Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **8**(2)(2019), 154-161

ADAMS COMPLETION AND EXTERIOR ALGEBRA

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ABSTRACT. Deleanu, Frei and Hilton have developed the notion of generalized Adams completion in a categorical context. In this note, it is shown that given an algebra, its exterior algebra is isomorphic to the Adams completion of the algebra with respect to a chosen set of morphisms in a suitable category.

Keywords:Category of fraction, Calculus of left fraction, Exterior algebra, Adams completion.

2000 Mathematics subject classification: 15A75, 55P60

1. ADAMS COMPLETION

The notion of (generalized) Adams completion arose from a general categorical completion process, suggested by Adams [1, 2]. Originally, this was considered for admissible categories and generalized homology (or cohomology) theories. Subsequently, this notion has been considered in a more general framework by Deleanu, Frei and Hilton [5] where an arbitrary category and an arbitrary set of morphisms of the category are considered.

It is to be emphasized that many algebric and geometrical constructions in algebra, general topology, algebric topology can be viewed as Adams completions or cocompletions of objects in suitable categories, with respect to carefully chosen sets of morphisms. The current work is also in the same direction. The central idea of this note is to investigate a case that given an algebra, its

[1]Corresponding author: mitaray8@gmail.com
Received: 23 February 2015
Revised: 19 April 2016
Accepted: 29 July 2018

clifford algebra is expressed in terms of Adams completion of the given algebra.

Let $\mathscr{C}$ be a category and S a set of morphisms of $\mathscr{C}$. Let $\mathscr{C}[S^{-1}]$ denote the category of fractions of $\mathscr{C}$ with respect to S and $F : \mathscr{C} \to \mathscr{C}[S^{-1}]$, the canonical functor. Let S denote the category of sets and functions. Then for a given object Y of $\mathscr{C}, \mathscr{C}[S^{-1}](\text{-}, Y) : \mathscr{C} \to S$ defines a contravariant functor. If this functor is representable by an object Y_s of $\mathscr{C}$, i.e., $\mathscr{C}[S^{-1}](\text{-}, Y) \cong \mathscr{C}(\text{-}, Y_S)$, then Y_S is called the *generalized Adams completion of* Y with respect to the set of morphisms S or simply the S-*completion* of Y. We shall often refer to Y_S as the *completion* of Y [5].

2. Exterior algebra

Let A be an associative algebra with unit element e and let F be a linear mapping of V into A such that $F(v)^2 = 0$ for all $v \in V$. Then F extends uniquely to an associative algebra homomorphism $\widetilde{F}$ from a unique associative algebra homomorphism $\widetilde{F} : \wedge(V) \to A$ such that $\widetilde{F}(1) = e$ and $\widetilde{F} i_A$ where i_A is the natural inclusion mapping of $V = \wedge^1(V)$ into $\wedge(V)$ [10]

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "F"'] & \wedge(V) \arrow[dl, "\widetilde{F}"] \\ A & \end{tikzcd} .$$

For any R-module M the tensor algebra $T(M)$ is defined to be the R-module $T(M) = \bigoplus_{i=0}^{\infty} M^{\otimes i} = R \oplus M \oplus (M \otimes M) \oplus \cdots$ The map $M \to T(M)$ defined by $m \mapsto (0, m, 0 \cdots)$ is a morphism of R-modules, which gives an isomorphism of R-modules of M with its image $T^1(M)$[8] .

Theorem 2.1. *Let V, W be K-modules and let $f : V \to W$ be module isomorphism of K- modules. Then f has the following property: given a module isomorphism $g : V \to T^1(V)$, there exists a unique module isomorphism such that $g = \theta f$.*

Proof. For $w \in W$, define $\theta : W \to T^1(V)$ by the rule $\theta(w) = gf^{-1}(w)$. Clearly θ is well-defined and is a homomorphism. We show that θ is one-one. If $\theta(w) = \theta(w')$ then $gf^{-1}(w) = gf^{-1}(w')$ impling $g(f^{-1}(w)) = g(f^{-1}(w'))$; thus $f^{-1}(w) = f^{-1}(w')$ i.e., $w = w'$. Next we show that θ is onto. Since g, f are surjective, we have $T^1(V) = g(V) = g(f^{-1}(W)) = \theta(W)$. Furthermore

$\theta f(v) = \theta(f(v)) = gf^{-1}f(v) = g(v)$ implying $\theta f = g$, i.e., the following diagram is commutative.

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "f"'] & T^1(V) \\ W \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

We show that θ is unique. Let there exist another $\theta' : W \to T^1(V)$ such that $\theta' f = g$. Consider $\theta(w) = gf^{-1}(w) = \theta' f f^{-1}(w) = \theta'(w)$ for each $w \in W$. This completes the proof . □

Theorem 2.2. *Let V, W be K-modules and let $f : V \to W$ be injective morphism of K-modules and $f(v)^2 = 0$ for all $v \in V$. Then f has the following property: given an injective K- module homorphism $i_v : V \to \wedge(V)$, there exists a unique injective module such that $\theta' f = i_v$, i.e., the following diagram is commutative.*

$$\begin{tikzcd} V \arrow[r, "i_v"] \arrow[d, "f"'] & \wedge(V) \\ W \arrow[ur, dashed, "\theta'"'] & \end{tikzcd}$$

Proof. By Theorem 2.1 , there exists a unique $\theta : W \to T^1(V)$ such that $\theta f = g$. Let $g' = i_v g^{-1}$. Consider the diagram

$$\begin{tikzcd} & T^1(V) \arrow[dr, "g'"] & \\ V \arrow[ur, "g"] \arrow[r, "f"'] & W \arrow[u, dashed, "\theta"] \arrow[r, dashed, "\theta'"'] & \wedge(V) \end{tikzcd}$$

For $w \in W$, define $\theta' : W \to \wedge(V)$ by the rule $\theta'(w) = g'\theta(w)$. Now consider $\theta'(f(v)) = g'\theta(f(v))$. For $v \in V$ let $f(v) = w$ and we have $\theta' f(v) = \theta'(w) = g'\theta(w) = i_V g^{-1}\theta(w) = i_V f^{-1}\theta^{-1}\theta(w) = i_V f^{-1}(w) = i_V(v)$, showing $\theta' f = i_V$. We show that θ' is one-one. Now consider the commutative diagram

$$\begin{tikzcd} V \arrow[r, "i_V"] \arrow[d, "f"'] & \wedge(V) \arrow[d, "\wedge(f)"] \\ W \arrow[ur, dashed, "\theta'"] \arrow[r, "i_W"'] & \wedge(W) \end{tikzcd} .$$

Since $\wedge(f)\theta' : W \to \wedge(W)$ is injective we have that θ' is injective. Next we show that $\theta'(w)^2 = 0$. We have $\theta'(w) = g'g(v)$. Since g is isomorphism, θ' is well defined. Thus $\theta'(w)^2 = g'g(v)^2 = 0$.

We show that θ' is unique. Let there exist $\theta'' : W \to \wedge(V)$ such that $\theta'' f = g'g$ and $g'\theta = \theta', g'\theta = \theta''$. Here $\theta'(f(v)) = g'\theta(f(v)) = \theta''(f(v))$ implies $\theta' = \theta''$. This completes the proof . $\square$

3. THE CATEGORY $\mathscr{A}$

Let $\mathscr{U}$ be a fixed Grothendieck universe [9]. Let $\mathscr{A}$ denote the category of all K- module and module homomorphisms. We assume that the underlying sets of the elements of $\mathscr{A}$ are elements of $\mathscr{U}$. Let S denote the set of all maps $f : M \to N$ such that f is injective and $f(m)^2 = 0$ for all $m \in M$.

Proposition 3.1. *Let $s_i : P \to Q$ lies in S, for each $i \in I$, where the index set I is an element of $\mathscr{U}$. Then $\underset{i\in I}{\vee} S_i : \underset{i\in I}{\vee} P \to \underset{i\in I}{\vee} Q$ lies in S.*

Proof. Coproducts in $\mathscr{A}$ are direct sums equipped with a collection of projection maps. Here $P = \underset{i\in I}{\vee} P_i = \underset{i\in I}{\oplus} P_i = \prod\limits_{i\in I} P_i, Q = \underset{i\in I}{\vee} Q_i = \underset{i\in I}{\oplus} Q_i = \prod\limits_{i\in I} Q_i$. Define $s = \underset{i\in I}{\vee} s_i : P \to Q$ by the rule $s(p) = (s_i(p_i))_{i\in I}$. Clearly, s is well defined and is also a homomorphism. In order to show s is injective, take $p, p' \in P$ and consider $s(p) = s(p')$. Then $(s_i(p_i))_{i\in I} = (s_i(p_i'))i \in I$ for each $i \in I$ (since s_i is injective for each $i \in I$) showing $p = p'$. Hence s is injective. Now we have to show that $s(p)^2 = 0$. Consider $s(p)^2 = s(p) \wedge s(p) = -(s(p) \wedge s(p))$ implying $2(s(p) \wedge s(p)) = 0$. Thus $s(p) \wedge s(p) = 0$. This competes the proof. $\square$

We will show that the set of morphisms S of the category $\mathscr{A}$ of K-modules and homomorphisms admits a calculus of left fraction.

Proposition 3.2. *S admits a calculus of left fractions.*

Proof. Let M, N, P be in $\mathscr{A}$. Let $s : M \to N$ and $t : N \to P$ be two morphisms of the category $\mathscr{A}$. We have to show that $ts(m)^2 = 0$ for all $m \in M$. Since s and t are in S, we have $s(m)^2 = 0, t(n)^2 = 0$. Consider $ts(m)^2 = t(s(m))^2 = 0$ for all $m \in M$. So S is a closed family of morphisms of category $\mathscr{A}$.We shall verify conditions (i) and (ii) of Theorem 1.3 ([5], P.67). Let s, t be two morphisms as described above of the category $\mathscr{A}$. We show that if $ts \in S$ and $s \in S$, then $t \in S$ i.e., $t(n)^2 = 0$ and t is injective. Consider the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, "i_M"'] & N \arrow[r, "t"] \arrow[d, "i_N"'] & P \arrow[d, "i_P"] \arrow[dl, "\varphi'"'] \\
\wedge(M) \arrow[r, "\wedge(s)"'] & \wedge(N) \arrow[r, "\wedge(t)"'] & \wedge(P)
\end{tikzcd}$$

From the above diagram we have a diagram

$$\begin{tikzcd}
 & \wedge(M) \arrow[dr, "\wedge(s)"] & \\
M \arrow[ur, "i_M"] \arrow[r, "ts"'] & P \arrow[u, dashed, "\varphi"] \arrow[r, dashed, "\varphi'"'] & \wedge(N)
\end{tikzcd}$$

By Theorem 2.2, φ is one-one and $\varphi ts = i_M$. Again we have $\varphi' = \wedge(s)\varphi$ implying φ' is injective. From $\varphi' t = i_N$ we conclude t is injective. Now consider $t(n)^2 = t(n)\wedge t(n) = -(t(n)\wedge t(n))$ implying $2(t(n)\wedge t(n)) = 0$. Thus $t(n)\wedge t(n) = 0$. Hence the condition (i) of Theorem 1.3 ([5],P.67) holds.

In order to prove conditions (ii) of 1.3 ([5],P.67) consider the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \\
C &
\end{tikzcd}$$

in $\mathscr{A}$ with $s \in S$. We assert that the above diagram can be embedded to a weak push-out diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, dashed, "t"] \\
C \arrow[r, dashed, "g"'] & D
\end{tikzcd}$$

in $\mathscr{A}$ with $t \in S$. Let $D = (B\oplus C)/N$ where N is a sub module of $B\oplus C$ generated by $\{(f(a), -s(a) : a \in A\}$. Define $t : B \to D$ by the rule $t(b) = (b,0)+N$ and $g : C \to D$ by the rule $g(c) = (0,c)+N$. Clearly, the two maps are well defined and homomorphisms. For any $a \in A$, $tf(a) = (f(a),0)+N = (0,s(a))+N = gs(a)$, impling that $tf = gs$. Hence the diagram is commutative.

Next we show $t \in S$, i.e., t is injective. Take $b \in B$ with $t(b) = N$; this implies $(b,0)+N = N$ i.e., $(b,0) \in N$. So $(b,0) = (f(a), -s(a))$ from which it follows that $a = 0$. Now we get $f(0) = (b) = 0$. Thus t is injective. Clearly $t(b)^2 = 0$ for all $b \in B$.

Next let $u : B \to X$ and $v : C \to X$ be in category $\mathscr{A}$ such that $uf = vs$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & B \arrow[d, "t"] \arrow[ddr, bend left, "u"] & \\ C \arrow[r, "g"'] \arrow[drr, bend right, "v"'] & D \arrow[dr, dashed, "\theta"] & \\ & & X \end{tikzcd}.$$

Define $\theta : D \to X$, by the rule $\theta((b,c)+N) = u(b)+v(c)$. It is easy to show that θ is well defined and also a homomorphism. Next we show the two triangles are commutative. For any $b \in B, \theta t(b) = \theta((b,0)+N) = u(b)$ and for any $c \in C, \theta g(c) = \theta(0,(c)+N) = v(c)$. So $\theta t = u$ and $\theta g = v$.

For showing the uniqueness of θ suppose that there exists another $\theta' : D \to X$ with $\theta' t = u$ and $\theta' g = v$. For any $d = (b,c), \theta(d+N) = u(b)+v(c) = \theta' t(b)+\theta' g(c) = \theta'(t(b)+\theta' g(c)) = \theta'(b),0)+N+(0,c+N) = \theta'((b,0)+(0,c)+N) = \theta'(b,c)+N = \theta'(d+N)$. Therefore, θ is unique. This completes the proof . ☐

Theorem 3.3. *The category $\mathscr{A}$ is cocomplete.*

From Theorems 3.1, 3.2 and 3.3 we see that all the conditions of the Theorem 1([7], P.32) are satisfied, hence we have the following result.

Theorem 3.4. *Every object V of the category $\mathscr{A}$ has an Adams completion V_S with respect to the set of morphisms S. Furthermore, there exists a morphism $e : V \to V_S$ in S which is couniversal with respect to the morphisms in S : given a morphism $s : V \to U$ in S there exists a unique morphism $t : U \to V_S$ in S such that $ts = e$. In other words the following diagram is commutative :*

$$\begin{tikzcd} V \arrow[r, "e"] \arrow[d, "s"'] & V_S \\ U \arrow[ur, dashed, "t"'] & \end{tikzcd}$$

Theorem 3.5. *The morphism $e : V \to V_S$ is in S.*

Proof. $S_1 = \{s : P \to Q$ in $\mathscr{A} \mid s$ is a injective and $\mathrm{s(p)}^2 = 0 \}$ $S_2 = \{s : P \to Q$ in $\mathscr{A} \mid s$ is a homomorphism$\}$. For S_1 and S_2, it easily follows that all the conditions of (Theorem 1.3 [4], P.533) are satisfied. Therefore, $e \in S$. This completes the proof. ☐

We show that the Exterior algebra $\wedge(V)$ of a K- module V, is precisely the Adams completion V_S of V.

Theorem 3.6. $\wedge(V) \cong V_S$.

Proof. Consider the following diagram :

$$
\begin{tikzcd}
V \arrow[r, "g"] \arrow[d, "e"'] & \wedge(V) \\
V_S \arrow[ur, dashed, "\varphi"'] &
\end{tikzcd}
$$

By Theorem 2.1, there exists a unique morphism $\varphi : V_S \to \wedge(V)$ in S such that $\varphi e = g$.

Next consider the following diagram :

$$
\begin{tikzcd}
V \arrow[r, "e"] \arrow[d, "g"'] & V_S \\
\wedge(V) \arrow[ur, dashed, "\psi"'] &
\end{tikzcd}
$$

By Theorem 3.4, there exists a unique morphism $\psi : \wedge(V) \to V_S$ in S such that $\psi g = e$.

Consider the following diagram :

$$
\begin{tikzcd}
V \arrow[rr, "e"] \arrow[dd, "e"'] & & V_S \\
& \wedge(V) \arrow[ur, "\psi"'] & \\
V_S \arrow[ur, "\varphi"'] \arrow[uurr, shift right=6ex, "1_{V_S}"'] & &
\end{tikzcd}
$$

We have $\psi\varphi e = \psi g = e$. By the uniqueness condition of the couniversal property of e, we conclude $\psi\varphi = 1_{V_S}$.

Next consider the following diagram :

$$
\begin{tikzcd}
V \arrow[rr, "g"] \arrow[dd, "g"'] & & \wedge(V) \\
& V_S \arrow[ur, "\varphi"'] & \\
\wedge(V) \arrow[ur, "\psi"'] \arrow[uurr, shift right=6ex, "1_{\wedge(V)}"'] & &
\end{tikzcd}
$$

We have $\varphi\psi g = \varphi e = g$. By the uniqueness condition of the couniversal property of g, we conclude $\varphi\psi = 1_{\wedge(V)}$.

Thus $\wedge(V) \cong V_S$. This completes the proof $\square$

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