RBF-Chebychev direct method for solving variational problems

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ABSTRACT. This paper establishes a direct method for solving variational problems via a set of Radial basis functions (RBFs) with Gauss-Chebyshev collocation centers. The method consists of reducing a variational problem into a mathematical programming problem. The authors use some optimization techniques to solve the reduced problem. Accuracy and stability of the multiquadric, Gaussian and inverse multiquadric RBF is examined and compared by some numerical experiments.

Keywords: Radial basis functions- direct method - variational problems - Gauss-Chebyshev centers.

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1. INTRODUCTION

The calculus of variations investigates methods that permit maximal or minimal values of functionals 6. Functional minimization problems naturally occur in engineering, mechanics, economics and so forth where minimization of functionals such as Lagrangian, potential and total energy, etc. give the laws governing the system behavior 1.

A direct method converts the variational problem into a mathematical programming problem. The idea of direct methods for solving variational problems consists in replacing the problem of searching for the extremum (usually, for the point of stationary) of a functional in the function space by a problem of searching for a

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1
Table 1. Some popular RBFs.

<table>
<thead>
<tr>
<th>Name</th>
<th>$\phi(r,c)$</th>
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</thead>
<tbody>
<tr>
<td>Gaussian (GA)</td>
<td>$\phi(r) = \exp(-cr^2)$</td>
</tr>
<tr>
<td>Multiquadric (MQ)</td>
<td>$\phi(r) = \sqrt{c^2 + r^2}$</td>
</tr>
<tr>
<td>Inverse multiquadric (IMQ)</td>
<td>$\phi(r) = (c^2 + r^2)^{-1/2}$</td>
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solution in a finite set of parameters. The Ritz method \[8\] usually based on the sub-
spaces of kinematically admissible complete functions, is the most commonly used
approach in direct methods for solving variational problems.

In the last decades, Radial Basis Functions (RBFs) have been widely applied
in different fields such as multivariate function interpolation and approximation,
neural networks and solution of differential and integral equations. The essence of
RBF interpolation is to use linear combination of $\phi(||x - x_j||)$, which is radially
symmetric about its center $x_j$ to approximate the unknown function $y(x)$. Common
choices for RBFs are listed in Table 1 where $r = ||x - x_j||$ and $||.||$ denotes the
Euclidean norm, and $x_j$s are the centers of RBFs and $c$ is a positive shape parameter
which controls the flatness (width) of the basis function.

Franke and Schaback \[7\] used RBFs for solving PDEs, Golbabai and Seifollahi
\[10\] used RBFs for solving the second kind integral equations (IEs). Moreover,
it has been observed that certain class of RBFs such as multiquadric (MQ) and
Gaussian(GA) RBFs exhibit superior error convergence properties \[19\], \[20\]. Some
advantages of the RBF-Based methods are their ease of implementation, accuracy
and efficiency, which are the reasons that this technique is getting popular.

In this paper we use GA-RBFs, MQ-RBFs and Inverse multiquadric(IMQ)-RBFs
for solving variational problems. For this purpose, we introduce the RBFs opera-
tional matrices of differentiation and the product of two RBF vectors. The RBFs
operational matrices can be used to solve problems in fields analysis, calculus of vari-
ations and optimal control. The method consists of reducing the variational problem
into a linear system of algebraic equations by first expanding the candidate functions
as a linear combination of RBFs with unknown coefficients, Gauss-Legendre quad-
rature is used to approximate integration. Finally we evaluate the coefficients of
RBF in such a way that the necessary conditions for extermination of a multivariate
function are imposed.
2. Main Results

Approximation of a function $y(x)$ may be expanded as

$$y(x) \approx y^N(x) = \sum_{i=0}^{N} a_i \varphi(||x - x_i||),$$

(2.1)

where $N$ is the number of RBFs and $a_i$s are the unknown coefficients and $x_i$ are the Gauss-Chebyshev centers.

**Proposition 1.** Let $y^N(x) = a^T \phi(x)$ where $a^T = [a_1, a_2, ..., a_N]$ and $\phi(x) = [\varphi(x), ..., \varphi_N(x)]^T$, $\phi_i(x) = (c^2 + ||x - x_i||^2)^{-\frac{1}{2}}$ for $i = 1, ..., N$. Then there exist symmetric matrices $A_{IMQ}, L_{IMQ}$ and vector $\tilde{\phi}(x)$ such that:

$$a) \ (y^N(x))^2 = a^T A_{IMQ} a,$$

(2.2)

$$b) \ \frac{d}{dx} y^N(x) = a^T \tilde{\phi}(x),$$

(2.3)

where $\tilde{\phi}(x) = [\tilde{\varphi}(x), ..., \tilde{\varphi}_N(x)]^T$ and $\tilde{\varphi}(x) = -(x - x_i)(c^2 + ||x - x_i||^2)^{-\frac{3}{2}}$.

$$c) \ (\frac{d}{dx} y^N(x))^2 = a^T L_{IMQ} a,$$

(2.4)

where $L_{IMQ} = (x - x_i)(x - x_j)(c^2 + ||x - x_i||^2)^{-\frac{3}{2}}(c^2 + ||x - x_j||^2)^{-\frac{3}{2}}$.

Proof.

The properties can be verified by the fact that if $a, b \in R^N$ then one can see that $(a^T b)^2 = a^T b(a^T b) = a^T b b^T a = a^T A$, where $A_{ij} = b_i b_j$ is a symmetric matrix.

Similar properties can be verified for MQ and GA RBFs.

2.1. RBF-Chebyshev direct method. Consider the problem of finding the extremum of the functional

$$\min \int_{x_a}^{x_b} L(x, y, \dot{y}) dx,$$

(2.5)

where $y : [x_a, x_b] \to R^N$ is a sufficiently smooth function, $y(x_a) = A$, $y(x_b) = B$ which two points $A$ and $B$ in $R^N$ are given.

Tonelli’s theorem \[4\] says under assumptions

1. Coercivity of rank $r > 1$ for certain constants $\alpha > 0$ and $\beta$ we have

   $$L(t, x, v) \geq \alpha ||v||^r + \beta. \text{for every } (t, x, v) \epsilon [a, b] \times R^N \times R^N$$

2. Convexity: $L_{vv}$ is everywhere positive semidefinite ($L_{vv} \geq 0$),

there exists a solution of the basic problem 2.5 relative to the class of absolutely continuous (A.C.) functions.
Euler-Lagrange equation leading to a necessary condition for \( y(x) \) on extremizing \( J(y) \) in the form of the well known Euler-Lagrange equation

\[
\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0,
\]

which can be easily solved only in a few classes. Thus it is more practical to use numerical and direct methods such as the Ritz method.

Using a Gauss-Legendre quadrature with \( m \) nodes for integrating yields:

\[
\int_{x_a}^{x_b} L(x, y, \dot{y}) dx \approx \frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k L(x_k, a^T \phi(\|x_k - x_i\|), a^T \frac{d\phi}{dx}(\|x_k - x_i\|))
\]

(2.7)

where \( x_k = \frac{x_a + x_b + x_k - x_a}{2}, z_k \) and \( w_k \) for \( k = 1, ..., m \) are nodes and coefficients of Gauss-Legendre quadrature and RBF centers \( x_i, i = 1, ..., N \) are Chebyshev nodes.

Finally the variational problem (2.5) will be reduced to a constrained optimization problem of finding

\[
\min J(a) = \frac{x_b - x_a}{2} \sum_{k=1}^{m} w_k L(x_k, a^T \phi(\|x_k - x_i\|), a^T \frac{d\phi}{dx}(\|x_k - x_i\|)),
\]

(2.8)

subject to

\[
a^T \phi(\|x_a - x_i\|) = A, \quad a^T \phi(\|x_b - x_i\|) = B.
\]

(2.9)

To solve the optimization problem (2.8)-(2.9) our choice is to use Lagrange multipliers method, so we establish Lagrangian

\[
J^*(a) = J(a) + \lambda_1(a^T \phi(\|x_a - x_i\|) - A) + \lambda_2(a^T \phi(\|x_b - x_i\|) - B),
\]

(2.10)

then we should solve the following algebraic system in order to calculate coefficients \( a^T = [a_0, a_1, ..., a_N]^T \)

\[
\frac{\partial J^*}{\partial a} = 0,
\]

(2.11)

\[
\frac{\partial J^*}{\partial \lambda_i} = 0 \quad i = 1, 2.
\]

(2.12)

Generally speaking, it is a complicated problem to solve the system of equations (2.11) and (2.12). This problem is simplified if the function \( J(y) \) is quadratic in the unknown function and its derivatives. For this case the equations (2.11) are linear in \( a_i \).s. It is easy to see that if Lagrangian in problem (2.8) is convex then objective function (2.8) will be convex function of unknown parameter \( a = [a_1, ..., a_N]^T \).
2.2. **Solving by BiCGSTAB.** For solving a general linear system two different classes are available: first, direct methods, including gaussian elimination and its variants such as LU, Cholskey. Second, iterative methods such as bicg, bicgstab, gmres, lsqr.

Sometimes solving the linear systems by a direct routin such as *linsolve* in MATLAB returns a warning:”Matrix is close to singular or badly scaled, Results may be inaccurate”, so there is warning that the results may be unreliable. To avoid this warning we suggest to use an iterative routin such as bicgstab to solve the system.

the biconjugate gradient stabilized method (BiCGSTAB) is an iterative method. It is a variant of the BiCG (biconjugate gradient method) and has faster and smoother convergence than the original BiCG as well as other variants such as the CGS (conjugate gradient square method).

2.3. **Chebyshev nodes.** The Chebychev nodes are important in approximation theory because they form a particularly good set of nodes for polynomial interpolation. For a given natural \( n \), chebyshev nodes in the interval \([-1, 1]\) are \( x_k = \cos\left(\frac{2k-1}{2n} \pi\right) \), \( k = 1, \ldots, n \). These are the roots of the chebychev polynomial of the first kind of degree \( n \). For nodes over an arbitrary interval \([a, b]\) an affine transform can be used

\[
x_k = \frac{x_b + x_a}{2} + \frac{x_b - x_a}{2} \cos\left(\frac{2k-1}{2n} \pi\right), \quad k = 1, \ldots, n.
\] (2.13)

3. **Examples**

**Example 3.1.** \([8], [3], [6]\) Find the extremal of the following functional

\[
J(y) = \int_0^1 y'^2(x) + xy'(x) + y(x)^2 \, dx.
\] (3.1)

The boundary conditions are

\[
y(0) = 0 , \quad y(1) = \frac{1}{4}.
\] (3.2)

The exact solution of the problem is \( y^*(x) = \frac{1}{2} + c_1 e^x + c_2 e^{-x} \), \( c_1 = \frac{2-e}{4(e^2-1)} \), \( c_2 = \frac{e-2e^2}{4(e^2-1)} \).

We used RBF direct method to solve the problem (3.1)-(3.2) with both equally spaced and Chebychev node centers. Using the described procedure in sec. 2.1 the
Figure 1. a) exact solution b) Absolute error vs. variable x for some RBFs.

Figure 2. a) Absolute error vs. shape parameter. b) Absolute error when Using Bicgstab method. c) Condition number vs. shape parameter.

The numerical solution for \( c = 1 \) and \( N = 36 \) (number of centers) is compared with [5], [9] and [18] (Table 3) by two criteria: relative error and RMS:
Relative error $\Delta(\%)$ for approximate solutions obtained here defined as:

$$\Delta(\%) = \frac{1}{m} \sum_{j=0}^{m-1} \|y^e_j - y^a_j\|^2 \sum_{j=0}^{m-1} \|y^e_j\|^2 \times 100\% \quad (3.3)$$

where $y^e_j$ and $y^a_j$ denote values of state $y$ at the $j$th center point of the analysed interval for respectively the exact solution and the approximate solution for different approximation degrees $m$ and different values of parameter $\gamma$ and different RBF bass (GA, MQ and IMQ) with $N = 36$ chebyshev center points.

The formula for the root mean square error $RMS_{\text{error}}$ is given by

$$RMS_{\text{error}} = \sqrt{\frac{1}{M} \sum_{j=1}^{M} \|y^e_j - y^a_j\|^2} \quad (3.4)$$

where $M$ is the number of selected test nodes and $y^e_j, y^a_j$ are as previous.

Figure 2 (a ,b) compares absolute error of GA, MQ and IMQ RBFs for various random shape parameteres between 0 and 10. It is obvious from the figure that solving the system by command linsolve makes the results more fluctuating than when solved by BiCGSTAB.

Figure 2 (c) compares condition number of the algebraic system in GA, MQ and IMQ cases. Obviously increasing in shape parameter yields decreasing in condition number of GA-RBF case. It is obvious from figure 3 that while solution of problem in IMQ-RBF case has desirable condition number and high absolute error , the solution in the GA-RBF case behaves more accurate. While using BiCGSTAB helps to MQ-RBF make better approximations.
Roundoff error is a fundamental part of any numerical computation. Numerically stable methods attempt to control roundoff error and stop it from accumulating too quickly. Figures 3, 4 demonstrates that increasing number of equally/Chebyshev spaced nodes does not yield uncontrolled increase in error, so it can be inferred that the proposed method is numerically stable. There is not significant difference in behaviour of solution in GA-RBF and MQ-RBF cases.

4. Conclusion

In this paper some properties of RBFs including IMQ-RBF were considered. Then a method that reduces the variational problem to a constrained optimization problem was described.

Numerically with the RMS and relative error criterias, it was proved that GA-RBF and MQ-RBF method accuracy is better than RH and Bernstein method.

The reduced linear algebraic system was proposed to solve by some kind of preconditioner including BICGSTAB that enhances the reliability of the results so there is no more WARNING from MATLAB to solve the corresponding linear system. To improve the efficiency of the method, Chebyshev nodes were used as center points of RBFs. Finally numerical stability of the method was examined and verified by some numerical experiments.

References


