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Construction of Toeplitz Matrices whose elements are the coefficients of univalent functions associated with q-derivative operator

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Abstract. In this paper, we find the coefficient bounds by using symmetric Toeplitz determinants for the functions belonging to the class $R(q)$.

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1. Introduction, Definitions and Notations

Let $\mathcal{A}$ indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

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Also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting functions of the form (1.1) which are also univalent, in Δ.

Let f be given by (1.1). Then $f \in \mathcal{R}$ if it satisfies the inequality

$$\Re\left(f'(z)\right) > 0, \qquad z \in \Delta.$$

The subclass $\mathcal{R}$ was studied systematically by MacGregor [6] who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part.

In the field of Geometric Functions Theory, various subclasses of analytic functions have been studied from different viewpoints. The fractional q-calculus is the important tools that are used to investigate subclasses of analytic functions. For example, the extension of the theory of univalent functions can be described by using the theory of q-calculus. Moreover, the q-calculus operators, such as fractional q-integral and fractional q-derivative operators, are used to construct several subclasses of analytic functions (see, e.g., [1, 2, 8, 13]). In a recent paper Purohit and Raina [10], investigated applications of fractional q-calculus operators to defined certain new classes of functions which are analytic in the open disk. Later, Mohammed and Darus [7] studied approximation and geometric properties of these q-operators in some subclasses of analytic functions in compact disk.

For the convenience, we provide some basic definitions and concept details of q-calculus which are used in this paper. We suppose throughout the paper that $0 < q < 1$. We shall follow the notation and terminology as in [3]. We recall the definitions of fractional q-calculus operators of complex valued function f.

Definition 1.1. Let $q \in (0, 1)$ and define

$$[n]_q = \frac{1 - q^n}{1 - q}$$

for $n \in \mathbb{N}$.

Definition 1.2. (See [5]) The q-derivative of a function f, defined on a subset of $\mathbb{C}$, is given by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & for \quad z \neq 0, \\ \\ f'(0) & for \quad z = 0. \end{cases}$$

We note that $\lim\limits_{q \to 1^-} (D_q f)(z) = f'(z)$ if f is differentiable at z. Additionally, in view of (1.1), we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q \, a_n z^{n-1}. \tag{1.2}$$

Toeplitz matrices frequently arise in many application areas and have been attracted much attention in recent years. They arise in pure mathematics: algebra, algebraic geometry, analysis, combinatorics, differential geometry, as well as in applied mathematics: approximation theory, compressive sensing, numerical integral equations, numerical integration, statistics, time series analysis, and among other areas (see, for example [14]).

The symmetric Toeplitz determinant of f for $n \geq 1$ and $q \geq 1$ is defined by Thomas and Halim [12], as follows:

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix} \qquad (a_1 = 1).$$

Note that

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix},$$

and

$$T_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

Very recently, the estimates of the Toeplitz determinant $|T_q(n)|$ for functions in $\mathcal{R}$ have been studied in [11].

2. Preliminaries

Let $\mathcal{P}$ be the class of functions with positive real part consisting of all analytic functions $p : \Delta \to \mathbb{C}$ satisfying $p(0) = 1$ and $\Re(p(z)) > 0$. The class $\mathcal{P}$ is called the class of Carathéodory function.

The following results will be required for proving our results.

Lemma 2.1. *(See [9]) If the function $p \in \mathcal{P}$, then*

$$|p_n| \leq 2 \qquad (n \in \mathbb{N} = \{1, 2, \ldots\})$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}.$$

Lemma 2.2. *(See [4]) If the function $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + x(4 - p_1^2)$$

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

Definition 2.3. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(q)$, if the following condition holds

$$\Re\left(D_q f\right)(z) > 0, \qquad z \in \Delta.$$

We note that

$$\lim_{q\to 1^-} \mathcal{R}(q) = \left\{ f \in \mathcal{A} : \lim_{q\to 1^-} \Re\left(D_q f\right)(z) > 0, \quad z \in \Delta \right\} = \mathcal{R}.$$

The aim of this work is to obtain the coefficient bounds using symmetric Toeplitz determinants $T_2(2)$, $T_2(3)$, $T_3(2)$ and $T_3(1)$ for the functions belonging to the subclass $\mathcal{R}(q)$.

3. Main Results and Their Consequences

Theorem 3.1. *Let f given by (1.1) be in the class $\mathcal{R}(q)$. Then*

$$|T_2(2)| \leq \frac{4q^2(q^2+2q+2)}{(1+2q+2q^2+q^3)^2}.$$

Proof. Let $f \in \mathcal{R}(q)$. Then there exists a function $p \in \mathcal{P}$ such that

$$\left(D_q f\right)(z) = p(z). \tag{3.1}$$

By equating the coefficients, we obtain

$$a_2 = \frac{p_1}{[2]_q}, \tag{3.2}$$

$$a_3 = \frac{p_2}{[3]_q}, \tag{3.3}$$

$$a_4 = \frac{p_3}{[4]_q}. \tag{3.4}$$

It follows from (3.2), (3.3) and (3.4) that

$$|T_2(2)| = \left|a_3^2 - a_2^2\right| = \left|\frac{p_2^2}{[3]_q^2} - \frac{p_1^2}{[2]_q^2}\right|.$$

Making use of Lemma 2.2 to express p_2 in terms of p_1, we obtain

$$\left|a_3^2 - a_2^2\right| = \left|\frac{p_1^4}{4\,[3]_q^2} - \frac{p_1^2}{[2]_q^2} + \frac{xp_1^2(4-p_1^2)}{2\,[3]_q^2} + \frac{x^2(4-p_1^2)^2}{4\,[3]_q^2}\right|.$$

Since the functions $p(z)$ and $p(e^{i\theta}z)$, $\theta \in \mathbb{R}$ are members of the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $p_1 > 0$. For convenience of notation, we take $p_1 = p$ ($p \in [0,2]$). Applying the triangle inequality with $P = (4-p^2)$, we get

$$\left|a_3^2 - a_2^2\right| \leq \left|\frac{p^4}{4\,[3]_q^2} - \frac{p^2}{[2]_q^2}\right| + \frac{|x|\,p^2 P}{2\,[3]_q^2} + \frac{|x|^2\,P^2}{4\,[3]_q^2} =: F(|x|).$$

Differentiating $F(|x|)$, one can see clearly that $F'(|x|) > 0$ on $[0,1]$, and so $F(|x|) \leq F(1)$. Hence

$$F(|x|) \leq F(1) = \left| \frac{p^4}{4\,[3]_q^2} - \frac{p^2}{[2]_q^2} \right| + \frac{1}{[3]_q^2}\left(4 - \frac{p^4}{4}\right).$$

Treating the cases when the absolute term is either positive or negative, we can show that this expression $F(|x|)$ has a maximum value $\frac{p^2}{[2]_q^2} - \frac{p^4}{4[3]_q^2}$ on $[0,2]$, when $p = 2$. □

Theorem 3.2. *Let f given by (1.1) be in the class $\mathcal{R}(q)$. Then*

$$\left|a_4^2 - a_3^2\right| \leq \frac{4}{(1+q+q^2)^2}.$$

Proof. Using (3.2), (3.3), (3.4) and Lemma 2.2 to express p_2 and p_3 in terms of p_1, we obtain, with $P = (4-p^2)$ and $R = (1-|x|^2)z$,

$$\begin{aligned}
\left|a_4^2 - a_3^2\right| \quad = \quad & \frac{1}{4}\left\{ \left(\frac{p^2P^2}{4\,[4]_q^2} + \frac{P^2}{[4]_q^2} - \frac{pP^2}{[4]_q^2} \right)|x|^4 + \left(\frac{p^2P^2}{[4]_q^2} - \frac{2pP^2}{[4]_q^2} \right)|x|^3 \right. \\
& + \left(\frac{p^2P^2}{[4]_q^2} - \frac{2P^2}{[4]_q^2} + \frac{P^2}{[3]_q^2} + \frac{p^4P}{2\,[4]_q^2} - \frac{p^3P}{[4]_q^2} + \frac{pP^2}{[4]_q^2} \right)|x|^2 \\
& + \left(\frac{p^4P}{[4]_q^2} + \frac{2pP^2}{[4]_q^2} + \frac{2p^2P}{[3]_q^2} \right)|x| \\
& \left. + \frac{P^2}{[4]_q^2} + \frac{p^3P}{[4]_q^2} + \left| \frac{p^6}{4\,[4]_q^2} - \frac{p^4}{[3]_q^2} \right| \right\} \\
& := G(p,|x|).
\end{aligned}$$

Differentiating and using a simple calculus shows that $\frac{\partial G(p,|x|)}{\partial |x|} \geq 0$ for $|x| \in [0,1]$ and fixed $p \in [0,2]$. It follows that $G(p,|x|)$ is an increasing function of $|x|$. So $G(p,|x|) \leq G(p,1)$. Upon letting $|x| = 1$, a simple algebraic manipulation yields

$$\left|a_4^2 - a_3^2\right| \leq \frac{4}{(1+q+q^2)^2}.$$

□

Theorem 3.3. *Let f given by (1.1) be in the class $\mathcal{R}(q)$. Then*

$$|T_3(2)| \leq \frac{16q^2}{(1+q+q^2+q^3)(1+q+q^2)^2}.$$

Proof. Using the same techniques as in Theorem 3.1 and Theorem 3.2, we obtain

$$|a_2 - a_4| \le \left| \frac{p}{[2]_q} - \frac{p^3}{4\,[4]_q} \right| + \frac{|x|\,pP}{2\,[4]_q} + \frac{|x|^2\,(p-2)P}{4\,[4]_q} + \frac{P}{2\,[4]_q}.$$

It is easy exercise to show that $|a_2 - a_4| \le \frac{p}{[2]_q} - \frac{p^3}{4[4]_q}$ on $[0,2]$, when $p = 2$. Thus,

$$|a_2 - a_4| \le \frac{2q^2}{1+q+q^2+q^3}. \tag{3.5}$$

Using the same techniques as above, one can obtain with simple computations that

$$\left|a_2^2 - 2a_3^2 + a_2 a_4\right| \le \frac{8}{(1+q+q^2)^2}. \tag{3.6}$$

Finally, from (3.5) and (3.6) we obtain

$$|T_3(2)| = \left|(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2 a_4)\right| \le \frac{16q^2}{(1+q+q^2+q^3)(1+q+q^2)^2}.$$

□

Theorem 3.4. *Let f given by (1.1) be in the class $\mathcal{R}(q)$. Then*

$$|T_3(1)| \le 1 + \frac{4}{(1+q+q^2)^2}.$$

Proof. Expanding the determinant by using (3.2), (3.3) and Lemma 2.2, we obtain

$$\begin{aligned} \left|1 + 2a_2^2(a_3 - 1) - a_3^2\right| &= \left|1 + 2\frac{p_1^2}{[2]_q^2}\left(\frac{p_2}{[3]_q} - 1\right) - \frac{p_2^2}{[3]_q^2}\right| \\ &= \left|1 + \left(\tfrac{1}{[2]_q^2[3]_q} - \tfrac{1}{4[3]_q^2}\right)p_1^4 - 2\tfrac{p_1^2}{[2]_q^2}\right. \\ &= \left. + \left(\tfrac{1}{[2]_q^2[3]_q} - \tfrac{1}{2[3]_q^2}\right)p_1^2 xP - \tfrac{x^2P^2}{4[3]_q^2}\right|. \end{aligned}$$

As before, without loss in generality we can assume that $p_1 = p$, where $p \in [0,2]$. Then, by using the triangle inequality and the fact that $|x| \le 1$ we obtain

$$\begin{aligned} |T_3(1)| \;\le\; & \left|1 + \left(\tfrac{1}{[2]_q^2[3]_q} - \tfrac{1}{4[3]_q^2}\right)p^4 - 2\tfrac{p^2}{[2]_q^2}\right| + \left(\tfrac{1}{[2]_q^2[3]_q} - \tfrac{1}{2[3]_q^2}\right)p^2(4-p^2) \\ & + \tfrac{(4-p^2)^2}{4[3]_q^2}. \end{aligned}$$

It is now a simple exercise in elementary calculus to show that this expression has a maximum value when $p = 2$, which completes the proof. □

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