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Intuitionistic fuzzy G-modules relative with a t-norm

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Abstract. In this paper we study about intuitionistic fuzzy G-modules and some properties of them. The relationship between t-norms and intuitionistic fuzzy G-modules will be investigated. Let G be a group and M be a $G-$module over K, which is a subfield of $\mathbb{C}$. Then an *intuitionistic fuzzy G-module* (IF$G-$module) on M is an intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of M such that following conditions are satisfied:

1) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$
$\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$, for every $a, b \in K$ and $x, y \in M$;
2) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, for every $g \in G$ and $m \in M$.

We investigate the nature of this kind of intuitionistic fuzzy G-modules under some algebraic operators.

Keywords: IF G-modules, t-norms, t-conorm, C-annihilation.

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1. Introduction

Fuzzy sets were defined by Zadeh [12] at first. Then many authors defined some algebraic fuzzy structures. Pushkav defined t-norm based

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fuzzy submodules [11]. Consequently, $(\alpha, \beta)-$fuzzy submodule with respect to a t-norm was introduced by Rahman and Saikia in [6].

The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. Algebraic structures play a vital role in mathematics and numerous applications of these structures are seen in many disciplines such as computer sciences, information sciences, theoretical physics, control engineering and so on. This inspires researchers to study and carry out research in various concepts of abstract algebra in fuzzy setting.

Biswas [2] applied the concept of intuitionistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group.

Fuzzy submodules of a module M over a ring R were first introduced by Naevoita and Ralescu [9]. Since then different types of fuzzy submodules were investigated in the last two decades. Many more results have been obtained by other researchers on intuitionistic fuzzy modules (see [4, 7, 10]).

In This paper, we introduce the notion of intuitionistic fuzzy $G-$modules with respect to a $t-$norm.

Throughout article R means a commutative ring with unity and M denotes a unitary left module.

$\vee$ and $\wedge$ denote respectively the maximum and minimum in the unit interval [0, 1].

2. Intuitionistic fuzzy sets and submodules

By R we mean a ring and M denotes a left R-module. The zero elements of R and M are 0 and θ, respectively.

Definition 2.1. Let X be a set, a map $\mu : X \to [0, 1]$ is called a *fuzzy subset* of X. The complement of μ, denoted by μ^c, is a fuzzy subset of X denoted by $\mu^c(x) = 1 - \mu(x)$ for every $x \in X$.

Definition 2.2. Let G be a group. A fuzzy subset μ of G is called a *fuzzy subgroup* of G if the following conditions hold for all $x, y \in G$
1) $\mu(xy) \geq \mu(x) \wedge \mu(y)$;
2) $\mu(x^{-1}) \geq \mu(x)$.

Definition 2.3. Let R be a ring and $I : R \to [0, 1]$. Then I is called a $fuzzy\ ideal$ of R if I satisfies the following:
1) $I(x - y) \geq \min\{I(x), I(y)\}$; $\forall x, y \in R$;
2) $I(xy) \geq \max\{I(x), I(y)\}$; $\forall x, y \in R$.

For our convenience, throughout the paper by a module we mean a left module.

Definition 2.4. Let M be an $R-$module and μ be a fuzzy subset of M. Then A is called a *fuzzy submodule* of M if A satisfies the following:
1) $\mu(0)=1$;
2) $\mu(x+y)\geq \mu(x)\wedge\mu(y)$; $\forall x,y\in M$;
3) $\mu(rx)\geq\mu(x)$; $\forall x\in M, r\in R$.

Definition 2.5. Let X be a non-empty set. *An intuitionistic fuzzy set* (IFS) A of X is an object of the form $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in X\}$, where

$$\mu_A : X\rightarrow[0,1]$$

and

$$\nu_A : X\rightarrow[0,1]$$

define the degree of membership and the degree of non-membership of the element $x\in X$, respectively, and for any $x\in X$, we have

$$\mu_A(x)+\nu_A(x)\leq 1$$

.

Definition 2.6. For any (IFS) $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in X\}$ of X, if

$$\pi_A(x)=1-\mu_A(x)-\nu_A(x)$$

for all $x\in X$. Then $\pi_A(x)$ is called the degree of indeterminacy of x in A.

Definition 2.7. Let $A=\{\langle x,\mu_A(x),\nu_A(x)\rangle : x\in X\}$ and
$B=\{\langle x,\mu_B(x),\nu_B(x)\rangle : x\in X\}$ any two IFS of X, then
1) $A\subseteq B$ if and only if $\mu_A(x)\leq\mu_B(x)$ and $\nu_A(x)\geq\nu_B(x)$ for all $x\in X$;
2) $A=B$ if and only if $\mu_A(x)=\mu_B(x)$ and $\nu_A(x)=\nu_B(x)$ for all $x\in X$;
3) $A^c=\{\langle x,(\mu_{A}^{c}),(\nu_{A}^{c})(x)\rangle : x\in X\}$ where $(\mu_{A}^{c})(x)=\nu_A(x)$ and $(\nu_{A}^{c})(x)=\mu_A(x)$ for all $x\in X$;
4) $A\cap B=\{\langle x,(\mu_A\cap\mu_B)(x),(\nu_A\cap\nu_B)(x)\rangle : x\in X\}$ where $(\mu_A\cap\mu_B)(x)=\min\{\mu_A(x),\mu_B(x)\}=\mu_A(x)\wedge\mu_B(x)$ and $(\nu_A\cap\nu_B)(x)=\max\{\nu_A(x),\nu_B(x)\}=\nu_A(x)\vee\nu_B(x)$;
5) $A\cup B=\{\langle x,(\mu_A\cup\mu_B)(x),(\nu_A\cup\nu_B)(x)\rangle : x\in X\}$ where $(\mu_A\cup\mu_B)(x)=\max\{\mu_A(x),\mu_B(x)\}=\mu_A(x)\vee\mu_B(x)$ and $(\nu_A\cup\nu_B)(x)=\min\{\nu_A(x),\nu_B(x)\}=\nu_A(x)\wedge\nu_B(x)$

Definition 2.8. Let G be a group. An IFS $A=(\mu_A(x),\nu_A(x))$ of G is called an *intuitionistic fuzzy subgroup* of G if the following conditions hold for all $x,y\in G$
1) $\mu_A(xy)\geq\mu_A(x)\wedge\mu_A(y)$;
2) $\nu_A(xy)\leq\nu_A(x)\vee\nu_A(y)$;
3) $\mu_A(x^{-1})\geq\mu_A(x)(consequently)\mu_A(x^{-1})=\mu_A(x)$;
4) $\nu_A(x^{-1})\geq\nu_A(x)(consequently)\nu_A(x^{-1})=\nu_A(x)$.

Definition 2.9. Let R be a ring and $A = (\mu_A(x), \nu_A(x))$ an IFS of R. Then A is called an *intuitionistic fuzzy ideal* of R if A satisfies the following

1) $\mu_A(x - y) \geq \mu_A(x) \wedge \mu_A(y)$, for every $x, y \in R$;
2) $\mu_A(xy) \geq \mu_A(y)$, for every $x, y \in R$;
3) $\nu_A(x - y) \leq \nu_A(x) \vee \nu_A(y)$, for every $x, y \in R$;
4) $\nu_A(xy) \leq \nu_A(y)$, for every $x, y \in R$.

Definition 2.10. Let M be an R-module and $A = (\mu_A(x), \nu_A(x))$ an IFS of M. Then A is called an *intuitionistic fuzzy submodule* (IFM) of M if A satisfies the following

1) $\mu_A(0) = 1$, $\nu_A(0) = 0$
2) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$, for all $x, y \in M$;
$\nu_A(x + y) \leq \nu_A(x) \vee nu_A(y)$, for all $x, y \in M$;
3) $\mu_A(rx) \geq \mu_A(x)$, for all $x, y \in M$ and $r \in R$;
$\nu_A(rx) \leq \nu_A(x)$, for all $x, y \in M$ and $r \in R$.

Definition 2.11. Let $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$ any two IFM's of X, then $(A + B) = (\mu_{A+B}(x), \nu_{A+B}(x))$, where

$$\mu_{A+B}(x) = \bigvee\{\mu_A(y) \wedge \mu_B(z)|_{x=y+z}; x, y, z \in M\}$$
$$\nu_{A+B}(x) = \bigwedge\{\nu_A(y) \vee \nu_B(z)|_{x=y+z}; x, y, z \in M\}$$

Definition 2.12. Let $A = (\mu_A, \nu_A)$ be an IFS of X. Define

$$\Box A = \{(x, \mu_A(x), \mu_A^c(x))|x \in X\}$$

and

$$\Diamond A = \{(x, \nu_A^c(x), \nu_A(x))|x \in X)\}.$$

Clearly $\Box A$ and $\Diamond A$ are IFS's of X. For an ordinary fuzzy set $A = \{(x, \mu(x), \mu^c(x))|x \in X\}$,

$$\Box A = A = \Diamond A.$$

If $A = \{(x, \mu_A(x), \nu_A(x))|x \in X\}$ is an IFS such that $0 \leq \mu_A(x) + \nu_A(x) < 1$ for some $x \in X$, then

$$\Box A \subset A \subset \Diamond A.$$

However If $A = \{(x, \mu_A(x), \nu_A(x))|x \in X\}$ is an IFS such that $\mu_A(x) + \nu_A(x) = 1$ for all $x \in X$.

Then A is merely an ordinary fuzzy set whose membership function is μ_A.

3. Fuzzy and intuitionistic fuzzy $G-$ modules

Definition 3.1. Let G be a group and M be a vector space over a field K. Then M is called a $G-module$ if for every $g \in G$ and $m \in M$, there exists a product (called the action of G on M), $g.m \in M$ satisfies the following axioms
1) $1_G.m = m, \forall m \in M$ ($1G$ being the identity of G);
2) $(gh).m = g.(h.m), \forall m \in M,\ g, h \in G$;
3) $g.(k_1 m_1 + k_2 m_2) = k_1(g.m_1) + k_2(g.m_2), \forall k_1, k_2 \in$K; $\forall m_1, m_2 \in M$ and $\forall g \in$G.

Definition 3.2. Let G be a group and K be a subset of M. Then K is called a G-submodule of M if K is a submodule of M and also a G-module.

Definition 3.3. Let G be a group and M be a $G-$module over K, which is a subfield of $\mathbb{C}$. Then an intuitionistic fuzzy G-module (IF$G-$module)on M is an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ of M such that following conditions are satisfied
1) $\mu_A(ax + by) \geq \mu_A(x) \wedge \mu_A(y)$
$\nu_A(ax + by) \leq \nu_A(x) \vee \nu_A(y)$, for every $a, b \in K$ and $x, y \in M$;
2) $\mu_A(gm) \geq \mu_A(m)$ and $\nu_A(gm) \leq \nu_A(m)$, for every $g \in G$ and $m \in M$.

Example 3.4. Let $G = \{-1, 1\}$, $M = \mathbb{R}^n$ over $\mathbb{R}$. Then M is a G-module. Define the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ on M by

$$\mu_A(x) = \begin{cases} 1, & if x = 0; \\ 0.5\ \ , & if x \neq 0. \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0, & if x = 0; \\ 0.25, & if x \neq 0. \end{cases}$$

Where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Then A is an intuitionistic fuzzy G-module on M.

Example 3.5. Consider the G-module $M = \mathbb{R}(i) = \mathbb{C}$ over the field $\mathbb{R}$ where $G = \{-1, 1\}$. Define the intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ by

$$\mu_A(z) = \begin{cases} 1, & z = 0; \\ 0.5, & z \in \mathbb{R} - \{0\}; \\ 0.25, & z \in \mathbb{R}(i) - \mathbb{R}. \end{cases}$$

and

$$\nu_A(z) = \begin{cases} 0, & z = 0; \\ 0.25, & z \in \mathbb{R} - \{0\}; \\ 0.5, & z \in \mathbb{R}(i) - \mathbb{R}. \end{cases}$$

Then A is an intuitionistic fuzzy G-module on M.

Remark 3.6. In Example (3.4), if we take the group $G = \{1, -1, i, -i\}$, then A is not IF G-module on M. Because the candition $\mu_A(gm) \geq \mu_A(m)$,

is not an satisfied, e. g. , take $m = 3$ and $g = i$, then

$$\mu_A(3i) = 0.25 \ngeq 0.5 = \mu_A(3)$$

also

$$\nu_A(3i) = 0.5 \nleq 0.25 = \nu_A(3).$$

4. t-norms and t-conorms

Definition 4.1. By a triangular norm or $(t-norm)$ we mean a mapping $T : [0, 1] \times [0, 1] \to [0, 1]$,

which satisfies the following axioms for every $a, b, c \in [0, 1]$

$(T1)$ $T(a, 1) = a$ (boundary condition);
$(T2)$ $b \leq c$ implies $T(a, b) \leq T(a, c)$ (monotonicity);
$(T3)$ $T(a, b) = T(b, a)$ (commutativity);
$(T4)$ $T(a, T(b, c)) = T(T(a, b), c)$ (associativity).

Definition 4.2. A fuzzy union or $(t - conorm)$ S is a mapping
$S : [0, 1] \times [0, 1] \to [0, 1]$;
which satisfies the following axioms for every $a, b, c \in [0, 1]$
$(S1)$ $S(a, 0) = a$ (boundary condition);
$(S2)$ $b \leq c$ implies $S(a, b) \leq S(a, c)$ (monotonicity);
$(S3)$ $S(a, b) = S(b, a)$ (commutativity);
$(S4)$ $S(a, S(b, c)) = S(S(a, b), c)$ (associativity).

Definition 4.3. Let M be a module over a ring R and μ a fuzzy subset of M. Then μ is called a *fuzzy submodule of M with respect to a t-norm T* if for every $x, y \in M$ and $r \in R$ the following conditions hold:
1) $\mu(\theta) = 1$;
2) $\mu(x + y) \geq T(\mu(x), \mu(y))$;
3) $\mu(rx) \geq \mu(x)$.

If T is the standard t-norm min, then μ is called a fuzzy submodule of M.

A t-norm T and a t-conorm S_T are called dual with respect to standard fuzzy complement if

(D_1) $1 - T(a, b) = S_T(1 - a, 1 - b)$,
(D_2) $1 - S_T(a, b) = T(1 - a, 1 - b)$,
for every $a, b \in [0, 1]$.

Definition 4.4. Let M be a module over a ring R, $A = (\mu_A, \nu_A)$ be an (IFS) of M T be a $t-$norm and S_T be its dual $t-$conorm. Then A

is said to be an *intuitionistic fuzzy submodule of* M *with respect to the* $t-norm$ T if and only if for every $x, y \in M$ and $r \in R$; the following axioms hold:
(M_1) $\mu_A(\theta) = 1$;
(M_2) $\mu_A(x+y) \geq T(\mu_A(x), \mu_A(y))$;
(M_3) $\mu_A(rx) \geq \mu_A(x)$;
(M_4) $\nu_A(x+y) \leq S_T(\nu_A(x), \nu_A(y))$;
(M_5) $\nu_A(rx) \leq \nu_A(x)$.

Since $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in M$ and $\mu_A(\theta) = 1$, we must have $\nu_A(\theta) = 0$.

If T and S_T are the standard $t-norm$ min and t-conorm max, then every IFS A satisfied to above conditions is an intuitionistic fuzzy submodule of M.

Example 4.5. Consider $M = \mathbb{Z}_8$ over $\mathbb{Z}$. Define $\mu_A, \nu_A \in [0,1]^M$ as follows:

$$\mu_A(x) = \begin{cases} 1, & x \in \{0,4\}; \\ 0.7, & x \in \{2,6\}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\nu_A(x) = \begin{cases} 0, & x \in \{0,4\}; \\ 0.2, & x \in \{2,6\}; \\ 1, & \text{otherwise.} \end{cases}$$

Then $A = (\mu_A, \nu_A)$ is an IF fuzzy submodule of M with respect to the following pairs of t-norms and t-conorms:
1) $\min(a,b), \max(a,b)$;
2) $ab, a+b-ab$;
3) $\max(0, a+b-1), \min(1, a+b)$.

Definition 4.6. [4] The *nilpotent minimum t-norm* T_{mo} is defined for every $a, b \in [0,1]$ as follows:

$$T_{mo}(a,b) = \left\{ \begin{array}{cc} 0, & b \leq 1-a; \\ min(a,b), & \text{otherwise.} \end{array} \right\}$$

.

Definition 4.7. The C-annihilation of T is denoted by $T_{(c)}$ and it is defined as follows:

$$T_{(c)}(a,b) : [0,1] \times [0,1] \to [0,1]$$

$$T_{(c)}(a,b) = \begin{cases} 0, & a \leq 1-b; \\ T(a,b), & \text{otherwise.} \end{cases}$$

for every $a, b \in [0,1]$.

Theorem 4.8. *For every continuous t-norm T, $T_{(c)}$ is a t-norm if only if $T_{(c)}$ is isomorphic either to $T_{(mo)}$*
or to $T_L(a,b) = \max(0, a+b-1)$ (Lukasiewics $t-norm$) or to

$$T_J(a,b) = \begin{cases} 0, & a \leq 1-b; \\ \frac{1}{3}+a+b-1, & a,b \in [\frac{1}{3}, \frac{2}{3}] \text{ and } a > 1-b; \\ \min(a,b) & \text{otherwise.} \end{cases}$$

for every $a, b \in [0,1]$.

Proof. It follows immediacy from [7]. □

Theorem 4.9. *Let R be a field, M be a module over R and $A = (\mu_A, \nu_A)$ be an IFS of M. Then for every $0 \neq \alpha \in R$, $\mu_A(\alpha x) = \mu_A(x)$ and $\nu_A(\alpha x) = \nu_A(x)$.*

Proof. Let $\alpha \neq 0 \in R$. Then $\nu_A(\alpha x) \leq \nu_A(x) = \nu_A(\alpha^{-1}(\alpha x) \leq \nu_A(\alpha x)$. Therefore, $\nu_A(\alpha x) = \nu_A(x)$. Similarly, $\mu_A(\alpha x) = \mu_A(x)$. □

5. Intuitionistic fuzzy $G-$ modules with respect to a $t-$norm

Definition 5.1. Let M be a $G-$ module over K, which is a subfield of $\mathbb{C}$. Let

$$A = \{(x, \mu_A(x), \nu_A(x)); x \in M\}$$

be an (IFS) of M, T be a $t-$norm and S_T be its dual $t-$conorm. Then A is said to be an *intuitionistic fuzzy $G-$submodule of M with respect to the $t-$norm T* if for every $x, y \in M$ and $a, b \in K$ and $g \in G$, the following axioms hold:
(M_1) $\mu_A(\theta) = 1$, $\nu_A(\theta) = 0$
(M_2) $\mu_A(ax+by) \geq T(\mu_A(ax), \mu_A(by))$;
(M_4) $\nu_A(ax+by) \leq S_T(\nu_A(ax), \nu_A(by))$;
(M_5) $\mu_A(gm) \geq \mu_A(m)$;
(M_6) $\nu_A(gm) \leq \nu_A(m)$.

By above definition we have the following corollary.

Corollary 5.2. *A fuzzy subset μ of M is a fuzzy $G-$submodule of M with respect to a $t-$norm T if and only if $A = (\mu, \mu^c)$ is an IF $G-$submodule of M with respect to the $t-$norm T.*

Theorem 5.3. *Let A be a non-empty subset of M. Then an $\bar{A} = (\mu_A, \nu_A)$ is defined by*

$$\mu_A(x) = \begin{cases} 1 & x \in A; \\ \alpha & \text{otherwise.} \end{cases}$$

$$\nu_A(x) = \begin{cases} 0 & x \in A; \\ \beta & \text{otherwise.} \end{cases}$$

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $\alpha+\beta \leq 1$ is an IF $G-$ submodule of M with respect to a $t-$ norm T if and only if A is a $G-$ submodule of M.

Proof. Let A be a $G-$submodule of M. Then $\theta \in A$ and so, $\mu_A(\theta)=1$. Let $x, y \in M$.
$Case(1)$:
if $x, y \in A$, then $ax+by \in A$ and so $\mu_A(ax+by)=1 \geq 1 \geq T(1,1)=T(\mu_A(ax), \mu_A(by))$.
Also $\nu_A(ax+by)=0 \leq 0=S_T(0,0)=S_T(\nu_A(ax), \nu_A(by))$
$Case(2)$:
if $x \in A, y \notin A$, then
$\mu_A(ax+by)=\alpha \geq \alpha=T(\alpha,1)=T(\mu_A(ax), \mu_A(by))$and
$\nu_A(ax+by)=\beta \leq \beta=S_T(0,\beta)=S_T(\nu_A(ax), \nu_A(by))$
$Case(3)$:
if $x \notin A, y \notin A$, $\mu_A(ax{+}by) \geq \alpha=T(1,\alpha) \geq T(\alpha,\alpha)=T(\mu_A(ax), \mu_A(by))$ and $\nu_A(ax+by) \leq \beta=S_T(\beta,\beta)=S_T(\nu_A(ax), \nu_A(by))$.
Thus for all cases $\mu_A(ax+by) \geq T(\mu_A, \mu_A)$ and so $\mu_A(rx)=1 \geq 1=\mu_A(x)$ and
$\nu_A(rx)=0 \leq 0=\nu_A(x)$. If $x \notin A$, then $\mu_A(x) \geq \alpha=\mu_A(x)$ and $\nu_A(rx) \leq \beta=\nu_A(x)$.
Therefore, $\overline{A}=(\mu_A, \nu_A)$ is an IF $G-$submodule of M with respect to a $t-$norm T. The converse is obvious. □

Corollary 5.4. *Let A be a non-empty subset of a $G-$module M. Then (χ_A, χ_A^c) is an IF $G-$submodule of M with respect to a $t-$norm T if and only if A is a $G-$submodule of M.*

Proof. It follows immediately from Theorem 5.3. □

Theorem 5.5. *Let R be a field, M be a G-module over R and $A=(\mu_A, \nu_A)$ be an IFS of M. If there exists $x \in M$ such that, $A(x)=(1,0)$, then $\mu_A(\theta)=1(\nu_A(\theta)=0)$.*

Proof. $\mu_A(\theta)=\mu_A(ax-ax) \geq T(\mu_A(ax), \mu_A(-ax))=T(1, \mu_A(-ax))=\mu_A(-ax) \geq \mu_A(x)=1$. Therefore, $\mu_A(\theta)=1$. □

Theorem 5.6. *An IF subset $A=(\mu_A, \nu_A)$ of a G-module M is an IF G-submodule of M with respect to a t-norm T if and only if $\Box A$ and $\Diamond A$ both are IF G-submodules of M with respect to the t-norm T.*

Proof. Let $A=(\mu_A, \nu_A)$ be an IF G-submodule of M with respect to a t-norm T. Let $x, y \in M$, $a, b \in K$ and $g \in G$. Then $\mu_A^c(ax+by)=1-\mu_A(ax+by) \leq 1-T(\mu_A(ax), \mu_A(by))=S_T(1-\mu_A(ax), 1-\mu_A(by))$. Therefore, $\mu_A^c(ax+by) \leq S_T(\mu_A^c(ax), \mu_A^c(by))$. Moreover, $\mu_A^c(gx)=1-\mu_A(gx) \leq 1-\mu_A(x)=\mu_A^c(x)$. The other three conditions can be obtained from our assumption that $A = (\mu_A, \nu_A)$ is an $IF\ G-submodule$ of M with respect to a t-norm T. Therefore, $\square A$ is an $IF\ G-submodule$ of M with respect to a t-norm T.Also, $\nu_A(\theta) = 0$ implies that $\nu_A^c(\theta) = 1$ Now, $\nu_A^c(ax + by) = 1 - \nu_A(ax + by) \geq 1 - S_T(\nu_A(ax), \nu_A(by)) = T(1 - \nu_A(ax), 1 - \nu_A(by))$. Therefore, $\nu_A^c(ax + by) \geq T(\nu_A^c(ax), \nu_A^c(by))$. Moreover, $\nu_A^c(gx) = 1 - \nu_A(gx) \geq 1 - \nu_A(x) = \nu_A^c(x)$. The other two conditions follow from our assumption that $A = (\mu_A, \nu_A)$ is an $IF\ G-submodule$ of M with respect to a t-norm T. Therefore, $\Diamond A$ is an $IF\ G - submodule$ of M with respect to a t-norm T. The converse part of the theorem is obvious. $\square$

Theorem 5.7. *Let $A = A_1, A_2, ..., A_n$, where $A_i = (\mu_i, \nu_i)$, $i = 1, 2, ..., n$ be $IF\ G-submodules$ of M with respect to a $t-norm$ T. Then $A_1 \cap A_2 \cap ... \cap A_n$ is also an $IF\ G-submodule$ of M with respect to the $t-norm$ T.*

Proof. Let $A = A_1 \cap A_2 \cap ... \cap A_n$. To show that A is an intuitionistic fuzzy $G-$submodule of M with respect to the $t-$norm T, we will use induction on n.

If $n = 1$, then $A = A_1$, and so, A is an $IF\ G-$submodule of M with respect to the $t-$norm T.

We assume that the intersection of $n - 1$ (or less) $IF\ G-$submodule of M with respect to the $t-$norm T is $IF\ G-$submodule of M with respect to the $t-$norm T. By the induction hypothesis,

$A_2 \cap A_3 \cap ... \cap A_n$ is an $IF\ G-$module of M with respect to the $t-$norm T.

Therefore,

$$(\mu_2 \cap \mu_3 ... \cap \mu_n(g\theta)) = T_{n-1}(\mu_2(\theta), \mu_3(\theta)...\mu_n(\theta) = 1.$$

Further

$$(\mu_1 \cap \mu_2 \cap ... \cap \mu_n)(g\theta) = T_n(\mu_1(g\theta), ...\mu_n(g\theta))$$

$$= T(\mu_1(g\theta)), T_{n-1}(\mu_2(g\theta), ...\mu_n(g\theta) = T(1, 1) = 1 \text{ (since } \mu_1(g\theta) = 1).$$

Then,

$$(\mu_1 \cap ... \cap \mu_n)(g\theta) = T_n(\mu_1(g\theta), ..., \mu_n(g\theta)) = 1.$$

Let $x, y \in M$ and $r \in R, g \in G$. Then

$$(\mu_1 \cap ... \cap \mu_n)(ax + by) = T_n(\mu_1(ax + by), ..., \mu_n(ax + by)) =$$

$$T(\mu_1(ax + by), T_{n-1}(\mu_2)(ax + by), ..., \mu_n(ax + by))) \geq$$

$$T(T(\mu_1)(ax), \mu_1(by), T(T_{n-1})(\mu_2)(ax), ..., \mu_n(ax)), T_{n-1}(\mu_2(by), ..., \mu_n(by))))$$

(since $A_2 \cap ... \cap A_n$ is an $IF\ G-$submodule of M).

$$= T(T(\mu_1)(by), \mu_1(ax)), T(T_{n-1}(\mu 2(ax), ..., \mu_n(ax)), T_{n-1}(\mu_2(by), ..., \mu_n(by)))) =$$

$$T(\mu_1(by), T(T(\mu_1(ax)), T_{n-1}(\mu_2(ax), ..., \mu_n(ax))), T_{n-1}(\mu 2(by), ..., \mu_n(by))))$$

$$= T(\mu_1(by), T(T_n(\mu_1(ax), ..., \mu_n(ax)), T_{n-1}(\mu_2(by), ..., \mu_n(by))))$$

$$= T(\mu_1(by), T(T_{n-1}(\mu_2(by), ..., \mu_n(by)), T_n(\mu_1(ax), ..., \mu_n(ax))))$$

$$= T(T(\mu_1(by), T_{n-1}(\mu_2(by), ..., \mu_n(by))), T_n(\mu_1(ax), ..., \mu_n(ax))) =$$

$$T(T_n(\mu_1(by), ..., \mu_n(by)), T_n(\mu_1(ax), ..., \mu_n(ax))) =$$

$T(T_n(\mu 1(ax), ..., \mu_n(ax)), T_n(\mu_1(by), ..., \mu_n(by))) = T((\mu_1 \cap ... \cap \mu_n)(ax), (\mu_1 \cap ... \cap \mu_n)(by))$

Thus,

$(\mu_1 \cap ... \cap \mu_n)(ax + by) \geq T((\mu_1 \cap ... \cap \mu_n)(ax), (\mu_1 \cap ... \cap \mu_n)(by))$.

Let S_T be the dual $t-$conorm of T.

Then

$$(\nu_1 \cup ... \cup \nu_n)(ax + by) = S_T(\nu_1(ax + by), ..., \nu_n(ax + by)) =$$

$$S_T(\nu_1(ax + by), S_{T_{n-1}}(\nu_2(ax + by), ..., \nu_n(ax + by))) \leq$$

$$S_T(S_T(\nu_1(ax), \nu_1(by), S_T(S_{T_{n-1}}(\nu_2(ax), ..., \nu_n(ax)), S_{T_{n-1}}(\nu_2(by), ..., \nu_n(by))))$$

(since $A_2 \cap ... \cap A_n$ is an IF $G-$submodule of M)

$$= S_T(S_T(\nu_1(by), \nu_1(ax)), S_T(S_{T_{n-1}}(\nu_2(ax), ..., \nu_n(ax)), S_{T_{n-1}}(\nu_2, ..., \nu_n(by)))) =$$

$S_T(\nu_1(by), S_T(S_T(\nu_1(ax), S_{T_{n-1}}(\nu_2(ax), ..., \nu_n(ax))), S_{T_{n-1}}(\nu_2(by), ..., \nu_n(by))))$.

Then,$(\nu_1 \cup ... \cup \nu_n)(ax + by) \leq S_T((\nu_1 \cup ... \cup \nu_n)(ax), (\nu_1 \cup ... \cup \nu_n)(by)$

$$\begin{aligned}(\mu_1 \cap ... \cap \mu_n)(gx) &= T_n(\mu_1(gx), ..., \mu_n(gx)) \\ &= T(\mu_1(gx), T_{n-1}(\mu_2(gx), ..., \mu_n(gx)) \\ &\geq T(\mu_1(x), T_{n-1}(\mu_2(x), ..., \mu_n(x)) \\ &= T_n(\mu_1(x), ..., \mu_n(x)) = (\mu_1 \cap ... \cap \mu_n)(x).\end{aligned}$$

$$(\nu_1 \cup ... \cup \nu_n)(gx) = S_{T_n}(\nu_1(gx), ..., \nu_n(gx)) =$$
$$S_T(\nu_1(gx), S_{T_{n-1}}(\nu_2(gx), ..., \mu_n(gx))$$
$$\leq S_T(\nu_1(x), S_{T_{n-1}}(\nu_2(x), ..., \nu_n(x))(\nu_1 \cup ... \cup \nu_n)(x).$$

Hence, $A = A_1 \cap .... \cap A_n$ is IF $G-$submodule of M with respect to the $t-$norm T. □

Theorem 5.8. *Suppose that the C-annihilation $T(c)$ of the $t-norm$ T provides a $t-norm$. If $A = (\mu_A; \nu_A)$ is an IF G-submodule of M with respect to the t-norm T, then $\Box A$ and $\Diamond A$ are IF G-submodules of M with respect to the t-norm T(c).*

Proof. Let $T(c)$ be the C-annihilation of T. Then the dual $S_{T_{(c)}}$ of $T_{(c)}$ is given by

$$S_{T_{(c)}}(a,b) = \begin{cases} 1, & 1-a \leq b; \\ 1 - T(1-a, 1-b), & \text{otherwise} \end{cases}$$

for every $a, b \in [0, 1]$. Since $A = (\mu_A, \nu_A)$ is an IF $G - submodule$ of M with respect to the t-norm T.

For every $x, y \in M$ and $r \in R$ we have
(M_1) $\mu_A(\theta) = 1$;
(M_2) $\mu_A(ax + by) \geq T(\mu_A(ax), \mu_A(by)) \geq T(c)(\mu_A(ax), \mu_A(by))$;
(M_3) $\mu_A(gx) \geq \mu_A(x)$;
(M_4) Now we get

$$S_{T_{(c)}}(\mu_A^c(ax), \mu_A^c(by)) = \begin{cases} 1, & \mu_A(ax) \leq 1 - \mu_A(by); \\ 1 - T(1-\mu_A^c(ax), 1-\mu_A^c(by)), & \text{otherwise.} \end{cases}$$

$S_{T_{(c)}}(\mu_A^c(ax), \mu_A^c(by)) \geq 1 - T(1-\mu_A^c(ax), 1-\mu_A^c(by))$
$= 1 - T(\mu_A(ax), \mu_A(by)) \geq 1 - \mu_A(ax + by)$
(since $\mu(ax + by) \geq T(\mu_A(ax), \mu_A(by))$)
$= \mu_A^c(ax + by)$.

Then, we have, $\mu_A^c(ax + by) \leq S_{T_{(c)}}(\mu_A^c(ax), \mu_A^c(by))$.
(M_5)
$\mu_A^c(gx) = 1 - \mu_A(gx) \leq 1 - \mu_A(x) = \mu_A^c(x)$.

Hence, $\Box A$ is an IF G-submodule of M with respect to $T_{(c)}$.

Next we claim that $\Diamond A$ is an IF G-submodule of M.
(M_1) Since $\lambda_A(\theta) = 0, \lambda_A^c(\theta) = 1$.
(M_2) Now

$$T_{(c)}(\lambda_A^c(ax), \lambda_A^c(by)) = \begin{cases} 0 & \lambda_A(ax) \leq 1 - \lambda_A(by); \\ T(\lambda_A^c(ax), \lambda_A^c(by)) & \text{otherwise.} \end{cases}$$

This implies

$T_{(c)}(\lambda^c_A(ax), \lambda^c_A(by)) \leq T(\lambda^c_A(ax), \lambda^c_A(by)) = 1 - S_T(\lambda_A(ax), \lambda_A(by))$ (*by duality*)

$\leq 1 - \lambda_A(ax + by)$, (since $\lambda_A(ax + by) \leq S_T(\lambda_A(ax), \lambda_A(by))) = \lambda^c_A(ax + by)$.

Thus, we have,

$\lambda^c_A(ax + by) \geq T_{(c)(\lambda^c_A(ax), \lambda^c_A(by)}$.

(M_3) $\lambda^c_A(gx) = 1 - \lambda_A(gx) \geq 1 - \lambda_A(x) = \lambda^c_A(x)$.

(M_4)

$\lambda_A(ax+by) \leq S_T(\lambda_A(ax), \lambda_A(by)) = 1 - T(1 - \lambda_A(ax), 1 - \lambda_A(by))$(*by duality*) $\leq S_{T_{(c)}}(\lambda_A(ax), \lambda_A(by))$.

Thus, we have $\lambda_A(ax + by) \leq S_{T_{(c)}}(\lambda_A(ax), \lambda_A(by))$.

(M_5) $\lambda_A(gx) \leq \lambda_A(x)$.

Hence, $\Diamond A$ is an IF G-submodule of M with respect to $T_{(c)}$.

□

Corollary 5.9. *Let $A = (\mu_A, \lambda_A)$ be an IF G-submodule of M with respect to the t-norm minimum.*

Then $\square A, \Diamond A$ and A are IF G-submodules of M with respect to the t-norm T_{mo} (the nilpotent minimum t-norm)

Proof. Follows from Theorem (3.8) and (4.9). □

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