Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **7**(2)(2018), 144-151

Existence of solution for semilinear elliptic equations via sub-super solutions method

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Abstract. In this paper, we consider a class of semilinear elliptic equations and extend some results about the method of sub-supper solutions. We obtain new results for the generalized semilinear elliptic equations using Schauder's fixed point Theorem.

Keywords: sub-super solution, semilinear problem, weak solution

2000 Mathematics subject classification: 35Pxx; Secondary 46Txx.

1. Introduction

We consider the following semilinear problem:

$$\begin{cases} -\Delta u = \lambda g(x) f(x,u) + \mu h(u), & x \in \Omega, \\ \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

where, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $\lambda, \mu > 0$ are parameters.

The classical method of sub-supper solutions (see [11, 13, 14, 15]) asserts that if f is smooth and if one can find smooth sub-super solutions $v_1 \leq v_2$ of (1.1), then there exists a classical solution u of (1.1), such

[1]Corresponding author: gh_karamali@azad.ac.ir
Received: xx Month 201x
Revised: xx Month 201x
Accepted: xx Month 201x

that $v_1 \leq u \leq v_2$. The classical proof is based on the monotone iteration scheme. This requires f be Lipschitz (or locally Lipschitz) function. The existence of a smallest and a largest solutions $u_1 \leq u_2$, in the interval $[v_1, v_2]$, is implied by this argument. Another proof, based on Schauder's fixed point theorem can be found in Akô [1, 9]. In this case, the existence of a smallest and a largest solution is proved separately, via a Perron-type argument. Using Akô's strategy, Clément and Sweers [8] have implemented the method of sub-super solutions by the assumptions that $v_1, v_2 \in C(\bar{\Omega})$ and f is continuous. Other study of this problem can also be found in [2, 3, 6, 7, 12, 16], specially, in Deuel-Hess [10] for H^1-solutions and in Brezis-Marcus-Ponce [4, 5] for L^1-solutions when f is continuous and nondecreasing. In this paper, we extend the method of sub-super solutions in order to establish existence of the solutions of (1.1) in the sense of L^1-solution. We follow the strategy of [1, 8], based on the Schauder's fixed point theorem. Substantially, some of the details be modified. We assume throughout the paper that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Definition 1.1. We say that u is an L^1-solution of (1.1) if

(a): $u \in L^1(\Omega)$ and $f(., u)\rho_0, h\rho_0 \in L^1(\Omega)$;
(b): for every $v \in C_0^2(\bar{\Omega})$,

$$-\int_{\Omega} u \triangle v = \int_{\Omega} (\lambda f(x, u) + \mu h(u)) v dx. \tag{1.2}$$

Here, $\rho_0(x) = d(x, \partial\Omega)$ for any $x \in \Omega$ and $C_0^2(\bar{\Omega}) = \{v \in C^2(\bar{\Omega}) : \ v = 0 \ on \ \partial\Omega\})$.

Definition 1.2. Let $u \in L^1(\Omega)$ and $f(., u)\rho_0, h\rho_0 \in L^1(\Omega)$ be given functions. Then we say that

(i): u is an L^1-sub solution of (1.1), if

$$-\int_{\Omega} u \triangle v \leq \int_{\Omega} (\lambda f(x, u) + \mu h(u)) v dx,$$

for every $v \in C_0^2(\bar{\Omega})$.

(ii): u is an L^1-super solution of (1.1) if

$$-\int_{\Omega} u \triangle v \geq \int_{\Omega} (\lambda f(x, u) + \mu h(u)) v dx,$$

for every $v \in C_0^2(\bar{\Omega})$.

2. Boundedness and equi-integrable

Definition 2.1. A set $B \subset L^1(\Omega;\rho_0 dx)$ is equi-integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $E \subset \Omega$ and

$$|E| < \delta \;\Rightarrow\; \int_E |g|\rho_0 dx < \epsilon \;\; \forall g \in B.$$

Lemma 2.2. [8] *Let $\{w_n\} \subset L^1$ and let $\{E_n\}$ be a sequence of measurable subsets of Ω such that*

$$|E_n| \to 0 \;\; \text{and} \int_{E_n} |w_n| \geq 1 \;\; \forall n \geq 1.$$

Then there exists a subsequence $\{w_{n_k}\}$ and a sequence of disjoint measurable sets $\{F_k\}$ such that

$$F_k \subset E_{n_k} \;\text{and} \int_{F_k} |w_n| \geq 1 \; \forall n \geq 1.$$

Proposition 2.3. [8] *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, $h \in C(\Omega)$ and $v_1, v_2 \in L^1(\Omega)$ such that $v_1 \leq V_2$ a.e. Suppose that*

$$f(.,v)\rho_0, h\rho_0 \in L^1(\Omega) \; \forall v \in L^1(\Omega) \text{ such that } v_1 \leq v \leq v_2 \text{ a.e.}$$

Then, the set

$$B = \{f(.,v) \in L^1(\Omega;\rho_0 dx) \;:\; v \in L^1(\Omega) \text{ and } v_1 \leq v \leq v_2 \text{ a.e.}\}$$

is bounded and equi-integrable in $L^1(\Omega;\rho_0 dx)$.

Theorem 2.4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $h \in C(\Omega)$ such that*

$$f(.,v)\rho_0, h\rho_0 \in L^1(\Omega) \; \forall v \in L^1(\Omega).$$

Then, the Nemytskii operator $F : L^1(\Omega;\rho_0 dx) \to L^1(\Omega)$ defined by

$$v \mapsto \lambda f(.,v) + \mu h(v)$$

is continuous.

Proof. Suppose that $v_n \to v$ in $L^1(\Omega)$. Let v_{n_k} be a subsequence such that $v_{n_k} \to v$ *a.e.* and $|v_{n_k}| \leq V$ *a.e.* For some function $V \in L^1(\Omega)$. In particular,

$$\lambda f(.,v_{n_k}) + \mu h(v_{n_k}) \to \lambda f(.,v) + \mu h(v) \text{ a.e.}$$

Moreover, by Proposition 2.3 the sequence $\{\lambda f(.,v_{n_k}) + \mu h(v_{n_k})\}$ is equi-integrable in $L^1(\Omega;\rho_0 dx)$. It then follows from Egorov's theorem that

$$\lambda f(.,v_{n_k}) + \mu h(v_{n_k}) \to \lambda f(.,v) + \mu h(v)$$

in $L^1(\Omega;\rho_0 dx)$. Since the limit is independent of the subsequence $\{v_{n_k}\}$ we deduce that

$$F(v_n)\to F(v) \ in\ L^1(\Omega;\rho_0 dx).$$

3. **Standard existence**

Theorem 3.1. *Suppose that $f(.,u),h\in L^1(\Omega;\rho_0 dx)$, there exists a unique $w\in L^1(\Omega)$ such that*

$$-\int_\Omega w\triangle v=\int_\Omega(\lambda f(x,w)+\mu h(w))v dx. \tag{3.1}$$

for every $v\in C_0^2(\bar\Omega)$. Moreover,

- **(i):** *For every $1\le p\le\frac{n}{n-1}$, $w\in L^p(\omega)$ and*

$$\|w\|_p\le M(\|\lambda\rho_0 f\|_{L^1}+\|\mu\rho_0 h\|_{L^1}), \tag{3.2}$$

  for some constant M.
- **(ii):** *Given $\{f_n\},\{h_n\}\in L^1(\Omega;\rho_0 dx)$ $n\ge 1$, let w_n be the solution of (3.1) associated to $\{f_n\},\{h_n\}$. If $\{f_n\},\{h_n\}$ are bounded in $L^1(\Omega)$, then $\{w_n\}$ is relatively compact in $L^1(\Omega)$ for every $1\le p\le\frac{n}{n-1}$.*

Proof. We prove (i) and (ii) and refer the reader to [5] for the existence and uniqueness of w.

Proof of (i). Note that w satisfies

$$\begin{aligned}|\int_\Omega w\triangle v|&=|\int_\Omega(\lambda f(.,w)+\mu h(w))v|\\&\le\int_\Omega\|\lambda\rho_0 f(.,w)+\mu\rho_0 h(w)\|_{L^1}\|\frac{v}{\rho_0}\|_{L^\infty}\\&\le M_1[\|\lambda\rho_0 f(.,w)\|_{L^1}+\|\mu\rho_0 h(w)\|_{L^1}]\|\frac{v}{\rho_0}\|_{L^\infty}\ \forall v\in C_0^2(\bar\Omega).\end{aligned} \tag{3.3}$$

Suppose that $f,h\in C_0^\infty(\bar\Omega)$ and $v\in C_0^2(\bar\Omega)$ be the solution of

$$\begin{cases}-\Delta v=\lambda f+\mu h, & x\in\Omega,\\ v(x)=0, & x\in\partial\Omega\ .\end{cases} \tag{3.4}$$

By standard Caldrón-Zygmund estimates [13]

$$\|v\|_{W^{2,q}}\le M_2(\|\lambda f\|_{L^q}+\|\mu h\|_{L^q}), \tag{3.5}$$

where, $\frac{1}{p}+\frac{1}{q}=1$. On the other hand, since $q>n$ it follows from Morrey's embedding [13] that

$$\|\frac{v}{\rho_0}\|_{L^\infty} \leq M_3(\|v\|_{L^\infty}+\|\nabla v\|_{L^\infty}) \leq M_3\|v\|_{W^{2,q}}. \tag{3.6}$$

According to conclusions above we get

$$\begin{aligned} |\int_\Omega(\lambda f(.,w)+\mu h(w))v| &\leq M_1[\|\lambda\rho_0 f\|_{L^1}+\|\mu\rho_0 h\|_{L^1}]\|\frac{v}{\rho_0}\|_{L^\infty} \\ &\leq M_1[\|\lambda\rho_0 f\|_{L^1}+\|\mu\rho_0 h\|_{L^1}]M_3\|v\|_{W^{2,q}} \\ &\leq M_1M_2M_3(\|\lambda\rho_0 f\|_{L^1}+\|\mu\rho_0 h\|_{L^1}). \end{aligned} \tag{3.7}$$

By duality, one deduces that $w\in L^p(\Omega)$ and 3.2 holds. Proof of (ii). Let $U\subset\Omega$ be a smooth domain, $v_n\in L^1(U)$ be the solution of the problem

$$\begin{cases} -\Delta v_n=\lambda f_n+\mu h_n, & x\in U, \\ v_n(x)=0, & x\in\partial U\ . \end{cases} \tag{3.8}$$

By standard elliptic estimates [17] for every $1\leq p\leq\frac{n}{n-1}$,

$$\|v_n\|_{W1,p(U)} \leq M_2(\|\lambda f\|_{L^q}+\|\mu h\|_{L^q}) \leq M_2(|\lambda|C_1+|\mu|C_2), \tag{3.9}$$

where, $\|f_n\|_{L^q}\leq C_1$ and $\|h_n\|_{L^q}\leq C_2$. On the other hand, since w_n-v_n is harmonic in U so for every $Y\subset U$ we have

$$\begin{aligned} \|w_n-v_n\|_{C^1(\bar{Y})} &\leq K_Y\|w_n-v_n\|_{L^1(U)} \\ &\leq K_Y\|\lambda f\rho_0+\mu h\rho_0\|_{L^1} \\ &\leq K_Y(|\lambda|\|f\rho_0\|_{L^1}+|\mu|\|h\rho_0\|_{L^1}) \\ &\leq K_Y(|\lambda\rho_0|K_1+|\mu\rho_0|K_2) \end{aligned} \tag{3.10}$$

where, K_1,K_2 are constants that $\|f\|_{L^1}\leq K_1$ and $\|h\|_{L^1}\leq K_2$ respectively. Therefore, there exists a subsequence $\{w_{n_k}\}$ such that $w_{n_k}\to w$ *a.e.* in Ω. On the other hand by (i) the sequence $\{w_n\}$ is bounded in $L^p(\Omega)$ for every $1\leq p\frac{n}{n-1}$. By Egorov's theorem $\{w_n\}$ is converges in $L^1(\Omega)$ so this proof complete.

Proposition 3.2. *Let $w\in L^1(\Omega)$ and $f,h\in L^1(\Omega;\rho_0dx)$ be such that*

$$-\int_\Omega w\triangle v \geq \int_\Omega(\lambda f+\mu h)v \tag{3.11}$$

for every $v\in C_0^2(\bar\Omega)$ and $v\geq 0$ in Ω. Then,

$$-\int_\Omega w^-\triangle v \geq \int_{[0\geq w]}(\lambda f+\mu h) \tag{3.12}$$

for every $v\in C_0^2(\bar\Omega)$ and $v\geq 0$ in Ω, where, $w^-=max\{-w,0\}$.

Proof. It is straightforward.

Corollary 3.3. *If u, v are solutions of problem* (1) *then* $\min\{u, v\}$ *is a super solution.*

Proof. We set $w = v-u$ and $\varphi := [\lambda f(.,v)+\mu h(v)]-[\lambda f(.,u)+\mu h(u)]$, then

$$-\int_{\Omega}(v-u)^{-}\triangle s \geq \int_{[v\leq u]}([\lambda f(x,v)+\mu h(v)]-[\lambda f(x,u)+\mu h(u)]sdx,$$

for every $s \in C_0^2(\bar{\omega})$ and $s \geq 0$ in Ω. Since $\min\{u, v\} = u + (v-u)^{-}$ so the result follows.

Now, we can state the main result.

Theorem 3.4. *Let v_1, v_2 be a sub and a super solution of problem* (1), *respectively. Suppose that $v_1 \leq v_2$ a.e. and*

$$f(.,v)\rho_0 \in L^1(\Omega), \text{ and } h(v)\rho_0 \in L^1(\Omega) \tag{3.13}$$

for every $v \in L^1(\Omega)$ such that $v_1 \leq v_2$ a.e. Then there exist solutions $u_1 \leq u_2$ of problem (1) *in $[v_1, v_2]$ such that solution u of problem (1.1) in the interval $[v_1, v_2]$ satisfies*

$$v_1 \leq u_1 \leq u \leq u_2 \leq v_2 \quad a.e. \tag{3.14}$$

Proof. Let be $(x,t) \in \Omega \times \mathbb{R}$, we define $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x,t) = \begin{cases} v_1(x), & t < v_1(x), \\ t, & v_1(x) \leq t \leq v_2(x), \\ v_2(x), & v_2(x) < t. \end{cases}$$

Then f is a *Carathéodory* function and by (3.13) $f\rho_0, h\rho_0 \in L^1(\Omega)$ for every $v \in L^1(\Omega)$. We set $G : L^1(\Omega) \to L^1(\Omega; \rho_0 dx)$ defined by $v \mapsto (\lambda f(.,v)) + \mu h(v)$ and $K : L^1(\Omega; \rho_0 dx) \to L^1(\Omega)$ defined by $s \mapsto w$, where, w is the unique solution of the problem

$$\begin{cases} -\Delta w = s, & x \in \Omega, \\ w = 0, & x \in \partial\Omega\ . \end{cases}$$

By Theorem 2.4 and 3.1 $KG : L^1(\Omega) \to L^1(\Omega)$ is continuous. Moreover, by Proposition 2.3 $G(L^1(\Omega))$ is a bounded subset of $L^1(\Omega; \rho_0 dx)$. Therefore, by Theorem 3.1 KG is compact and there exists $C > 0$ such that

$$\|KG(v)\|_{L^1} \leq C_1 \|G(v)\|_{L^1} \leq C$$

for every $v \in L^1(\Omega)$. It follows from Schauder's fixed point theorem that KG has a fixed point $u \in L^1(\Omega)$. In other words, u satisfies

$$\begin{cases} -\Delta u = \lambda f(x,u) + \mu h(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

We will show that u is a solution of problem (1.1) and satisfies $v_1 \leq u \leq v_2$ $a.e.$. To do this, we show that $v_1 \leq u$ $a.e.$, the proof of the inequality $u \leq v_2$ $a.e.$ is similar. Note that

$$\lambda f(., v_1) + \mu h(v_1) = \lambda f(., u) + \mu h(u) \ a.e.$$

on the set $[v_1 \leq u]$. Therefore, by proposition 3.2 and with $w = v_1 - u$, we get

$$-\int_\Omega w^- \triangle v \geq \int_{[v_1 \leq u]} [(\lambda f(x.v_1) + \mu h(v_1)) - (\lambda f(x.u) + \mu h(u))] v dx = 0$$

for every $v \in C_0^2(\bar{\Omega})$ and $v \geq 0$ in Ω. since $w^- \geq 0$ $a.e.$ so $w = 0$ $a.e.$, this implies that $v_1 \leq u$ $a.e.$ Now, we show that there exist a smallest and largest solution $u_1 \leq u_2$ of problem (1.1) in the interval $[v_1, v_2]$. We prove the existence of the smallest solution u_1, the existence of u_2 is similar. Let

$$A = \inf\{\int_\Omega w \ ; \ v_1 \leq w \leq v_2 \ a.e. \ \},$$

where, w is a solution of problem (1.1).

By definition of solution for problem (1.1) implies that $A < \infty$. If w_1, w_2 are two solutions of problem (1.1) and v_1, v_2 are sub-super solution of problem (1.1), respectively, such that $v_1 \leq w_1, w_2 \leq v_2$ $a.e.$ Then, the problem (1.1) has a solution w such that

$$v_1 \leq w \leq \min\{w_1, w_2\} \leq v_2 \ a.e. \ . \tag{3.15}$$

For proof of this claim we use corollary 3.3, where, $\min\{w_1, w_2\}$ is a super solution of problem (1.1). Similarly, By applying above arguments with $v_2 = \min\{w_1, w_2\}, v_1$, (without loss of generality), one finds a solution w of problem (1.1) satisfies (3.15). Therefore, it follows from the claim above that one finds a non-increasing sequence of solutions $\{w_n\}$ of problem (1.1) such that

$$v_1 \leq w_n \leq v_2 \ a.e. \ and \ \int_\Omega w_n \to A.$$

On the other hand, by Proposition 2.3 the sequence $\{f(., w_n)\}$ is equi-integrable in $L^1(\Omega; \rho_0 dx)$. It then follows from Egorov's theorem that

$$\lambda f(., w_n) + \mu h(w_n) \to \lambda f(., w) + \mu h(w)$$

in $L^1(\Omega; \rho_0 dx)$. Therefore, w is a solution of problem (1.1) and $\int_\Omega w = A$. By the claim above, w is the largest solution of problem (1.1) in the interval $[v_1, v_2]$. This completes the proof.

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