Existence of solution for semilinear elliptic equations via sub-super solutions method

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\textbf{Abstract.} In this paper, we consider a class of semilinear elliptic equations and extend some results about the method of sub-supper solutions. We obtain new results for the generalized semilinear elliptic equations using Schauder’s fixed point Theorem.

Keywords: sub-supper solution, semilinear problem, weak solution


\section{1. Introduction}

We consider the following semilinear problem:

\[
\begin{cases}
-\Delta u = \lambda g(x)f(x, u) + \mu h(u), & x \in \Omega, \\
u(x) = 0, & x \in \partial \Omega.
\end{cases}
\]  

\text{(1.1)}

where, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), and \( \lambda, \mu > 0 \) are parameters.

The classical method of sub-supper solutions (see \([11, 13, 14, 15]\)) asserts that if \( f \) is smooth and if one can find smooth sub-supper solutions \( v_1 \leq v_2 \) of \text{(1.1)}, then there exists a classical solution \( u \) of \text{(1.1)}, such
that $v_1 \leq u \leq v_2$. The classical proof is based on the monotone iteration scheme. This requires $f$ be Lipschitz (or locally Lipschitz) function. The existence of a smallest and a largest solutions $u_1 \leq u_2$, in the interval $[v_1, v_2]$, is implied by this argument. Another proof, based on Schauder’s fixed point theorem can be found in Akô [1, 9]. In this case, the existence of a smallest and a largest solution is proved separately, via a Perron-type argument. Using Akô’s strategy, Clément and Sweers [8] have implemented the method of sub-super solutions by the assumptions that $v_1, v_2 \in C(\bar{\Omega})$ and $f$ is continuous. Other study of this problem can also be found in [2, 3, 6, 7, 12, 16], specially, in Deuel-Hess [10] for $H^1$-solutions and in Brezis-Marcus-Ponce [4, 5] for $L^1$-solutions when $f$ is continuous and nondecreasing. In this paper, we extend the method of sub-super solutions in order to establish existence of the solutions of (1.1) in the sense of $L^1$-solution. We follow the strategy of [1, 8], based on the Schauder’s fixed point theorem. Substantially, some of the details be modified. We assume throughout the paper that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

**Definition 1.1.** We say that $u$ is an $L^1$-solution of (1.1) if

(a): $u \in L^1(\Omega)$ and $f(., u)\rho_0, h\rho_0 \in L^1(\Omega)$;

(b): for every $v \in C^2_0(\bar{\Omega})$,

$$-\int_{\Omega} u \Delta v = \int_{\Omega} (\lambda f(x, u) + \mu h(u)) v dx.$$ 

(1.2)

Here, $\rho_0(x) = d(x, \partial \Omega)$ for any $x \in \Omega$ and $C^2_0(\bar{\Omega}) = \{ v \in C^2(\bar{\Omega}) : v = 0 \text{ on } \partial \Omega \}$.

**Definition 1.2.** Let $u \in L^1(\Omega)$ and $f(., u)\rho_0, h\rho_0 \in L^1(\Omega)$ be given functions. Then we say that

(i): $u$ is an $L^1$-sub solution of (1.1), if

$$-\int_{\Omega} u \Delta v \leq \int_{\Omega} (\lambda f(x, u) + \mu h(u)) v dx,$$

for every $v \in C^2_0(\bar{\Omega})$.

(ii): $u$ is an $L^1$-super solution of (1.1) if

$$-\int_{\Omega} u \Delta v \geq \int_{\Omega} (\lambda f(x, u) + \mu h(u)) v dx,$$

for every $v \in C^2_0(\bar{\Omega})$. 

2. Boundedness and equi-integrable

**Definition 2.1.** A set $B \subseteq L^1(\Omega; \rho_0 dx)$ is equi-integrable if for every $\epsilon > 0$ there exists $\delta > 0$ such that $E \subset \Omega$ and

$$|E| < \delta \Rightarrow \int_E |g| \rho_0 dx < \epsilon \quad \forall g \in B.$$ 

**Lemma 2.2.** [8] Let $\{w_n\} \subset L^1$ and let $\{E_n\}$ be a sequence of measurable subsets of $\Omega$ such that

$$|E_n| \to 0 \quad \text{and} \quad \int_{E_n} |w_n| \geq 1 \quad \forall n \geq 1.$$ 

Then there exists a subsequence $\{w_{n_k}\}$ and a sequence of disjoint measurable sets $\{F_k\}$ such that

$$F_k \subset E_{n_k} \quad \text{and} \quad \int_{F_k} |w_n| \geq 1 \quad \forall n \geq 1.$$ 

**Proposition 2.3.** [8] Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, $h \in C(\Omega)$ and $v_1, v_2 \in L^1(\Omega)$ such that $v_1 \leq V_2$ a.e. Suppose that

$$f(., v) \rho_0, h \rho_0 \in L^1(\Omega) \quad \forall v \in L^1(\Omega)$$ 

such that $v_1 \leq v \leq v_2$ a.e.

Then, $v \mapsto \lambda f(., v) + \mu h(v)$ is bounded and equi-integrable in $L^1(\Omega; \rho_0 dx)$.

**Theorem 2.4.** Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and $h \in C(\Omega)$ such that

$$f(., v) \rho_0, h \rho_0 \in L^1(\Omega) \quad \forall v \in L^1(\Omega).$$

Then, the Nemytskii operator $F : L^1(\Omega; \rho_0 dx) \to L^1(\Omega)$ defined by

$$v \mapsto \lambda f(., v) + \mu h(v)$$

is continuous.

**Proof.** Suppose that $v_n \to v$ in $L^1(\Omega)$. Let $v_{n_k}$ be a subsequence such that $v_{n_k} \to v$ a.e. and $|v_{n_k}| \leq V$ a.e. For some function $V \in L^1(\Omega)$. In particular,

$$\lambda f(., v_{n_k}) + \mu h(v_{n_k}) \to \lambda f(., v) + \mu h(v) \quad \text{a.e.}$$

Moreover, by Proposition 2.3 the sequence $\{\lambda f(., v_{n_k}) + \mu h(v_{n_k})\}$ is equi-integrable in $L^1(\Omega; \rho_0 dx)$. It then follows from Egorov’s theorem that

$$\lambda f(., v_{n_k}) + \mu h(v_{n_k}) \to \lambda f(., v) + \mu h(v)$$
in \(L^1(\Omega; \rho_0 dx)\). Since the limit is independent of the subsequence \(\{v_{n_k}\}\) we deduce that
\[
F(v_n) \to F(v) \text{ in } L^1(\Omega; \rho_0 dx).
\]

3. Standard existence

**Theorem 3.1.** Suppose that \(f(.,u), h \in L^1(\Omega; \rho_0 dx)\), there exists a unique \(w \in L^1(\Omega)\) such that
\[
- \int_{\Omega} w \Delta v = \int_{\Omega} (\lambda f(x,w) + \mu h(w))vdx.
\]
for every \(v \in C^2_0(\bar{\Omega})\). Moreover,

(i): For every \(1 \leq p \leq \frac{n}{n-1}\), \(w \in L^p(\omega)\) and
\[
\|w\|_p \leq M(\|\lambda \rho_0 f\|_{L^1} + \|\mu \rho_0 h\|_{L^1}),
\]
for some constant \(M\).

(ii): Given \(\{f_n\}, \{h_n\} \in L^1(\Omega; \rho_0 dx)\) \(n \geq 1\), let \(w_n\) be the solution of (3.1) associated to \(\{f_n\}, \{h_n\}\). If \(\{f_n\}, \{h_n\}\) are bounded in \(L^1(\Omega)\), then \(\{w_n\}\) is relatively compact in \(L^1(\Omega)\) for every \(1 \leq p \leq \frac{n}{n-1}\).

**Proof.** We prove (i) and (ii) and refer the reader to [5] for the existence and uniqueness of \(w\).

Proof of (i). Note that \(w\) satisfies
\[
| \int_{\Omega} w \Delta v | = | \int_{\Omega} (\lambda f(.,w) + \mu h(w))v | \\
\leq \int_{\Omega} \|\lambda \rho_0 f(.,w) + \mu \rho_0 h(w)\|_{L^1} \|\frac{v}{\rho_0}\|_{L^\infty} \\
\leq M_1(\|\lambda \rho_0 f(.,w)\|_{L^1} + \|\mu \rho_0 h(w)\|_{L^1}) \|\frac{v}{\rho_0}\|_{L^\infty} \forall v \in C^2_0(\bar{\Omega}).
\]

(3.3)

Suppose that \(f, h \in C^\infty_0(\bar{\Omega})\) and \(v \in C^2_0(\bar{\Omega})\) be the solution of
\[
\begin{align*}
-\Delta v &= \lambda f + \mu h, \quad x \in \Omega, \\
v(x) &= 0, \quad x \in \partial \Omega.
\end{align*}
\]
(3.4)

By standard Calderón-Zygmund estimates [13]
\[
\|v\|_{W^{2,q}} \leq M_2(\|\lambda f\|_{L^q} + \|\mu h\|_{L^q}),
\]
(3.5)
where, $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, since $q > n$ it follows from Morrey’s embedding [13] that
\[ \|v\|_{L^\infty} \leq M_3(\|v\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \leq M_3\|v\|_{W^{2,q}}. \] (3.6)

According to conclusions above we get
\[ |\int_\Omega (\lambda f(\cdot, w) + \mu h(w))v| \leq M_1(\|\lambda\rho_0f\|_{L^1} + \|\mu\rho_0h\|_{L^1})\|v\|_{L^\infty} \]
\[ \leq M_1(\|\lambda\rho_0f\|_{L^1} + \|\mu\rho_0h\|_{L^1})\|v\|_{W^{2,q}} \]
\[ \leq M_1M_2M_3(\|\lambda\rho_0f\|_{L^1} + \|\mu\rho_0h\|_{L^1}). \] (3.7)

By duality, one deduces that $w \in L^p(\Omega)$ and 3.2 holds. Proof of (ii). Let $U \subset \Omega$ be a smooth domain, $v_n \in L^1(U)$ be the solution of the problem
\[ \begin{cases} -\Delta v_n = \lambda f + \mu h, & x \in U, \\ v_n(x) = 0, & x \in \partial U. \end{cases} \] (3.8)

By standard elliptic estimates [17] for every $1 \leq p \leq \frac{n}{n-\tau}$,
\[ \|v_n\|_{W^{1,p}(U)} \leq M_2(\|\lambda f\|_{L^q} + \|\mu h\|_{L^q}) \leq M_2(\|\lambda\|C_1 + \|\mu\|C_2), \] (3.9)
where, $\|f_n\|_{L^q} \leq C_1$ and $\|h_n\|_{L^q} \leq C_2$. On the other hand, since $w_n - v_n$ is harmonic in $U$ so for every $Y \subset U$ we have
\[ \|w_n - v_n\|_{C^1(Y)} \leq K_1\|w_n - v_n\|_{L^1(U)} \]
\[ \leq K_1\|\lambda f\rho_0 + \mu h\rho_0\|_{L^1} \]
\[ \leq K_1(\|\lambda\|\rho_0\|_{L^1} + \|\mu\|\rho_0\|_{L^1}) \]
\[ \leq K_1(\|\lambda\rho_0\|K_1 + \|\mu\rho_0\|K_2) \] (3.10)

where, $K_1, K_2$ are constants that $\|f\|_{L^1} \leq K_1$ and $\|h\|_{L^1} \leq K_2$ respectively. Therefore, there exists a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \to w$ a.e. in $\Omega$. On the other hand by (i) the sequence $\{w_n\}$ is bounded in $L^p(\Omega)$ for every $1 \leq p \leq \frac{n}{n-\tau}$. By Egorov’s theorem $\{w_n\}$ is converges in $L^1(\Omega)$ so this proof complete.

**Proposition 3.2.** Let $w \in L^1(\Omega)$ and $f, h \in L^1(\Omega; \rho_0 dx)$ be such that
\[ -\int_\Omega w\triangle v \geq \int_\Omega (\lambda f + \mu h)v \] (3.11)
for every $v \in C^2_0(\Omega)$ and $v \geq 0$ in $\Omega$. Then,
\[ -\int_\Omega w^-\triangle v \geq \int_{[0 \geq w]} (\lambda f + \mu h) \] (3.12)
for every $v \in C^2_0(\Omega)$ and $v \geq 0$ in $\Omega$, where, $w^- = \max\{-w, 0\}$. 
Proof. It is straightforward.

**Corollary 3.3.** If \( u, v \) are solutions of problem (1) then \( \min\{u, v\} \) is a super solution.

**Proof.** We set \( w = v - u \) and \( \varphi := [\lambda f(., v) + \mu h(v)] - [\lambda f(., u) + \mu h(u)] \), then

\[
\int_{\Omega} (v - u)^- \Delta s \geq \int_{[v \leq u]} ([\lambda f(x, v) + \mu h(v)] - [\lambda f(x, u) + \mu h(u)]) s \, dx,
\]

for every \( s \in C^2_0(\bar{\Omega}) \) and \( s \geq 0 \) in \( \Omega \). Since \( \min\{u, v\} = u + (v - u)^- \) so the result follows.

Now, we can state the main result.

**Theorem 3.4.** Let \( v_1, v_2 \) be a sub and a super solution of problem (1), respectively. Suppose that \( v_1 \leq v_2 \) a.e. and

\[
f(., v)\rho_0 \in L^1(\Omega), \text{ and } h(v)\rho_0 \in L^1(\Omega)
\]

for every \( v \in L^1(\Omega) \) such that \( v_1 \leq v_2 \) a.e. Then there exist solutions \( u_1 \leq u_2 \) of problem (1) in \([v_1, v_2]\) such that solution \( u \) of problem (1.1) in the interval \([v_1, v_2]\) satisfies

\[
v_1 \leq u_1 \leq u \leq u_2 \leq v_2 \text{ a.e.}
\]

**Proof.** Let be \((x, t) \in \Omega \times \mathbb{R}\), we define \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) such that

\[
f(x, t) = \begin{cases} v_1(x), & t < v_1(x), \\ t, & v_1(x) \leq t \leq v_2(x), \\ v_2(x), & v_2(x) < t. \end{cases}
\]

Then \( f \) is a Carathéodory function and by (3.13) \( f\rho_0, h\rho_0 \in L^1(\Omega) \) for every \( v \in L^1(\Omega) \). We set \( G : L^1(\Omega) \to L^1(\Omega; \rho_0 dx) \) defined by \( v \mapsto (\lambda f(., v)) + \mu h(v) \) and \( K : L^1(\Omega; \rho_0 dx) \to L^1(\Omega) \) defined by \( s \mapsto w \), where, \( w \) is the unique solution of the problem

\[
\begin{aligned}
-\Delta w &= s, & x \in \Omega, \\
w &= 0, & x \in \partial \Omega.
\end{aligned}
\]

By Theorem 2.4 and 3.1 \( KG : L^1(\Omega) \to L^1(\Omega) \) is continuous. Moreover, by Proposition 2.3 \( G(L^1(\Omega)) \) is a bounded subset of \( L^1(\Omega; \rho_0 dx) \). Therefore, by Theorem 3.1 \( KG \) is compact and there exists \( C > 0 \) such that

\[
\|KG(v)\|_{L^1} \leq C\|G(v)\|_{L^1} \leq C
\]

for every \( v \in L^1(\Omega) \). It follows from Schauder's fixed point theorem that \( KG \) has a fixed point \( u \in L^1(\Omega) \). In other words, \( u \) satisfies

\[
\begin{aligned}
-\Delta u &= \lambda f(x, u) + \mu h(u), & x \in \Omega, \\
u &= 0, & x \in \partial \Omega.
\end{aligned}
\]
We will show that $u$ is a solution of problem (1.1) and satisfies $v_1 \leq u \leq v_2$ a.e. To do this, we show that $v_1 \leq u$ a.e., the proof of the inequality $u \leq v_2$ a.e. is similar. Note that

$$\lambda f(., v_1) + \mu h(v_1) = \lambda f(., u) + \mu h(u) \text{ a.e.}$$

on the set $[v_1 \leq u]$. Therefore, by proposition 3.2 and with $w = v_1 - u$, we get

$$-\int_{\Omega} w^- \nabla v \geq \int_{[v_1 \leq u]} [(\lambda f(x.v_1) + \mu h(v_1)) - (\lambda f(x.u) + \mu h(u))]vdx = 0$$

for every $v \in C^2_0(\bar{\Omega})$ and $v \geq 0$ in $\Omega$. Since $w^- \geq 0$ a.e. so $w = 0$ a.e., this implies that $v_1 \leq u$ a.e. Now, we show that there exist a smallest and largest solution $u_1 \leq u_2$ of problem (1.1) in the interval $[v_1, v_2]$. We prove the existence of the smallest solution $u_1$, the existence of $u_2$ is similar. Let

$$A = \inf\{ \int_{\Omega} w ; v_1 \leq w \leq v_2 \text{ a.e. } \},$$

where, $w$ is a solution of problem (1.1).

By definition of solution for problem (1.1) implies that $A < \infty$. If $w_1, w_2$ are two solutions of problem (1.1) and $v_1, v_2$ are sub-super solution of problem (1.1), respectively, such that $v_1 \leq w_1, w_2 \leq v_2$ a.e. Then, the problem (1.1) has a solution $w$ such that

$$v_1 \leq w \leq \min\{w_1, w_2\} \leq v_2 \text{ a.e.} \quad (3.15)$$

For proof of this claim we use corollary 3.3, where, $\min\{w_1, w_2\}$ is a super solution of problem (1.1). Similarly, By applying above arguments with $v_2 = \min\{w_1, w_2\}, v_1$, (without loss of generality), one finds a solution $w$ of problem (1.1) satisfies (3.15). Therefore, it follows from the claim above that one finds a non-increasing sequence of solutions $\{w_n\}$ of problem (1.1) such that

$$v_1 \leq w_n \leq v_2 \text{ a.e. and } \int_{\Omega} w_n \rightarrow A.$$ 

On the other hand, by Proposition 2.3 the sequence $\{f(., w_n)\}$ is equi-integrable in $L^1(\Omega; \rho_0 dx)$. It then follows from Egorov’s theorem that

$$\lambda f(., w_n) + \mu h(w_n) \rightarrow \lambda f(., w) + \mu h(w)$$

in $L^1(\Omega; \rho_0 dx)$. Therefore, $w$ is a solution of problem (1.1) and $\int_{\Omega} w = A$.

By the claim above, $w$ is the largest solution of problem (1.1) in the interval $[v_1, v_2]$. This completes the proof.
REFERENCES


