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Existence of solution for semilinear elliptic equations via sub-super solutions method

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Abstract. In this paper, we consider a class of semilinear elliptic equations and extend some results about the method of sub-supper solutions. We obtain new results for the generalized semilinear elliptic equations using Schauder's fixed point Theorem.

Keywords: sub-super solution, semilinear problem, weak solution

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1. Introduction

We consider the following semilinear problem:

$$\begin{cases} -\Delta u = \lambda g(x) f(x,u) + \mu h(u), & x \in \Omega, \\ \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

where, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and $\lambda, \mu > 0$ are parameters.

The classical method of sub-supper solutions (see [11, 13, 14, 15]) asserts that if f is smooth and if one can find smooth sub-super solutions $v_1 \le v_2$ of (1.1), then there exists a classical solution u of (1.1), such

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that $v_1 \le u \le v_2$. The classical proof is based on the monotone iteration scheme. This requires f be Lipschitz (or locally Lipschitz) function. The existence of a smallest and a largest solutions $u_1 \le u_2$, in the interval $[v_1, v_2]$, is implied by this argument. Another proof, based on Schauder's fixed point theorem can be found in Akô [1, 9]. In this case, the existence of a smallest and a largest solution is proved separately, via a Perron-type argument. Using Akô's strategy, Clément and Sweers [8] have implemented the method of sub-super solutions by the assumptions that $v_1, v_2 \in C(\bar{\Omega})$ and f is continuous. Other study of this problem can also be found in [2, 3, 6, 7, 12, 16], specially, in Deuel-Hess [10] for H^1-solutions and in Brezis-Marcus-Ponce [4, 5] for L^1-solutions when f is continuous and nondecreasing. In this paper, we extend the method of sub-super solutions in order to establish existence of the solutions of (1.1) in the sense of L^1-solution. We follow the strategy of [1, 8], based on the Schauder's fixed point theorem. Substantially, some of the details be modified. We assume throughout the paper that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Definition 1.1. We say that u is an L^1-solution of (1.1) if

(a): $u \in L^1(\Omega)$ and $f(., u)\rho_0, h\rho_0 \in L^1(\Omega)$;
(b): for every $v \in C_0^2(\bar{\Omega})$,

$$-\int_\Omega u \triangle v = \int_\Omega (\lambda f(x,u) + \mu h(u)) v dx. \tag{1.2}$$

Here, $\rho_0(x) = d(x, \partial\Omega)$ for any $x \in \Omega$ and $C_0^2(\bar{\Omega}) = \{v \in C^2(\bar{\Omega}) : \ v = 0 \ on \ \partial\Omega\})$.

Definition 1.2. Let $u \in L^1(\Omega)$ and $f(., u)\rho_0, h\rho_0 \in L^1(\Omega)$ be given functions. Then we say that

(i): u is an L^1-sub solution of (1.1), if

$$-\int_\Omega u \triangle v \le \int_\Omega (\lambda f(x,u) + \mu h(u)) v dx,$$

for every $v \in C_0^2(\bar{\Omega})$.

(ii): u is an L^1-super solution of (1.1) if

$$-\int_\Omega u \triangle v \ge \int_\Omega (\lambda f(x,u) + \mu h(u)) v dx,$$

for every $v \in C_0^2(\bar{\Omega})$.

2. Boundedness and equi-integrable

Definition 2.1. A set $B \subset L^1(\Omega; \rho_0 dx)$ is equi-integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $E \subset \Omega$ and

$$|E| < \delta \;\Rightarrow\; \int_E |g|\rho_0 dx < \epsilon \;\; \forall g \in B.$$

Lemma 2.2. [8] *Let $\{w_n\} \subset L^1$ and let $\{E_n\}$ be a sequence of measurable subsets of Ω such that*

$$|E_n| \to 0 \;\; \text{and} \int_{E_n} |w_n| \geq 1 \;\; \forall n \geq 1.$$

Then there exists a subsequence $\{w_{n_k}\}$ and a sequence of disjoint measurable sets $\{F_k\}$ such that

$$F_k \subset E_{n_k} \;\text{and} \int_{F_k} |w_n| \geq 1 \; \forall n \geq 1.$$

Proposition 2.3. [8] *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, $h \in C(\Omega)$ and $v_1, v_2 \in L^1(\Omega)$ such that $v_1 \leq V_2$ a.e. Suppose that*

$$f(.,v)\rho_0, h\rho_0 \in L^1(\Omega) \; \forall v \in L^1(\Omega) \text{ such that } v_1 \leq v \leq v_2 \text{ a.e.}$$

Then, the set

$$B = \{f(.,v) \in L^1(\Omega;\rho_0 dx) \;:\; v \in L^1(\Omega) \text{ and } v_1 \leq v \leq v_2 \text{ a.e.}\}$$

is bounded and equi-integrable in $L^1(\Omega; \rho_0 dx)$.

Theorem 2.4. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $h \in C(\Omega)$ such that*

$$f(.,v)\rho_0, h\rho_0 \in L^1(\Omega) \; \forall v \in L^1(\Omega).$$

Then, the Nemytskii operator $F : L^1(\Omega; \rho_0 dx) \to L^1(\Omega)$ defined by

$$v \mapsto \lambda f(.,v) + \mu h(v)$$

is continuous.

Proof. Suppose that $v_n \to v$ in $L^1(\Omega)$. Let v_{n_k} be a subsequence such that $v_{n_k} \to v$ *a.e.* and $|v_{n_k}| \leq V$ *a.e.* For some function $V \in L^1(\Omega)$. In particular,

$$\lambda f(., v_{n_k}) + \mu h(v_{n_k}) \to \lambda f(.,v) + \mu h(v) \text{ a.e.}$$

Moreover, by Proposition 2.3 the sequence $\{\lambda f(., v_{n_k}) + \mu h(v_{n_k})\}$ is equi-integrable in $L^1(\Omega; \rho_0 dx)$. It then follows from Egorov's theorem that

$$\lambda f(., v_{n_k}) + \mu h(v_{n_k}) \to \lambda f(.,v) + \mu h(v)$$

in $L^1(\Omega;\rho_0 dx)$. Since the limit is independent of the subsequence $\{v_{n_k}\}$ we deduce that

$$F(v_n) \to F(v) \ in \ L^1(\Omega;\rho_0 dx).$$

3. **Standard existence**

Theorem 3.1. *Suppose that $f(.,u), h \in L^1(\Omega;\rho_0 dx)$, there exists a unique $w \in L^1(\Omega)$ such that*

$$-\int_\Omega w\triangle v = \int_\Omega (\lambda f(x,w) + \mu h(w)) v dx. \tag{3.1}$$

for every $v \in C_0^2(\bar{\Omega})$. Moreover,

(i): *For every $1 \leq p \leq \frac{n}{n-1}$, $w \in L^p(\omega)$ and*

$$\|w\|_p \leq M(\|\lambda\rho_0 f\|_{L^1} + \|\mu\rho_0 h\|_{L^1}), \tag{3.2}$$

for some constant M.

(ii): *Given $\{f_n\},\{h_n\} \in L^1(\Omega;\rho_0 dx)$ $n \geq 1$, let w_n be the solution of (3.1) associated to $\{f_n\},\{h_n\}$. If $\{f_n\},\{h_n\}$ are bounded in $L^1(\Omega)$, then $\{w_n\}$ is relatively compact in $L^1(\Omega)$ for every $1 \leq p \leq \frac{n}{n-1}$.*

Proof. We prove (i) and (ii) and refer the reader to [5] for the existence and uniqueness of w.

Proof of (i). Note that w satisfies

$$\begin{aligned} |\int_\Omega w\triangle v| &= |\int_\Omega (\lambda f(.,w) + \mu h(w)) v| \\ &\leq \int_\Omega \|\lambda\rho_0 f(.,w) + \mu\rho_0 h(w)\|_{L^1} \|\frac{v}{\rho_0}\|_{L^\infty} \\ &\leq M_1[\|\lambda\rho_0 f(.,w)\|_{L^1} + \|\mu\rho_0 h(w)\|_{L^1}] \|\frac{v}{\rho_0}\|_{L^\infty} \ \forall v \in C_0^2(\bar{\Omega}). \end{aligned} \tag{3.3}$$

Suppose that $f, h \in C_0^\infty(\bar{\Omega})$ and $v \in C_0^2(\bar{\Omega})$ be the solution of

$$\begin{cases} -\Delta v = \lambda f + \mu h, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega \ . \end{cases} \tag{3.4}$$

By standard Caldrón-Zygmund estimates [13]

$$\|v\|_{W^{2,q}} \leq M_2(\|\lambda f\|_{L^q} + \|\mu h\|_{L^q}), \tag{3.5}$$

where, $\frac{1}{p}+\frac{1}{q}=1$. On the other hand, since $q>n$ it follows from Morrey's embedding [13] that

$$\|\frac{v}{\rho_0}\|_{L^\infty} \le M_3(\|v\|_{L^\infty}+\|\nabla v\|_{L^\infty}) \le M_3\|v\|_{W^{2,q}}. \tag{3.6}$$

According to conclusions above we get

$$\begin{aligned} |\int_\Omega (\lambda f(.,w)+\mu h(w))v| &\le M_1[\|\lambda\rho_0 f\|_{L^1}+\|\mu\rho_0 h\|_{L^1}]\|\frac{v}{\rho_0}\|_{L^\infty} \\ &\le M_1[\|\lambda\rho_0 f\|_{L^1}+\|\mu\rho_0 h\|_{L^1}]M_3\|v\|_{W^{2,q}} \\ &\le M_1M_2M_3(\|\lambda\rho_0 f\|_{L^1}+\|\mu\rho_0 h\|_{L^1}). \end{aligned} \tag{3.7}$$

By duality, one deduces that $w \in L^p(\Omega)$ and 3.2 holds. Proof of (ii). Let $U \subset \Omega$ be a smooth domain, $v_n \in L^1(U)$ be the solution of the problem

$$\begin{cases} -\Delta v_n = \lambda f_n + \mu h_n, & x \in U, \\ v_n(x) = 0, & x \in \partial U\ . \end{cases} \tag{3.8}$$

By standard elliptic estimates [17] for every $1 \le p \le \frac{n}{n-1}$,

$$\|v_n\|_{W1,p(U)} \le M_2(\|\lambda f\|_{L^q}+\|\mu h\|_{L^q}) \le M_2(|\lambda|C_1+|\mu|C_2), \tag{3.9}$$

where, $\|f_n\|_{L^q} \le C_1$ and $\|h_n\|_{L^q} \le C_2$. On the other hand, since $w_n - v_n$ is harmonic in U so for every $Y \subset U$ we have

$$\begin{aligned} \|w_n - v_n\|_{C^1(\bar{Y})} &\le K_Y\|w_n - v_n\|_{L^1(U)} \\ &\le K_Y\|\lambda f\rho_0 + \mu h\rho_0\|_{L^1} \\ &\le K_Y(|\lambda|\|f\rho_0\|_{L^1}+|\mu|\|h\rho_0\|_{L^1}) \\ &\le K_Y(|\lambda\rho_0|K_1+|\mu\rho_0|K_2) \end{aligned} \tag{3.10}$$

where, K_1, K_2 are constants that $\|f\|_{L^1} \le K_1$ and $\|h\|_{L^1} \le K_2$ respectively. Therefore, there exists a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \to w$ *a.e.* in Ω. On the other hand by (i) the sequence $\{w_n\}$ is bounded in $L^p(\Omega)$ for every $1 \le p\frac{n}{n-1}$. By Egorov's theorem $\{w_n\}$ is converges in $L^1(\Omega)$ so this proof complete.

Proposition 3.2. *Let $w \in L^1(\Omega)$ and $f, h \in L^1(\Omega; \rho_0 dx)$ be such that*

$$-\int_\Omega w\triangle v \ge \int_\Omega (\lambda f + \mu h)v \tag{3.11}$$

for every $v \in C_0^2(\bar{\Omega})$ and $v \ge 0$ in Ω. Then,

$$-\int_\Omega w^-\triangle v \ge \int_{[0\ge w]} (\lambda f + \mu h) \tag{3.12}$$

for every $v \in C_0^2(\bar{\Omega})$ and $v \ge 0$ in Ω, where, $w^- = max\{-w, 0\}$.

Proof. It is straightforward.

Corollary 3.3. *If u, v are solutions of problem* (1) *then* $\min\{u, v\}$ *is a super solution.*

Proof. We set $w = v - u$ and $\varphi := [\lambda f(., v) + \mu h(v)] - [\lambda f(., u) + \mu h(u)]$, then

$$-\int_{\Omega} (v-u)^{-} \triangle s \geq \int_{[v \leq u]} ([\lambda f(x,v) + \mu h(v)] - [\lambda f(x,u) + \mu h(u)] s dx,$$

for every $s \in C_0^2(\bar{\omega})$ and $s \geq 0$ in Ω. Since $\min\{u, v\} = u + (v-u)^{-}$ so the result follows.

Now, we can state the main result.

Theorem 3.4. *Let v_1, v_2 be a sub and a super solution of problem* (1), *respectively. Suppose that $v_1 \leq v_2$ a.e. and*

$$f(., v)\rho_0 \in L^1(\Omega), \text{ and } h(v)\rho_0 \in L^1(\Omega) \tag{3.13}$$

for every $v \in L^1(\Omega)$ such that $v_1 \leq v_2$ a.e. Then there exist solutions $u_1 \leq u_2$ of problem (1) *in $[v_1, v_2]$ such that solution u of problem (1.1) in the interval $[v_1, v_2]$ satisfies*

$$v_1 \leq u_1 \leq u \leq u_2 \leq v_2 \quad a.e. \tag{3.14}$$

Proof. Let be $(x, t) \in \Omega \times \mathbb{R}$, we define $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x,t) = \begin{cases} v_1(x), & t < v_1(x), \\ t, & v_1(x) \leq t \leq v_2(x), \\ v_2(x), & v_2(x) < t. \end{cases}$$

Then f is a *Carathéodory* function and by (3.13) $f\rho_0, h\rho_0 \in L^1(\Omega)$ for every $v \in L^1(\Omega)$. We set $G : L^1(\Omega) \to L^1(\Omega; \rho_0 dx)$ defined by $v \mapsto (\lambda f(., v)) + \mu h(v)$ and $K : L^1(\Omega; \rho_0 dx) \to L^1(\Omega)$ defined by $s \mapsto w$, where, w is the unique solution of the problem

$$\begin{cases} -\Delta w = s, & x \in \Omega, \\ w = 0, & x \in \partial\Omega . \end{cases}$$

By Theorem 2.4 and 3.1 $KG : L^1(\Omega) \to L^1(\Omega)$ is continuous. Moreover, by Proposition 2.3 $G(L^1(\Omega))$ is a bounded subset of $L^1(\Omega; \rho_0 dx)$. Therefore, by Theorem 3.1 KG is compact and there exists $C > 0$ such that

$$\|KG(v)\|_{L^1} \leq C_1 \|G(v)\|_{L^1} \leq C$$

for every $v \in L^1(\Omega)$. It follows from Schauder's fixed point theorem that KG has a fixed point $u \in L^1(\Omega)$. In other words, u satisfies

$$\begin{cases} -\Delta u = \lambda f(x,u) + \mu h(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

We will show that u is a solution of problem (1.1) and satisfies $v_1 \leq u \leq v_2$ *a.e.* . To do this, we show that $v_1 \leq u$ *a.e.*, the proof of the inequality $u \leq v_2$ *a.e.* is similar. Note that

$$\lambda f(.,v_1) + \mu h(v_1) = \lambda f(.,u) + \mu h(u) \ a.e.$$

on the set $[v_1 \leq u]$. Therefore, by proposition 3.2 and with $w = v_1 - u$, we get

$$-\int_{\Omega} w^{-} \triangle v \geq \int_{[v_1 \leq u]} [(\lambda f(x.v_1) + \mu h(v_1)) - (\lambda f(x.u) + \mu h(u))] v dx = 0$$

for every $v \in C_0^2(\bar{\Omega})$ and $v \geq 0$ in Ω. since $w^{-} \geq 0$ *a.e.* so $w = 0$ *a.e.*, this implies that $v_1 \leq u$ *a.e.* Now, we show that there exist a smallest and largest solution $u_1 \leq u_2$ of problem (1.1) in the interval $[v_1, v_2]$. We prove the existence of the smallest solution u_1, the existence of u_2 is similar. Let

$$A = \inf\{\int_{\Omega} w \ ; \ v_1 \leq w \leq v_2 \ a.e. \ \},$$

where, w is a solution of problem (1.1).

By definition of solution for problem (1.1) implies that $A < \infty$. If w_1, w_2 are two solutions of problem (1.1) and v_1, v_2 are sub-super solution of problem (1.1), respectively, such that $v_1 \leq w_1, w_2 \leq v_2$ *a.e.* Then, the problem (1.1) has a solution w such that

$$v_1 \leq w \leq \min\{w_1, w_2\} \leq v_2 \ a.e. \ . \tag{3.15}$$

For proof of this claim we use corollary 3.3, where, $\min\{w_1, w_2\}$ is a super solution of problem (1.1). Similarly, By applying above arguments with $v_2 = \min\{w_1, w_2\}, v_1$, (without loss of generality), one finds a solution w of problem (1.1) satisfies (3.15). Therefore, it follows from the claim above that one finds a non-increasing sequence of solutions $\{w_n\}$ of problem (1.1) such that

$$v_1 \leq w_n \leq v_2 \ a.e. \ and \ \int_{\Omega} w_n \to A.$$

On the other hand, by Proposition 2.3 the sequence $\{f(., w_n)\}$ is equi-integrable in $L^1(\Omega; \rho_0 dx)$. It then follows from Egorov's theorem that

$$\lambda f(., w_n) + \mu h(w_n) \to \lambda f(., w) + \mu h(w)$$

in $L^1(\Omega; \rho_0 dx)$. Therefore, w is a solution of problem (1.1) and $\int_{\Omega} w = A$. By the claim above, w is the largest solution of problem (1.1) in the interval $[v_1, v_2]$. This completes the proof.

References

[1] K.Akô, On the Dirichlet problem for quasi-linear elliptic differential equations of the second order. J. Math. Soc. Japan.**13**, (1961) 45–62.

[2] J. Ali, R. Shivaji, Existence results for classes of Laplacian systems with sign-changing weight. Appl. Math. Lett. **20**, (2007) 558–562.

[3] J. Ali, R. Shivaji, Positive solutions for a class of p-Laplacian systems with multiple parameters. J. Math. Anal. Appl. **335**, (2007) 1019–1037.

[4] H. Brezis, M. Marcus and A. C. Ponce, Nonlinear elliptic equations with measures revisited. C. R. Acad. Sci. Paris, Ser. **I 339**, (2004) 169–174.

[5] H. Brezis, T. Cazenave, Y. Martel, and A. Ramiandrisoa, Blow up for $u_t - \triangle u = g(u)$ revisited. Adv. Diff. Eqns. **1**, (1996) 73–90.

[6] C. H. Chen, On positive weak solutions for a class of quasilinear elliptic systems. Nonl Anal. **62**, (2005) 751–756.

[7] M. Chhetri, D. D. Hai and R. Shivaji, On positive solutions for class of p-Laplacian semipositone systems. On positive solutions for class of p-Laplacian semipositone systems. **9**(4), (2003) 1063–1071.

[8] P. Clément and G. Sweers, Getting a solution between sub- and supersolutions without monotone iteration. Rend. Istit. Mat. Univ. Trieste. textbf19, (1987) 189–194.

[9] P. Dravak and J. Hernandez, Existence uniquenness of positive solution for some qusilinear elliptic problems. Nonl. Anal. **44**, (2001) 189–204.

[10] J. Deuel and P. Hess, A criterion for the existence of solutions of non-linear elliptic boundary value problems. Proc. Roy. Soc. Edinburgh Sect. **A74A**, (1974) 49–54.

[11] L. C. Evens, Partial Differential Equation (Graduate Studies in MathISSN). Amer. Math. Soc. **19**, (1998) 1065–7339.

[12] D.D Hai , R. Shivaji, An existence result on positive solutions for a class of p-Laplacian systems. Nonl. Anal.**56**, (2004) 1007–1010.

[13] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order. Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin.**24**(1983).

[14] D. A. Kandilakis and M. Magiropoulos, A subsolution-supersolution method for quasilinear systems. Elec. J.Diff. Eqns. **97**, (2005) 1–5.

[15] T.Kura, The weak supersolution-subsolution method for second order quasilinear elliptic equations. Hiroshima Math. J. **19**, (1989) 1–36.

[16] C. Maya, S. Oruganti and R. Shivaji, Positive solutions for classes of p-Laplacian Equations. Diff. Int. Eqns. **(16)**(6), (2003)757–768.

[17] G. Stampacchia, Équations elliptiques du second ordre á coefficients discontinus. Śeminaire de Mathématiques Supérieures. été, Les Presses de ĺUniversité de Montŕeal, Montŕeal, Qúebec. **16** (1966).