2 \widetilde{\in} P'$$

is an equivalent relation in which gP' *is the equivalence class of* g, *for all* $g \in P$. *Moreover, we have* $(g_1 g_2) g_3 \overset{P'}{\sim} g_1 (g_2 g_3)$ *for any* $g_1, g_2, g_3 \in P$.

Note that by the above results the set of all left cosets of P' is a partition for P. Also by Lemma 3.3 (ii) $g_1 \overset{P'}{\sim} g_2$ if and only if $g_1P' = g_2P'$. Moreover, we can define the relation $\overset{P'}{\backsim}$ on P by

$$g_1 \overset{P'}{\backsim} g_2 \Leftrightarrow g_1g_2^{-1}\widetilde{\in}P'$$

which is an equivalent relation in which $P'g$ is the equivalence class of g, for all $g \in P$.

Definition 3.6. For a topological space X, we call a subset A of X *path saturated* if for each $x \in A$ the path component of X which contains x is a subset of A. If $Y \subseteq X$, then we define the path saturation of Y in X as $\widetilde{Y} = \{x \in X \mid x\widetilde{\in}Y\}$.

Note that the notion of path saturated guarantees that homotopies remain in the subsets and also, if A and B are subset of P and $g \in A$, then using Lemma 2.10, it is easy to see that gA, Ag and AB are path saturated.

Lemma 3.7. *If A is a path saturated subset of P, then for any $g \in P$ we have*

$$|\pi_0(A)| = |\pi_0(gA)| = |\pi_0(Ag)|.$$

Proof. We claim that if $g_1, g_2 \in A$ lie in different path components of P', then gg_1 and gg_2 also lie in different path components of gA, for all $g \in P$. By contrary, if gg_1 and gg_2 lie in the same path component of gA, then $gg_1 \widetilde{=} gg_2$. By Lemma 2.10 (v) we have $g_1 \widetilde{=} g_2$ which is a contradiction. Thus $|\pi_0(A)| \leq |\pi_0(gA)|$. Similarly, if $g_1, g_2 \in gA$ do not lie in the same path component, then $g^{-1}g_1$, $g^{-1}g_2\widetilde{\in}A$ do not lie in the same path component of A. Hence $|\pi_0(gA)| \leq |\pi_0(A)|$. □

Lemma 3.8. *For a sub-H-group P' of H-group P, there are as many right cosets as left cosets.*

Proof. Define $\theta : \{gP'|g \in P\} \longrightarrow \{P'g|g \in P\}$ by $gP' \mapsto P'g^{-1}$. We show that θ is a one-to-one correspondence. Let $g_1P' = g_2P'$, then $g_2^{-1}g_1\widetilde{\in}P'$ and by Lemma 2.11 $(g_2^{-1}g_1)^{-1}\widetilde{\in}P'$. By Lemma 2.10 we have $(g_2^{-1}g_1)^{-1} \widetilde{=} g_1(g_2^{-1})^{-1}$. Thus $g_1(g_2^{-1})^{-1}\widetilde{\in}P'$ and so $P'g_1^{-1} = P'g_2^{-1}$. Hence θ is well-defined. Similarly θ is one to one. Since $g \widetilde{=} (g^{-1})^{-1}$, θ is onto. □

Definition 3.9. Let P' be a sub-H-group of P, then by the above lemma we can define the index of P' in P, denoted by $[P : P']$, to be the cardinal of the set of all left (or right) cosets of P' in P.

We have the following basic result which is analogues to Lagrange theorem in group theory.

Theorem 3.10. *If P' is a path saturated sub-H-group of P and $|\pi_0(P)|$ is finite, then*

$$|\pi_0(P)| = |\pi_0(P')|[P : P'].$$

Proof. There are $[P : P']$ left cosets of P' each of which with $|\pi_0(P')|$ path components by Lemma 3.7. If $g_1P' \neq g_2P'$, then by Lemma 3.3 (iii) g_1 and g_2 are not in the same path component. Hence the result holds. □

Proposition 3.11. *If P' is a path saturated sub-H-group of P, then $\pi_0(P')$ is a subgroup of $\pi_0(P)$.*

Proof. Note that since P' is a path saturated subset of P, every path component in P' is in fact a path component of P i.e. $\pi_0(P') \subseteq \pi_0(P)$. For any $g \in P$ we denote the path component of P containing g by $[g]$. Let $[g_1], [g_2] \in \pi_0(P')$, then $g_1, g_2 \in P'$ and by Lemma 2.11 $g_1g_2^{-1} \widetilde{\in} P'$. Hence $[g_1][g_2]^{-1} = [g_1g_2^{-1}] \in \pi_0(P')$. □

Example 3.12. Note that in the above proposition the hypothesis "path saturatedness" is essential, for if $X = \mathbb{R}^2$ and Y is as in Example 3.4, then $\pi_0(\Omega(X)) = 1$ and $\pi_0(\Omega(Y)) = \mathbb{Z} * \mathbb{Z}$, where 1 is trivial group and $Z * Z$ is the free product of two copies of $\mathbb{Z}$.

Theorem 3.13. *A path saturated pointed subset (A, p_0) of P which is closed under inherited multiplication and inversion is a sub-H-group of P.*

Proof. Let $\mu_A = \mu|_{A\times A}$ and $\eta_A = \eta|_A$ be as multiplication and inversion of A. By Remark 2.7 and Theorem 2.8, it suffices to show that $i : A \hookrightarrow P$ is monic. Let $h_1, h_2 : Z \longrightarrow A$ such that $i \circ h_1 \simeq i \circ h_2$ by a homotopy $H : Z \times I \longrightarrow P$. Since $H(z, 0), H(z, 1) \in A$ and path components of A and P coincide, $H(z, t) \in A$, for all $z \in Z, t \in I$. Hence the result holds. □

Theorem 3.14. *For every subgroup K of $\pi_0(P)$, there exists a sub-H-group P_K of P such that $\pi_0(P_K) = K$.*

Proof. Define $P_K = \{g \in P | \ [g] \in K\}$. We show that P_K is a sub-H-group of P. Let $x, y \in P_K$, then $[x], [y] \in K$. Since K is a subgroup of $\pi_0(P)$, $[x][y] = [xy] \in K$ and $[x]^{-1} = [x^{-1}] \in K$ which implies $xy,\ x^{-1} \in P_K$. Therefore P_K is closed under inherited multiplication and inversion. Hence by Theorem 3.13 P_K is a sub-H-group. □

The following corollary is a consequence of definitions.

Corollary 3.15. *Let P' be a path saturated sub-H-group of P and $K = \pi_0(P')$. Then using notation of Theorem 3.14 we have $P' = P_K$.*

Note that if P' and P'' are sub-H-groups of P, then $P'P''$ do not need to be an H-group since $(p'_1p''_1)(p'_2p''_2)$ is not necessarily connected to $(p'_1p'_2)(p''_1p''_2)$ by a path, where $p'_1, p'_2 \in P'$ and $p''_1, p''_2 \in P''$. For example, consider $Z = Im(\alpha_4)$ in Example 3.4, then ΩZ and ΩY are sub-H-groups of ΩX and $(\alpha_1 * \alpha_4) * (\alpha_3 * \alpha_4)$ is not homotopic to $(\alpha_1 * \alpha_3) * (\alpha_4 * \alpha_4)$. But if P is an Abelian H-group, then $P'P''$ will be an H-group. In the following proposition we have a useful generalization of this observation.

Proposition 3.16. *If P' and P'' are sub-H-groups of P, then $P'P''$ is a sub-H-group of P if and only if $P'P'' = P''P'$.*

Proof. First suppose that $P'P''$ is a sub-H-group of P. By definition we have $\pi_0(P'P'') = \{[g'] | g' \cong ab \text{ for some } a \in P', b \in P''\} = \{[a][b] | a \in P', b \in P''\} = \pi_0(P')\pi_0(P'')$. Put $H = \pi_0(P')$ and $K = \pi_0(P'')$, then $\pi_0(P'P'') = HK$ and similarly $\pi_0(P''P') = KH$. Since $P'P''$ is a path saturated sub-H-group of P, by Proposition 3.11 HK is a subgroup of $\pi_0(P)$ and hence $HK = KH$. Since $P'P''$ is a path saturated sub-H-group of P, by Corollary 3.15 $P'P'' = P_{HK}$. By the proof of Theorem 3.14 it is easy to see that $P_{HK} = P_{KH} = P''P'$. Hence $P'P'' = P''P'$. Conversely, suppose that $P'P'' = P''P'$. In order to show that $P'P''$ is a sub-H-group of P, by Theorem 3.13 it is enough to show that $P'P''$ is closed under inherited multiplication and inversion. For any $g'_1, g'_2 \in P'P''$ there are $a_1, a_2 \in P'$ and $b_1, b_2 \in P''$ such that $g'_1 \cong a_1b_1$ and $g'_2 \cong a_2b_2$. By Lemma 2.10 we have $g'_1g'_2 \cong (a_1b_1)(a_2b_2) \cong (a_1(b_1a_2))b_2$. Since $P'P'' = P''P'$, there are $a'_2 \in P'$ and $b'_1 \in P''$ such that $b_1a_2 \cong a'_2b'_1$. Now by Lemma 2.10 we have $g'_1g'_2 \cong (a_1(b_1a_2))b_2 \cong (a_1(a'_2b'_1))b_2 \cong (a_1a'_2)(b'_1b_2) \in P'P''$. Also by Lemma 2.10, we have $(g'_1)^{-1} \cong (a_1b_1)^{-1} \cong b_1^{-1}a_1^{-1} \in P''P' = P'P''$. Hence the result holds. □

Lemma 3.17. *If P' and P'' are sub-H-groups of P, then the following statements hold.*
(i) $P'P' = \widetilde{P'}$.
(ii) $gP' = g\widetilde{P'} (= \widetilde{gP'})$, *for each* $g \in P$.
(iii) $P'P'' = \widetilde{P'}\widetilde{P''} (= \widetilde{P'P''})$.
(iv) $(g_1P')(g_2P') = g_1((P'g_2)P') = (g_1(P'g_2))P'$, *for each* $g_1, g_2 \in P$.
(v) $g'P' = \widetilde{P'}$, *for each* $g' \widetilde{\in} P'$.

Proof. (i) For any $g' \in P'P'$ there are $a, b \in P'$ such that by Lemma 2.11 $g' \cong ab \cong \mu'(a, b) \in P'$, where μ' is the multiplication of P'. Hence $P'P' \subseteq \widetilde{P'}$. Conversely, let $g' \in \widetilde{P'}$, then $g' \widetilde{\in} P'$ and so there is $a \in P'$ such that $g' \cong a$. Hence $g' \cong a \cong ap_0 \in P'P'$.
(ii) Clearly $gP' \subseteq g\widetilde{P'}$. For any $g' \in g\widetilde{P'}$ there is $a \in \widetilde{P'}$ such that $g \cong ga$. Also there is $a' \in P'$ such that $a \cong a'$. Hence by Lemma 2.10

$g'\widetilde{=}ga\widetilde{=}ga' \in gP'$.
(iii) Clearly $P'P'' \subseteq \widetilde{P'}\widetilde{P''}$. For any $g' \in \widetilde{P'}\widetilde{P''}$ there are $a \in \widetilde{P'}$ and $b \in \widetilde{P''}$ such that $g'\widetilde{=}ab$. Also, there are $a' \in P'$ and $b' \in P''$ such that $a\widetilde{=}a'$ and $b'\widetilde{=}b$. Hence by Lemma 2.10 $g'\widetilde{=}ab\widetilde{=}a'b' \in P'P''$.
(iv) It follows by Remark 3.2.
(v) For any $g'' \in g'P'$ there is $a \in P$ such that $g''\widetilde{=}g'a$. Since $g'\widetilde{\in}P'$ there is $b \in P'$ such that $g'\widetilde{=}b$. Hence by Lemmas 2.10 and 2.11 $g''\widetilde{=}g'a\widetilde{=}ba\widetilde{=}\mu'(b,a) \in P'$ and so $g'' \in \widetilde{P'}$. Conversely, for any $g'' \in \widetilde{P'}$ there is $d \in P'$ such that $g''\widetilde{=}d\widetilde{=}p_0d\widetilde{=}g'(g')^{-1}d\widetilde{=}g'(b^{-1}d) \in g'P'$. □

Remark 3.18. If P' is a sub-H-group of P, then we can multiply g_1P' and g_2P' as $(g_1P')(g_2P')$ and it seems natural to hope that $(g_1P')(g_2P') = (g_1g_2)P'$. But this does not always happen. As an example, put $P' = \Omega Y$, $g_1 = \alpha_2$ and $g_2 = \alpha_4$ in Example 3.4, then $\alpha_2 * \alpha_1 * \alpha_4 * \alpha_3 \in (g_1P')(g_2P')$, but $\alpha_2 * \alpha_1 * \alpha_4 * \alpha_3 \notin (g_1.g_2)P'$. The following lemma gives us one possible criterion.

Lemma 3.19. *If P' is a sub-H-group of P, then the following two conditions are equivalent.*
(i) $(g_1P')(g_2P') = (g_1g_2)P'$, for all $g_1, g_2 \in P$;
(ii) $gP' = P'g$ (or equivalently $(g^{-1}P')g = \widetilde{P'} = g^{-1}(P'g)$), for all $g \in P$.

Proof. Let (ii) hold, then by Lemma 3.17 we have $(g_1P')(g_2P') = g_1((P'g_2)P')$ $= g_1((g_2P')P') = g_1(g_2(P'P')) = g_1(g_2P') = (g_1g_2)P'$. Conversely, let (i) hold. Then $(g^{-1}P')g \subseteq ((g^{-1}P')g)P' = (g^{-1}P')(gP') = (g^{-1}g)P' = \widetilde{P'}$. This implies that $gP' \subseteq \widetilde{P'}g = P'g$. Since this containment holds for all $g \in P$, we have $P'g \subseteq gP'$, and hence the result follows. □

Note that by Remark 3.2 $(g^{-1}P')g = g^{-1}(P'g)$, hence we can use $g^{-1}P'g$ instead of $(g^{-1}P')g$ or $g^{-1}(P'g)$. Also, note that if $g^{-1}P'g \subseteq \widetilde{P'}$, for all $g \in P$, then we have $gP'g^{-1} = \widetilde{P'}$, for all $g \in P$.

Definition 3.20. Let P' be a sub-H-group of P. Then we call P' a normal sub-H-group of P, denoted by $P' \trianglelefteq P$, if and only if $g^{-1}P'g \subseteq \widetilde{P'}$, for all $g \in P$ (or equivalently $(g^{-1}g')g\widetilde{=}g^{-1}(g'g)\widetilde{\in}P'$ for each $g \in P$ and $g' \in P'$). Also, we define the quotient of P by P', denoted by P/P' as follows:

$$P/P' = \{gP' \mid g \in P\}.$$

Theorem 3.21. *If P' is a normal sub-H-group of P, then P/P' is a group in which the coset $p_0P'(=\widetilde{P'})$ is the identity element.*

Proof. As a binary operation define $(g_1P')(g_2P') = (g_1g_2)P'$. If $g_1P' = g_2P'$, $h_1P' = h_2P'$, then by Lemma 3.3 $g_1^{-1}g_2\widetilde{\in}P'$, $h_1^{-1}h_2\widetilde{\in}P'$. Normality of P' and Lemmas 2.10 and 2.11 guaranties that

$$(g_1h_1)^{-1}(g_2h_2)\widetilde{=}(h_1^{-1}g_1^{-1})(g_2h_2)\widetilde{=}(h_1^{-1}(g_1^{-1}g_2))h_1\widetilde{=}(h_1^{-1}(g_1^{-1}g_2)h_1)(h_1^{-1}h_2)\widetilde{\in}P'$$

which implies that $(g_1h_1)P' = (g_2h_2)P'$. Therefore the above binary operation is well-defined. Associativity follows from Remark 3.2. By Lemmas 2.10 and 3.3 $(gP')(p_0P') = (gp_0)P' = gP' = (p_0g)P' = (p_0P')(gP')$, for all $g \in P$. Hence p_0P' is the identity element. Finally, By Lemmas 2.10 and 3.3 we have $(gg^{-1})P' = p_0P' = (g^{-1}g)P'$, for all $g \in P$. Hence $g^{-1}P'$ is the inverse of gP'. □

It is easy to that if P' is a normal sub-H-group of P, then so does $\widetilde{P'}$.

Lemma 3.22. *If P' is a normal sub-H-group of P, then $P/P' \cong P/\widetilde{P'}$.*

Proof. Using Lemma 3.17 (ii) and the fact that $p_0P' = \widetilde{P'}$ is identity element of P/P', the result holds. □

Theorem 3.23. *If P' is a sub-H-group of P and P'' is a sub-H-group of P', then the following statements hold.*
(i) If P' is a path saturated normal sub-H-group of P, then $\pi_0(P')$ is a normal subgroup of $\pi_0(P)$.
(ii) P'' is a sub-H-group of P.
(iii) If P'' is normal in P and P', then P'/P'' is a subgroup of P/P''. Also, P'/P'' is a normal subgroup of P/P'' if and only if P' is a normal sub-H-group of P.

Proof. Using definitions and Lemmas 2.10, 2.11 and 3.3 the results hold. □

Note that If P'' is normal in P, then it is not necessarily normal in P'. For example, by the notations of Example 3.4, ΩY is normal in $\Omega\mathbb{R}^2$, but not in ΩX.

Lemma 3.24. *The path component of P that contains p_0, named principle component of P which is denoted by P_0, is a normal sub-H-group of P and $\pi_0(P) \simeq P/P_0$.*

Proof. Clearly $P_0 = \widetilde{\{p_0\}}$ so the first claim follows by Lemma 2.10. For the second claim, define $\theta : \pi_0(P) \longrightarrow P/P_0$ by $\theta([g]) = gP_0$ which is easily a group isomorphism. □

4. H-homomorphisms

In this section, we assume that (P, μ_1, η_1, c_1) and (Q, μ_2, η_2, c_2) are two H-groups with based points p_0 and q_0, respectively. We also recall that $\varphi : P \longrightarrow Q$ is an H-homomorphism if $\mu_2 \circ (\varphi \times \varphi) \simeq \varphi \circ \mu_1$ rel$\{(p_0, p_0)\}$ and $\varphi \circ \eta_1 \simeq \eta_2 \circ \varphi$ rel$\{p_0\}$.

Lemma 4.1. *Let $\varphi : (P, \mu_1, \eta_1, c_1) \longrightarrow (Q, \mu_2, \eta_2, c_2)$ be an H-homomorphism, then $\mu_2(\varphi(a), \varphi(b)) \widetilde{=} \varphi(\mu_1(a, b))$ and $\varphi(\eta_1(a)) \widetilde{=} \eta_2(\varphi(a))$, for all $a, b \in P$.*

Proof. Since $\mu_2 \circ (\varphi \times \varphi) \simeq \varphi \circ \mu_1$ rel$\{(p_0, p_0)\}$, there is a continuous map $F : P \times P \times I \longrightarrow Q$ such that $F(a, b, 0) = \mu_2 \circ (\varphi \times \varphi)(a, b)$ and $F(a, b, 1) = \varphi \circ \mu_1(a, b)$. Hence $\lambda : I \longrightarrow Q$ defined by $\lambda(t) = F(a, b, t)$ is path in Q from $\mu_2(\varphi(a), \varphi(b))$ to $\varphi(\mu_1(a, b))$. Also, since $\varphi \circ \eta_1 \simeq \eta_2 \circ \varphi$ rel$\{p_0\}$, there is a continuous map $H : P \times I \longrightarrow Q$ such that $H(a, 0) = \varphi \circ \eta_1(a)$ and $H(a, 1) = \eta_2 \circ \varphi(a)$. Hence $\gamma : I \longrightarrow Q$ defined by $\gamma(t) = H(a, t)$ is a path in Q from $\varphi(\eta_1(a))$ and $\eta_2(\varphi(a))$. □

In order to simplify the notation we use ab instead of both $\mu_1(a, b)$ and $\mu_2(a, b)$, and a^{-1} instead of both $\eta_1(a)$ and $\eta_2(a)$ if there is no ambiguity. Using these notations and the above lemma, we have $\varphi(ab) \widetilde{=} \varphi(a)\varphi(b)$ and $\varphi(a^{-1}) \widetilde{=} (\varphi(a))^{-1}$, for all $a, b \in P$ and any H-homomorphism $\varphi : P \longrightarrow Q$.

Definition 4.2. Let $\varphi : P \longrightarrow Q$ be an H-homomorphism. We define the *kernel* of φ as

$$ker\varphi = \{g \in P |\ \varphi(g) \widetilde{=} q_0\},$$

where q_0 is the based point of Q.

Proposition 4.3. *Let $\varphi : (P, \mu_1, \eta_1, c_1) \longrightarrow (Q, \mu_2, \eta_2, c_2)$ be an H-homomorphism. Then $ker\varphi$ is a path saturated normal sub-H-group of P.*

Proof. Let $a, b \in ker\varphi$, then by Lemmas 2.10 and 4.1 we have $\varphi(ab) \widetilde{=} \varphi(a)\varphi(b) \widetilde{=} q_0 q_0 \widetilde{=} q_0$ and $\varphi(a^{-1}) \widetilde{=} (\varphi(a))^{-1} \widetilde{=} q_0^{-1} \widetilde{=} q_0$ which imply that $ker\varphi$ is closed under multiplication and inversion of P. By definition of $ker\varphi$, it is easy to see that it is path saturated. Now by Theorem 3.13 $ker\varphi$ is a sub-H-group of P. Finally by Lemma 2.10 we have $\varphi(g^{-1}(g'g)) \widetilde{=} (\varphi(g))^{-1}(\varphi(g')\varphi(g)) \widetilde{=} (\varphi(g))^{-1}(q_0\varphi(g)) \widetilde{=} (\varphi(g))^{-1}\varphi(g) \widetilde{=} q_0$, for all $g \in P$ and $g' \in ker\varphi$ which implies $ker\varphi \trianglelefteq P$. □

Let $\varphi : P \longrightarrow Q$ be an H-homomorphism, $A \subseteq P$ and $B \subseteq P'$, then by definition

$$\widetilde{\varphi(A)} = \{q \in Q \mid q \widetilde{\in} \varphi(A)\},$$
$$\widetilde{\varphi^{-1}(B)} = \{p \in P \mid \varphi(p) \widetilde{\in} B\}.$$

Now we can state the following useful lemma which is proved by a similar proof of Proposition 4.3.

Lemma 4.4. *Let $\varphi : (P, \mu_1, \eta_1, c_1) \longrightarrow (Q, \mu_2, \eta_2, c_2)$ be an H-homomorphism. Then*
(i) If $(P', \mu_1', \eta_1', c_1')$ is a sub-H-group of P, then $\widetilde{\varphi(P')}$ is a saturated sub-H-group of Q;

(ii) If $(Q', \mu_2', \eta_2', c_2')$ is a sub-H-group of Q, then $\widetilde{\varphi^{-1}(Q')}$ is a sub-H-group of P. If Q' is normal, then so is $\widetilde{\varphi^{-1}(Q')}$.

Suppose that N is a normal sub-H-group of P, φ is an H-homomorphism from P to Q and π is the natural map from P to P/N. We would like to find an induced H-homomorphism $\overline{\varphi} : P/N \longrightarrow Q$ such that $\overline{\varphi}(gN) = \varphi(g)$. But there is no meaning of H-homomorphism for $\overline{\varphi}$ because P/N is not necessarily an H-group related to P. Note that although we can assume every abstract group as a topological group by discrete topology, but it is prevalent that topology of P/N must be related to the topology of P. By using the functor π_0, we can overcome this problem and have some results as follow in the category of groups. In Section 5, we will endow P/N by the quotient topology induced from P and prove that P/N by this topology is a quasitopological group in the sense of [1] and the rest of results in this section hold in the category of quasitopological groups.

For the canonical map $\pi : P \longrightarrow P/N$, let $\overline{\pi} : \pi_0(P) \longrightarrow P/N$ defined by $\overline{\pi}([g]) = gN$. Here is the key result.

Theorem 4.5. *For any H-homomorphism $\varphi : P \longrightarrow Q$ whose kernel $ker\varphi = K$ contains a normal sub-H-group N of P, $\pi_0(\varphi)$ can be factored through P/N. In other words, there is a unique group homomorphism $\overline{\pi_0(\varphi)} : P/N \longrightarrow \pi_0(Q)$ such that $\overline{\pi_0(\varphi)} \circ \overline{\pi} = \pi_0(\varphi)$, i.e. the following diagram is commutative:*

$$\begin{array}{ccc} \pi_0(P) & \xrightarrow{\pi_0(\varphi)} & \pi_0(Q) \\ {\scriptstyle \overline{\pi}}\downarrow & \nearrow {\scriptstyle \overline{\pi_0(\varphi)}} & \\ \frac{P}{N}. & & \end{array}$$

Furthermore,
(i) $\overline{\pi_0(\varphi)}$ is an epimorphism if $\pi_0(\varphi)$ is onto;
(ii) $\overline{\pi_0(\varphi)}$ is a monomorphism if and only if $K = \widetilde{N}$.

Proof. Define $\overline{\pi_0(\varphi)} : P/N \longrightarrow \pi_0(Q)$ by $\overline{\pi_0(\varphi)}(gN) = [\varphi(g)]$. If $g_1N = g_2N$, then $g_2^{-1}g_1 \widetilde{\in} N \subseteq K$. Hence $\varphi(g_2^{-1}g_1) \cong q_0$ which implies that $\varphi(g_1) \cong \varphi(g_2)$ and so $[\varphi(g_1)] = [\varphi(g_2)]$. Also $\overline{\pi_0(\varphi)}((g_1N)(g_2N)) = [\varphi(g_1g_2)] = [\varphi(g_1)][\varphi(g_2)]$. Clearly the diagram is commutative and $\overline{\pi_0(\varphi)}$ is unique.
(i) It follows from commutativity of the diagram.
(ii) Assume $\overline{\pi_0(\varphi)}$ is monomorphism. Since K is path saturated and contains N, we have $\widetilde{N} \subseteq K$. Let $g \in K$, then $\varphi(g) \cong q_0$ and so $\overline{\pi_0(\varphi)}(gN) = [q_0]$. By injectivity of $\overline{\pi_0(\varphi)}$, $gN = \widetilde{N}$ and therefore

$K = \widetilde{N}$. The converse is trivial.

□

The factor theorem yields the following fundamental result .

Theorem 4.6. *(The First H-isomorphism Theorem). If $\varphi : P \longrightarrow Q$ is an H-homomorphism with kernel K, then $\pi_0(\widetilde{\varphi(P)})$ is isomorphic to P/K.*

Proof. Consider $\theta : P/K \longrightarrow \pi_0(\widetilde{\varphi(P)})$ by $\theta(gK) = [\varphi(g)]$. Since φ is an H-homomorphism, θ is well defined and homomorphism. For any $[q] \in \pi_0(\widetilde{\varphi(P)})$ there exist $p \in P$ such that $q \tilde{=} \varphi(p)$ and so $\theta(pK) = [\varphi(p)] = [q]$. Hence θ is onto. Also, if $\theta(gK) = [\varphi(g)] = [q_0]$, then $\varphi(g) \tilde{=} q_0$ and hence θ is injective. □

If M and N are path saturated sub-H-groups of P, $G_1 = \pi_0(M)$ and $G_2 = \pi_0(N)$, then using Theorem 3.15 $M \cap N = P_{G_1 \cap G_2}$ that is a sub-H-group of P.

Lemma 4.7. *Let M and N be path saturated sub-H-groups of P and $N \trianglelefteq P$. Then the following statements hold.*
(i) $MN = NM$ and MN is a sub-H-group of P;
(ii) N is a normal sub-H-group of MN;
(iii) $M \cap N$ is a normal sub-H-group of M.

Proof. Lemma 3.19, Proposition 3.16 and normality of N imply (i). Since N and MN are path saturated and $\pi_0(MN) = \pi_0(M)\pi_0(N)$, by Corollary 3.15 and (i) $\pi_0(N)$ is a normal subgroup of $\pi_0(M)\pi_0(N)$ which implies that N is a normal sub-H-group of MN. The proof of (iii) is similar to (ii). □

Theorem 4.8. *(The Second H-isomorphism Theorem). If M and N are path saturated sub-H-groups of P and $N \trianglelefteq P$, then*

$$M/M \cap N \cong MN/N.$$

Proof. Define $\theta : M/M \cap N \longrightarrow MN/N$ by $\theta(g(M \cap N)) = gN$. It is routine to check that θ is a well defined group homomorphism. If $\theta(g(M \cap N)) = gN = p_0N$, then $g \tilde{\in} N$ (equivalently $g \in N$ since N is path saturated) and hence θ is a monomorphism. Assume $gN \in MN/N$. By definition of MN, there exist $m \tilde{\in} M$ and $n \tilde{\in} N$ such that $g \tilde{=} mn$. Hence $\theta(m(M \cap N)) = mN = mnN = gN$ which implies that θ is an epimorphism. □

Theorem 4.9. *(The Third H-isomorphism Theorem). If N and M are path saturated normal sub-H-groups of P and N is contained in M, then*

$$P/M \cong \frac{P/N}{M/N}.$$

Proof. Define $\theta : P/N \longrightarrow P/M$ by $\theta(gN) = gM$ which is a group epimorphism with kernel M/N. □

Now suppose that N is a normal sub-H-group of P. If M is a path saturated sub-H-group of P containing N, then there is a natural analogue of M in the quotient H-group P/N, namely the subgroup M/N . In fact, we can make this correspondence precisely. Let Ψ be a map from the set of path saturated sub-H-groups of P containing N to the set of subgroups of P/N by $\Psi(M) = M/N$. We claim that Ψ is a bijection. For, if $M_1/N = M_2/N$, then for any $m_1 \in M_1$, we have $m_1N = m_2N$, for some $m_2 \in M_2$, so that $m_2^{-1}m_1 \widetilde{\in} N$ which is contained in M_2. Thus $M_1 \subseteq M_2$, and by symmetry the reverse inclusion holds, so that $M_1 = M_2$ and Ψ is injective. Now, if G is a subgroup of P/N and $\pi : P \longrightarrow P/N$ is the canonical map, then

$$\pi^{-1}(G) = \{p \in P \mid pN \in G\}$$

is a path saturated sub-H-group of P containing N, and $\Psi(\pi^{-1}(G)) = \{pN \mid pN \in G\} = G$ proving surjectivity of Ψ. The map Ψ has a number of other interesting properties, summarized in the following result.

Theorem 4.10. *(The Correspondence Theorem). If N is a normal sub-H-group of P, then the above map Ψ sets up a one-to-one correspondence between path saturated sub-H-groups of P containing N and subgroups of P/N. The inverse of Ψ is the map $\Phi : G \mapsto \pi^{-1}(G)$, where π is the canonical map of P to P/N. Moreover, the following statements hold.*
(i) M_1 is a sub-H-group of M_2 if and only if $M_1/N \leq M_2/N$, and in this case we have

$$[M_2 : M_1] = [M_2/N : M_1/N].$$

(ii) If M is a normal sub-H-group of P, then M/N is a normal subgroup of P/N.
(iii) M_1 is a normal sub-H-group of M_2 if and only if M_1/N is a normal subgroup of M_2/N , and in this case,

$$M_2/M_1 \cong \frac{M_2/N}{M_1/N}.$$

We introduced monics, epics and H-homomorphisms in $hTop_*$ in Section 2. Now we define H-morphisms.

Definition 4.11. (i) An *H-monomorphism* is a monic H-homomorphism.
(ii) An *H-epimorphism* is an epic H-epimorphism.
(iii) An *H-endomorphism* is an H-homomorphism of an H-group to itself.
(iv) An *H-automorphism* is an H-isomorphism of an H-group to itself.

5. Topological view

In this section (P, μ, η, c) is an H-group , (P', μ', η', c') is a sub-H-group of P and P/P' is the set of all left cosets of P' in P. We intend to topologized the set P/P' by the quotient topology induced by the canonical map $q : P \longrightarrow P/P'$ which makes it a quasitopological group. Also, we study the path component space of an H-group and find out some conditions for significance of semilocally 0-connectedness introduced in [3].

As introduced in [9], the path component space of a topological space X is $\pi_0(X)$ with the quotient topology with respect to the quotient map $q' : X \longrightarrow \pi_0(X)$, where $q'(x) = [x]$ which is denoted by $\pi_0^{qtop}(X)$. Also a continuous map $f : X \longrightarrow Y$ induces a continuous map $\pi_0^{qtop}(f) : \pi_0^{qtop}(X) \longrightarrow \pi_0^{qtop}(Y)$ taking the path component containing x in X to the path component containing $f(x)$ in Y.

Definition 5.1. A space X is semilocally 0-connected if for each point $x \in X$, there is an open neighborhood U of x such that the induced map $\pi_0^{qtop}(i) : \pi_0^{qtop}(U) \longrightarrow \pi_0^{qtop}(X)$ by the inclusion $i : U \longrightarrow X$ is a constant map (see [3, Definition 2.1]).

Proposition 5.2. *A space X is semilocally 0-connected if and only if each path component of X is open.*

Proof. Let $X = \bigsqcup_{i \in I} X_i$, where X_i's are path components of X. For an arbitrary x there is $j \in I$ such that $x \in X_j$. Since X is semilocally 0-connected, there exists an open neighborhood U of x such that $\pi_0^{qtop}(i) : \pi_0^{qtop}(U) \longrightarrow \pi_0^{qtop}(X)$ is a constant map, or equivalently U meets just one path component of X which implies $U \subseteq X_j$. Conversely, if each path component of X is open, then put U to be the path component containing x. □

Remark 5.3. Obviously locall path connectivity follows semilocally 0-connectedness. Also, X is semilocally 0-connected if and only if $\pi_0^{qtop}(X)$ has the discrete topology (see [3, Remark 2.2]).

Let P be an H-group with the based point p_0 and $P_0 = \widetilde{\{p_0\}}$ be the principal component of P. Then by Lemma 3.24, P_0 is a path saturated normal sub-H-group of P and $\theta : P/P_0 \longrightarrow \pi_0(P)$ defined by $\theta(gP_0) = [g]$ is a group isomorphism. By topologizing P/P_0 by the quotient topology induced by the canonical map $q : P \longrightarrow P/P'$ we can get more result as follows.

Theorem 5.4. *The group isomorphism $\theta : P/P_0 \longrightarrow \pi_0^{qtop}(P)$ is a homeomorphism.*

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{1_P} & P \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ P/P_0 & \xrightarrow{\theta} & \pi_0^{qtop}(P). \end{array}$$

Since q and q' are quotient maps, the result holds. □

Let P' be a sub-H-group of P and P/P' be the set of all left cosets of P' endowed with the quotient topology induced from P by $\pi : P \longrightarrow P/P'$. Some facts about the canonical map π are collected in the following.

Proposition 5.5. *With the above assumption we have*
(i) π is an onto continuous map.
(ii) If P is semilocally 0-connected, then π is open .

Proof. (i) It is obvious and follows by the definition of quotient topology. For (ii), let U be open in P. We must show that $\pi(U)$ is open in P/P' i.e. $\pi^{-1}(\pi(U))$ is open in P. We have $\pi^{-1}(\pi(U)) = \widetilde{U} = \bigcup_{\alpha\in J} O_\alpha$, where O_α's are path components of P that intersect U. Semilocally 0-connectivity of P implies that O_α's are open and hence $\pi^{-1}(\pi(U))$ is open, as desired. □

Theorem 5.6. *Let N be a normal sub-H-group of P. Then P/N is a homogeneous space.*

Proof. For any $a \in P$ define $L_{aN} : P/N \longrightarrow P/N$ by $L_{aN}(gN) = (ag)N$. Then it is easy to check that L_{aN} is well-defined mapping of P/N onto itself. Continuity of L_{aN} comes from the continuity of $L_a : P \longrightarrow P$ defined by $L_a(g) = ag$, the quotient map $\pi : P \longrightarrow P/N$ and the following commutative diagram.:

$$\begin{array}{ccc} P & \xrightarrow{L_a} & P \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ P/N & \xrightarrow{L_{aN}} & P/N. \end{array}$$

Applying the previous argument to a^{-1} we get ${L_{aN}}^{-1} = L_{a^{-1}N}$ which is continuous. Hence L_{aN} is a homeomorphism. Therefore P/N acts on itself by left and right translation $(R_{aN}(gN) = (ga)N)$ as a group of self homeomorphisms. Clearly these actions are both transitive, and hence the result holds. □

Note that L_a is not necessarily a homeomorphism because $L_a \circ (L_a)^{-1} = L_a \circ L_{a^{-1}}$ is homotopic to 1_P but is not equal to 1_P. However, fortunately L_a's are homotopy equivalence.

Theorem 5.7. *Let N be a normal sub-H-group of P. Then P/N is a quasitopological group.*

Proof. It was proved in the previous theorem that all translations are continuous. Continuity of the inversion $\overline{\eta} : P/n \longrightarrow P/N$ defined by $\overline{\eta}(gN) = g^{-1}N$ follows from the quotient map $q : P \longrightarrow P/N$, the continuity of the homotopy inversion $\eta : P \longrightarrow P$ and the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\eta} & P \\ \pi \downarrow & & \downarrow \pi \\ P/N & \xrightarrow{\overline{\eta}} & P/N. \end{array}$$

□

Corollary 5.8. π_0^{qtop} *is a functor from the category of H-groups to the category of quasitopological groups.*

Theorem 5.9. *Let P be a semilocally 0-connected H-group with N as normal sub-H-group. Then P/N is a topological group.*

Proof. By Proposition 5.5, π is a continuous open map which implies that $\pi \times \pi$ is a quotient map. Hence the following commutative diagram shows that the multiplication in P/N is continuous:

$$\begin{array}{ccc} P \times P & \xrightarrow{\mu} & P \\ \pi\times\pi \downarrow & & \downarrow \pi \\ P/N \times P/N & \xrightarrow{\overline{\mu}} & P/N. \end{array}$$

The result holds by Theorem 5.7. □

Theorem 5.10. *Let P be a semilocally 0-connected H-group with normal sub-H-group N, then P/N is a discrete topological group.*

Proof. Since P is semilocally 0-connected, by Proposition 5.2 $\widetilde{N}$ is open which implies that the identity element in the topological group P/N is open. Hence P/N has discrete topology. □

If we consider quotients of H-groups and path component spaces by quotient topology as described above, then all group homomorphisms and group isomorphisms in Section 4 hold in the category of quasitopological group and continuous homomorphism.

6. Revisiting of topological fundamental group

For $n \geq 1$, $\pi_n^{qtop}(X,x)$ is the familiar n-th homotopy group endowed with the quotient topology inherited from the path components of based n-loops in X with the compact-open topology [2, 3, 4, 5, 7, 8].

In this section we reprove some results in topological homotopy groups and topological fundamental groups by using advantages of Section 5 which can be found in [3] and [5].

Theorem 6.1. *If X is a path connected topological space, then $\pi_n^{top}(X,x) \cong \pi_n^{top}(X,y)$ as quasitopological groups, for each $x, y \in X$ and $n \geq 1$.*

Proof. By Example 2.2, α^+ is an H-isomorphism between $\Omega(X,x)$ and $\Omega(X,y)$, where α is a path from x to y. Since π_0^{qtop} is a functor, $\pi_0^{qtop}(\alpha^+)$ is an equivalence morphism in the category of quasitopological groups. Therefore $\pi_1^{qtop}(X,x) \cong \pi_1^{qtop}(X,y)$. Also α^+ is a homotopy equivalence, hence $\Omega(\alpha^+) : \Omega(\Omega(X,x),c_x) \longrightarrow \Omega(\Omega(X,y),c_y)$ is an H-isomorphism and therefore a homotopy equivalence, where c_z is the constant loop at $z \in X$. Consider Ω^n as the composition of Ω with itself n times for $n \in \mathbb{N}$. For $n > 1$, we can construct by induction H-isomorphisms $\Omega^n(\alpha^+) : \Omega(\Omega^n(X,x),c_x) \longrightarrow \Omega(\Omega^n(X,y),c_y)$, where $\Omega^n(\alpha^+)(\lambda) = \Omega^{n-1}(\alpha^+) \circ \lambda$. Since π_0^{qtop} is a functor, $\pi_0^{qtop}(\Omega^n(X,c_x)) \cong \pi_0^{qtop}(\Omega^n(X,c_y))$. Therefore $\pi_n^{qtop}(X,x) = \pi_0^{qtop}(\Omega^n(X,x)) \cong \pi_0^{qtop}(\Omega^n(X,y)) = \pi_n^{top}(X,y)$, as desired. □

Theorem 6.2. *For any homotopically equivalent topological spaces (X,x) and (Y,y), we have $\pi_n^{top}(X,x) \cong \pi_n^{top}(Y,y)$ as quasitopological groups, for all $n \geq 1$.*

Proof. We know that for each $n \in \mathbb{N}$, Ω^n is a functor from the category of pointed topological spaces, Top_*, to the category of H-groups and hence Ω^n sends equivalent objects to equivalent objects. Since π_0^{qtop} is also a functor from the category of H-groups to the category of quasitopological groups, $\pi_0^{qtop}(\Omega^n(X,x)) \cong \pi_0^{qtop}(\Omega^n(Y,y))$, as desired. □

Lemma 6.3. *For any locally path connected, semilocally simply connected space X, $\Omega(X,x)$ is locally path connected, for each $x \in X$.*

Proof. Use the proof of Lemma 3.2 in [5]. □

Theorem 6.4. *For any locally path connected space X, $\pi_1^{qtop}(X,x)$ is a discrete topological group, for each $x \in X$ if and only if X is semilocally simply connected.*

Proof. Assume X is semilocally simply connected. By Lemma 6.3 $\Omega(X,x)$ is locally path connected. Hence Remark 5.3 implies that $\pi_0^{qtop}(\Omega(X,x)) \cong$

$\pi_1^{qtop}(X,x)$ is a discrete topological group. For converse see [5, Theorem 1]. □

H. Wada in [14] showed that for every m-dimensional finite polyhedron Y and locally n-connected space X, the mapping space X^Y is locally (n-m)-connected. Therefore we have the following result.

Theorem 6.5. *For every locally n-connected pointed space (X,x), the loop space $\Omega(X,x)$ is locally (n-1)-connected.*

In [7] it is shown that the topological n-th homotopy group of every locally n-connected metric space is a discrete topological group. In the following theorem we prove this result in general case, in fact without metricness.

Theorem 6.6. *For every locally n-connected space X, $\pi_n^{top}(X,x)$ is a discrete topological group, for each $x \in X$.*

Proof. By Theorem 6.5, $\Omega(X,x)$ is locally (n-1)-connected space and so $\Omega^n(X,x)$ is locally 0-connected or equivalently a locally path connected H-group. Also $\pi_0^{qtop}(\Omega^n(X,x)) \cong \pi_n^{qtop}(X,x)$. Thus $\pi_n^{qtop}(X,x)$ is a discrete topological group by Remark 5.3. □

A topological space X is called n-semilocally simply connected if for each $x \in X$ there exists an open neighborhood U of x for which any n-loop in U is nullhomotopic in X. In [7] it is proved that for locally (n-1)-connected metric spaces, discreteness of $\pi_n^{top}(X,x)$ and n-semilocally connectivity of X are equivalent. By using this fact and Theorem 6.6 we have the same result without metricness.

Corollary 6.7. *Suppose that X is a locally (n-1)-connected space and $x \in X$. Then the following are equivalent.*
(i) $\pi_n^{top}(X,x)$ is discrete.
(ii) X is n-semilocally simply connected at x.

Definition 6.8. ([13]) A non-trivial loop $\alpha : (I,\partial I) \longrightarrow (X,x)$ is called *small* if there exists a representative of the homotopy class $[\alpha] \in \pi_1(X,x)$ in every open neighborhood U of x. A space X is called small loop at $x \in X$ if every non-trivial loop $\alpha : (I,\partial I) \longrightarrow (X,x)$ is small. A non-simply connected space X is called *small loop space* if X is small loop at every $x \in X$.

Biss in [2] showed that the topological fundamental group of the Harmonic Archipelago has indiscrete topology. Z. Virk in [13] introduced a class of spaces, named small loop spaces, and constructed an example of small loop spaces by using the Harmonic Archipelago. In the next theorem we will show that the topological fundamental group of an space which is small loop at least at one point has indiscrete topology and so is a topological group. A basic account of small loop spaces may be found in [13].

Theorem 6.9. *If X is small loop at x, then $\pi_1^{qtop}(X,x)$ has indiscrete topology.*

Proof. Let X be small at $x \in X$. If there exists an open subset U of $\pi_1^{qtop}(X,x)$ such that $\varnothing \neq U \neq \pi_1^{qtop}(X,x)$, then we can assume that U contains $[c_x]$, the identity element of $\pi_1^{qtop}(X,x)$, since $\pi_1^{qtop}(X,x)$ is a quasitopological group. Let $[\alpha] \in \pi_1^{qtop}(X,x)$ such that $[\alpha] \notin U$, then $q^{-1}(U)$ is an open neighborhood of c_x in $\Omega(X,x)$ that does not contain α. There is a basic open neighborhood of c_x like $\bigcap_{i=1}^{n} < K_i, U_i >$ such that $c_x \in \bigcap_{i=1}^{n} < K_i, U_i > \subseteq q^{-1}(U)$. Let $V = \bigcap_{i=1}^{n} U_i$, then $< I, V > \subseteq q^{-1}(U)$. Note that V is a non-empty open subset of X, since $x \in U_i$, for each i=1,2,...,n. By small loop property of X at x, there exists a loop $\alpha_V : I \longrightarrow V$ such that $[\alpha] = [\alpha_V]$. But $\alpha_V \in < I, V >$ implies that $[\alpha_V] = q(\alpha_V) \in U$. Hence $[\alpha] = [\alpha_V] \in U$ which is a contradiction. □

Remark 6.10. Brazas [4] introduced a new topology on fundamental groups made them topological groups and denoted this new functor by π_1^{τ}. For a topological space X, $\pi_1^{qtop}(X,x)$ and $\pi_1^{\tau}(X,x)$ has the same underlying set and algebraic structure but different topologies. In fact, the topology of $\pi_1^{\tau}(X,x)$ is obtained by removing some open subsets of $\pi_1^{qtop}(X,x)$ to make it a topological group. Note that since the topology of $\pi_1^{\tau}(X,x)$ is coarser than the one of $\pi_1^{qtop}(X,x)$, in fact $\pi_1^{\tau}(X,x)$ and $\pi_1^{qtop}(X,x)$ have the same open subgroups [4, Corollary 3.9], and $\pi_1^{\tau}(X,x)$ is always a topological group, Theorem 6.9 holds if we replace $\pi_1^{qtop}(X,x)$ with $\pi_1^{\tau}(X,x)$.

By an n-Hawaiian like space X we mean the natural inverse limit $\varprojlim(Y_i^{(n)}, y_i^*)$, where

$$(Y_i^{(n)}, y_i^*) = \bigvee_{j \leq i} (X_j^n, x_j^*)$$

is the wedge of $X_j^{(n)}$'s in which $X_j^{(n)}$'s are (n-1)-connected, locally (n-1)-connected, n-semilocally simply connected, and compact CW spaces (see [8]). The third author et.al. in [8] proved that the topological n-th homotopy group of an n-Hawaiian like space is prodiscrete metrizable topological group for all $n > 1$. Also, they proved in [7] that for a metric space X, $\pi_n^{top}(X,x) \cong \pi_1^{qtop}(\Omega^{n-1}(X,x), c_x)$. Since weak join of metric spaces is metric, n-Hawaiian like spaces are metric which implies that $\pi_1^{qtop}(Y,y) \cong \pi_n^{top}(X,x)$, where Y is $\Omega^{n-1}(X,x)$ and $y = c_x$ for n-Hawaiian like space X. Therefore we have a family of spaces with topological fundamental groups as topological groups.

Theorem 6.11. *If $Y = \Omega^{n-1}(X,x)$, for n-Hawaiian like space X and $n > 1$, then $\pi_1^{qtop}(Y,y)$ is a topological group. Moreover, it is a prodiscrete metric space.*

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