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Solving Inverse Sturm-Liouville Problems with Transmission Conditions on Two Disjoint Intervals

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Abstract. In the present paper, some spectral properties of boundary value problems of Sturm-Liouville type on two disjoint bounded intervals with transmission boundary conditions are investigated. Uniqueness theorems for the solution of the inverse problem are proved, then we study the reconstructing of the coefficients of the Sturm-Liouville problem by the spectrtal mappings method.

Keywords: Inverse Sturm-Liouville problem, Asymptotic behavior, Transmission conditions, Weyl-Titchmarsh m-function, Spectrtal mappings method.

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1. Introduction

In this study, we consider the boundary value problem $\mathcal{L} := \mathcal{L}(\chi(x), A_{11}, A_{12}, A_{21}, A_{22})$ consisting of the Sturm-Liouville differential equation

$$-z''(x,\lambda) + \chi(x)z(x,\lambda) = \lambda z(x,\lambda), \tag{1.1}$$

on two disjoint intervals $[a,0) \cup (0,b]$, with the boundary conditions

$$U(z) := A_{11}z(a,\lambda) + A_{12}z'(a,\lambda) = 0, \tag{1.2}$$

$$V(z) := A_{21}z(b,\lambda) + A_{22}z'(b,\lambda) = 0, \tag{1.3}$$

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and the transmission conditions at the interface point $x = 0$ as

$$\begin{cases} a_{11}z'(0+,\lambda) + a_{12}z(0+,\lambda) + a_{13}z'(0-,\lambda) + a_{14}z(0-,\lambda) = 0, \\ a_{21}z'(0+,\lambda) + a_{22}z(0+,\lambda) + a_{23}z'(0-,\lambda) + a_{24}z(0-,\lambda) = 0, \end{cases} \tag{1.4}$$

where $\lambda = \rho^2$ is a spectral parameter, $-\infty < a < 0 < b < \infty$, A_{ij}, $i, j = 1, 2$, are non-zero real numbers, a_{ik}, $k = 1, 2$, are real constants, $\chi(x)$ is a real function and continuous on $[a, 0) \cup (0, b]$, and $\chi(c\pm 0) < \infty$.

Differential equations of the Sturm-Liouville type (1.1) often appear in mathematics, physics, chemistry, mathematical physics, mathematical chemistry and other branches of natural sciences (for example, see [4, 5, 7, 12, 14, 15, 16, 17, 21] and references therein). Also, boundary value problems with transmission conditions can be applied in problems such as electronics, mechanics and physics [1, 22]. Inverse problems with discontinuity condition together with separated bondary conditions on the interval $(0, 1)$ or $(0, \pi)$ were studied in [7, 18, 20]. Further, Sturm-Liouville boundary value problems with a discontinuity in an interior point on the interval $(0, \pi)$, and eigen-parameter dependent boundary conditions were investigated in [6, 9, 10, 11].

The rest of this paper is organized as follows: In Section 2, we obtain the spectral properties of $\mathcal{L}$ on $[a, 0) \cup (0, b]$, and investigate the fundamental solutions. In Section 3, the asymptotic behavior of the eigenvalues and the eigenfunctions when $\mathcal{L}$ has a discontinuity at $x = 0$, are studied. In Section 4, we intruduce the Weyl-Titchmarsh m-function and solution of $\mathcal{L}$, and using the spectrtal mappings method on $(a, 0) \cup (0, b)$ we study the recovering of the solution of the inverse problem associated with $\mathcal{L}$.

2. Asymptotic solutions

In this section, for sufficiently large values of $|\rho|$, we provide some estimates for fundamental solutions of (1.1) under suitable initial conditions, which will be used in studing the asymptotic behavior of the eigenvalues.

First, for $i, j = 1, 2$, we define

$$\mathbf{d}_{ij} = \det \begin{bmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{bmatrix}.$$

Also, we assume that $\mathbf{d}_{12} > 0$ and $\mathbf{d}_{34} > 0$.

Now, let $z_1(x, \lambda), v_1(x, \lambda)$ and $z_2(x, \lambda), v_2(x, \lambda)$ be the solutions of (1.1) on the intervalse $[a, 0)$ and $(0, b]$ respectively, which satisfy the initial conditions

$$z_1(a, \lambda) = A_{12}, \quad z_1'(a, \lambda) = -A_{11}, \tag{2.1}$$

$$\begin{cases} z_2(0+,\lambda) = \mathbf{d}_{12}^{-1}\{\mathbf{d}_{23}z_1(0-,\lambda) + \mathbf{d}_{24}z_1'(0-,\lambda)\}, \\ z_2'(0+,\lambda) = -\mathbf{d}_{12}^{-1}\{\mathbf{d}_{13}z_1(0-,\lambda) + \mathbf{d}_{14}z_1'(0-,\lambda)\}, \end{cases} \tag{2.2}$$

$$\begin{cases} v_1(0-,\lambda) = -\mathbf{d}_{34}^{-1}\{\mathbf{d}_{14}v_2(0+,\lambda) + \mathbf{d}_{24}v_2'(0+,\lambda)\}, \\ v_1'(0-,\lambda) = \mathbf{d}_{34}^{-1}\{\mathbf{d}_{13}v_2(0+,\lambda) + \mathbf{d}_{23}v_2'(0+,\lambda)\}, \end{cases} \tag{2.3}$$

$$v_2(b,\lambda) = -A_{22}, \quad v_2'(b,\lambda) = A_{21}. \tag{2.4}$$

For each fixed x, it is known that the solutions $z_1(x,\lambda)$ and $v_2(x,\lambda)$ are entire functions of λ. Moreover, by the method used in [2], it can be shown that the solutions $z_2(x,\lambda)$ and $v_1(x,\lambda)$ are also entire in λ for each fixed x.

Using the method of variation of parameters [3] and from the initial conditions (2.1)-(2.4), we can write:
(a) for $x \in [a,0)$,

$$\begin{aligned} z_1(x,\lambda) &= A_{12}\cos(\rho(x-a)) - A_{11}\frac{\sin(\rho(x-a))}{\rho} \\ &\quad + \frac{1}{\rho}\int_a^x q(t)\sin(\rho(x-t))z_1(t,\lambda)dt, \end{aligned} \tag{2.5}$$

$$\begin{aligned} v_1(x,\lambda) &= \{-\mathbf{d}_{14}\mathbf{d}_{34}^{-1}v_2(0+,\lambda) - \mathbf{d}_{24}\mathbf{d}_{34}^{-1}v_2'(0+,\lambda)\}\cos(\rho x) \\ &\quad + \{\mathbf{d}_{13}\mathbf{d}_{34}^{-1}v_2(0+,\lambda) + \mathbf{d}_{23}\mathbf{d}_{34}^{-1}v_2'(0+,\lambda)\}\frac{\sin(\rho x)}{\rho} \\ &\quad + \frac{1}{\rho}\int_x^0 q(t)\sin(\rho(x-t))v_1(t,\lambda)dt, \end{aligned}$$

(b) for $x \in (0,b]$,

$$\begin{aligned} z_2(x,\lambda) &= \{\mathbf{d}_{12}^{-1}\mathbf{d}_{23}z_1(0-,\lambda) + \mathbf{d}_{12}^{-1}\mathbf{d}_{24}z_1'(0-,\lambda)\}\cos(\rho x) \\ &\quad - \{\mathbf{d}_{12}^{-1}\mathbf{d}_{13}z_1(0-,\lambda) + \mathbf{d}_{12}^{-1}\mathbf{d}_{14}z_1'(0-,\lambda)\}\frac{\sin(\rho x)}{\rho} \\ &\quad + \frac{1}{\rho}\int_0^x q(t)\sin(\rho(x-t))z_2(t,\lambda)dt, \end{aligned}$$

$$\begin{aligned} v_2(x,\lambda) &= -A_{22}\cos(\rho(b-x)) - A_{21}\frac{\sin(\rho(b-x))}{\rho} \\ &\quad + \frac{1}{\rho}\int_x^b q(t)\sin(\rho(x-t))v_2(t,\lambda)dt. \end{aligned}$$

Multiplying (2.5) by $\exp(-|\Im\rho|(x-a))$, we obtain

$$\max|z_1(x,\lambda)\exp(-|\Im\rho|(x-a))| = O(1)$$

as $|\lambda| \to \infty$. This gives us $z_1(x,\lambda) = O(\exp(|\Im\rho|(x-a)))$ as $|\lambda| \to \infty$. This together with (2.5) yields

$$z_1(x,\lambda) = A_{12}\cos(\rho(x-a)) + O(\frac{\exp(|\Im\rho|(x-a))}{|\rho|}).$$

Similarly, we can obtain the asymptotic form of the solutions $z_2(x,\lambda)$, $v_1(x,\lambda)$ and $v_2(x,\lambda)$ as the following lemma.

Lemma 2.1. *As $|\lambda| \to \infty$, the following asymptotic formulas hold:*

$$z_1(x,\lambda) = A_{12}\cos(\rho(x-a)) + O(\frac{\exp(|\Im\rho|(x-a))}{|\rho|}), \tag{2.6}$$

$$z_2(x,\lambda) = A_{12}\boldsymbol{d}_{12}^{-1}\boldsymbol{d}_{24}\rho\sin(\rho b)\cos(\rho x) + O(\exp(|\Im\rho|(x-a))), \tag{2.7}$$

$$v_1(x,\lambda) = -A_{22}\boldsymbol{d}_{24}\boldsymbol{d}_{34}^{-1}\rho\sin(\rho b)\cos(\rho x) + O(\exp(|\Im\rho|(b-x))), \tag{2.8}$$

$$v_2(x,\lambda) = A_{22}\cos(\rho(b-x)) + O(\frac{\exp(|\Im\rho|(b-x))}{|\rho|}). \tag{2.9}$$

3. The eigenvalues and the eigenfunctions

First, we denote the characteristic function

$$\Delta(\lambda) := W(z_1(x,\lambda), v_1(x,\lambda)) = z_1(x,\lambda)v_1'(x,\lambda) - z_1'(x,\lambda)v_1(x,\lambda),$$

since we know from [13] that the Wronskian z_1 and v_1 is independent of x. Hence,

$$\Delta(\lambda) = z_1(0-,\lambda)v_1'(0-,\lambda) - z_1'(0-,\lambda)v_1(0-,\lambda). \tag{3.1}$$

Theorem 3.1. *The eigenvalues of the boundary value problem $\mathcal{L}$ are real and simple.*

Proof. Let λ be any eigenvalue of $\mathcal{L}$, and $z(x,\lambda)$ be the eigenfunction corresponding to λ. Then,

$$\mathbf{d}_{34}\int_a^0 \lambda z(x,\lambda)\overline{z(x,\lambda)}dx + \mathbf{d}_{12}\int_0^b \lambda z(x,\lambda)\overline{z(x,\lambda)}dx =$$
$$\mathbf{d}_{34}\int_a^0 (-z''(x,\lambda)+\chi(x)z(x,\lambda))\overline{z(x,\lambda)}dx$$
$$+\mathbf{d}_{12}\int_0^b (-z''(x,\lambda)+\chi(x)z(x,\lambda))\overline{z(x,\lambda)}dx.$$

Thus, using Lagrange's identity [23], we have

$$\begin{aligned}
&\mathbf{d}_{34}\int_a^0 \lambda z(x,\lambda)\overline{z(x,\lambda)}dx+\mathbf{d}_{12}\int_0^b \lambda z(x,\lambda)\overline{z(x,\lambda)}dx=\\
&\quad \mathbf{d}_{34}\int_a^0 z(x,\lambda)\overline{\lambda z(x,\lambda)}dx+\mathbf{d}_{12}\int_0^b z(x,\lambda)\overline{\lambda z(x,\lambda)}dx\\
&\quad +\mathbf{d}_{34}W(z,\overline{z})(0-)-\mathbf{d}_{34}W(z,\overline{z})(a)\\
&\quad +\mathbf{d}_{12}W(z,\overline{z})(b)-\mathbf{d}_{12}W(z,\overline{z})(0+). \qquad (3.2)
\end{aligned}$$

On the other hand, since the eigenfunction $z(x,\lambda)$ is satisfied (1.1)-(1.4), therefore $W(z,\overline{z})(a)=0=W(z,\overline{z})(b)$, moreover,

$$\begin{aligned}
W(z,\overline{z})(0-)&=z(0_-,\lambda)\frac{\partial\overline{z}(0_-,\lambda)}{\partial x}-\overline{z}(0_-,\lambda)\frac{\partial z(0_-,\lambda)}{\partial x}\\
&=\frac{\mathbf{d}_{12}}{\mathbf{d}_{34}}\{z(0_+,\lambda)\frac{\partial\overline{z}(0_+,\lambda)}{\partial x}-\overline{z}(0_+,\lambda)\frac{\partial z(0_+,\lambda)}{\partial x}\}.
\end{aligned}$$

Consequently, we get

$$\begin{cases} W(z,\overline{z})(a)=W(z,\overline{z})(b)=0,\\ \mathbf{d}_{34}W(z,\overline{z})(0-)=\mathbf{d}_{12}W(z,\overline{z})(0+). \end{cases} \qquad (3.3)$$

Substituting (3.3) into (3.2), we get

$$(\lambda-\overline{\lambda})\{\mathbf{d}_{34}\int_a^0 z^2(x,\lambda)dx+\mathbf{d}_{12}\int_0^b z^2(x,\lambda)dx\}=0.$$

This together with $\mathbf{d}_{12}>0$ and $\mathbf{d}_{34}>0$ yields $\lambda=\overline{\lambda}$, and consequently, the eigenvalues of $\mathcal{L}$ are real.

Now, let $z(x,\lambda)$ and $\widetilde{z}(x,\lambda)$ be the eigenfunctions corresponding to the eigenvalue λ. According to (1.2), we have $z(a,\lambda)\widetilde{z}'(a,\lambda)-z'(a,\lambda)\widetilde{z}(a,\lambda)=0$, i.e. for some $0\neq\eta\in\mathbb{R}$,

$$z(a,\lambda)=\eta\widetilde{z}(a,\lambda),\quad z'(a,\lambda)=\eta\widetilde{z}'(a,\lambda).$$

Hence, it follows from the uniqueness theorem for the solution of (1.1) that there exist the non-zero constants η_1 and η_2 such that

$$z(x,\lambda)=\eta_1\widetilde{z}(x,\lambda),\quad x\in[a,0),\qquad z(x,\lambda)=\eta_2\widetilde{z}(x,\lambda),\quad x\in(0,b].$$

These together with substituting $z(x,\lambda)$ and $\widetilde{z}(x,\lambda)$ into (1.4) yield $\eta_1=\eta_2$. Consequently, $z(x,\lambda)$ and $\widetilde{z}(x,\lambda)$ are linearly dependent. This completes the proof. □

In the next theorem, we present the asymptotic forms of the eigenvalues and the eigenfunctions of $\mathcal{L}$.

Theorem 3.2. *If $a = -b$. Then:*
(a) the asymptotic form of the eigenvalues of the boundary value problem $\mathcal{L}$ is as follow:

$$\rho_n = \sqrt{\lambda_n} = \frac{(n-1)\pi}{2b} + O(\frac{1}{n}) \qquad as\ n \to \infty; \tag{3.4}$$

(b) the eigenfunctions of $\mathcal{L}$ have the following forms as $n \to \infty$:

$$z_1(x, \lambda_n) = A_{12} \cos(\rho_n(x + b)) + O(\frac{1}{n}), \qquad x \in [-b, 0),$$
$$z_2(x, \lambda_n) = A_{12} \boldsymbol{d}_{12}^{-1} \boldsymbol{d}_{24} \rho_n \sin(\rho_n b) \cos(\rho_n x) + O(1), \quad x \in (0, b].$$

Proof. It is well-known from [13] that the eigenvalues of $\mathcal{L}$ coincide with the zeros of the characteristic function $\Delta(\lambda)$. Substituting (2.6) and (2.8) into (3.1), we calculate

$$\Delta(\lambda) = A_{12}A_{22}\mathbf{d}_{24}^{-1}\mathbf{d}_{34}\rho^2 \sin(\rho a) \sin(\rho b) + O(|\rho| \exp(|\Im\rho|(b - a))). \tag{3.5}$$

On the other hand, it follows from Rouche's theorem [19] that $\Delta(\lambda)$ has the same number of zeros inside the appropriate large contours as the term $A_{12}A_{22}\mathbf{d}_{24}^{-1}\mathbf{d}_{34}\rho^2 \sin(\rho a) \sin(\rho b)$. So, the zeros of $\Delta(\lambda)$ are countable, can be numbered as $\lambda_1 < \lambda_2 < \dots$, and from the hypothesis $a = -b$, $\rho_n = \frac{(n-1)\pi}{2b} + \tau_n$, where $\tau_n < \frac{\pi^2}{4b}$ for sufficiently large n. From this and (3.5) we conclude $\tau_n = O(\frac{1}{n})$, and we arrive at (3.4). Moreover, since the eigenfunctions of $\mathcal{L}$ are

$$z(x, \lambda_n) = \begin{cases} z_1(x, \lambda_n) \\ z_2(x, \lambda_n) \end{cases},$$

substituting (3.4) into (2.6)-(2.7) we arrive at (b). □

4. Reconstruction of differential operator

In this section, we study two uniqueness theorems for the solution of the inverse problem associated with $\mathcal{L}$, first by the Weyl-Titchmarsh m-function, and second by nodes of $\mathcal{L}$. Then, we solve the inverse problem using the spectral mappings method. Here, we denote

$$m(\lambda) = \frac{-v_2(a, \lambda)}{A_{12}\Delta(\lambda)}. \tag{4.1}$$

$m(\lambda)$ is called the Weyl-Titchmarsh m-function of $\mathcal{L}$. According to (2.9) and (3.5),

$$m(\lambda) = \frac{\mathbf{d}_{34}}{A_{12}^2\mathbf{d}_{24}\lambda} + O(\frac{1}{|\lambda|} \exp(|\Im\sqrt{\lambda}|(b - a)))$$

as $|\rho| \to \infty$. Now, let $F(x,\lambda)$ be the solution of (1.1) satisfies (1.4) and the following initial conditions

$$F(a,\lambda) = 0, \quad F'(a,\lambda) = \frac{1}{A_{12}}.$$

We define

$$\Phi(x,\lambda) := \frac{v_2(x,\lambda)}{\Delta(\lambda)}, \tag{4.2}$$

which is called Weyl-solution of $\mathcal{L}$. Hence, it follows from (2.1), (4.1) and (4.2) that

$$\Phi(x,\lambda) = -F(x,\lambda) - m(\lambda) z_1(x,\lambda). \tag{4.3}$$

Now, let $\widetilde{\mathcal{L}}$ be the boundary value problem consisting of (1.1)-(1.4) but with coefficients $\widetilde{\chi}(x)$, $\widetilde{A}_{ij}$ and $\widetilde{a}_{ij} = a_{ik}$, $i,j = 1,2$, $k = 1,2,3,4$.

Theorem 4.1. *(Uniqueness Theorem 1) If $m(\lambda) = \widetilde{m}(\lambda)$, then $\mathcal{L} = \widetilde{\mathcal{L}}$, i.e. $\chi(x) = \widetilde{\chi}(x)$ and $A_{ij} = \widetilde{A}_{ij}$, $i,j = 1,2$.*

Proof. We define the matrix $\mathbf{M}(x,\lambda) = \left[\mathbf{M}_{ij}(x,\lambda)\right]_{i,j=1,2}$ by the formula

$$\mathbf{M}(x,\lambda) \begin{bmatrix} \widetilde{z}_1(x,\lambda) & \widetilde{\Phi}(x,\lambda) \\ \widetilde{z}'_1(x,\lambda) & \widetilde{\Phi}'(x,\lambda) \end{bmatrix} = \begin{bmatrix} z_1(x,\lambda) & \Phi(x,\lambda) \\ z'_1(x,\lambda) & \Phi'(x,\lambda) \end{bmatrix}.$$

Therefore, since $W(\widetilde{z}_1(x,\lambda), \widetilde{\Phi}(x,\lambda)) \equiv 1$, we have

$$\begin{cases} \mathbf{M}_{11}(x,\lambda) = z_1(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \widetilde{z}'_1(x,\lambda)\Phi(x,\lambda), \\ \mathbf{M}_{12}(x,\lambda) = \widetilde{z}_1(x,\lambda)\Phi(x,\lambda) - z_1(x,\lambda)\widetilde{\Phi}(x,\lambda), \end{cases} \tag{4.4}$$

$$\begin{cases} \mathbf{M}_{21}(x,\lambda) = z'_1(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \widetilde{z}'_1(x,\lambda)\Phi'(x,\lambda), \\ \mathbf{M}_{22}(x,\lambda) = \widetilde{z}_1(x,\lambda)\Phi'(x,\lambda) - z'_1(x,\lambda)\widetilde{\Phi}(x,\lambda). \end{cases}$$

These together with (4.1) and (4.3) conclude that for fixed x, the functions $\mathbf{M}_{ij}(x,\lambda)$ are meromorphic functions in λ, and have simple poles in the eigenvalues $\lambda_n, \widetilde{\lambda}_n$ of $\mathcal{L}, \widetilde{\mathcal{L}}$, respectively. Substituting (4.3) and $\widetilde{\Phi}(x,\lambda) = \widetilde{F}(x,\lambda) - \widetilde{m}(\lambda)\widetilde{z}_1(x,\lambda)$ into (4.4) we have

$$\begin{cases} \mathbf{M}_{11}(x,\lambda) = -z_1(x,\lambda)\widetilde{F}'(x,\lambda) + \widetilde{z}'_1(x,\lambda)F(x,\lambda) \\ \qquad\qquad +(m(\lambda) - \widetilde{m}(\lambda))z_1(x,\lambda)\widetilde{z}'_1(x,\lambda), \\ \\ \mathbf{M}_{12}(x,\lambda) = -\widetilde{z}_1(x,\lambda)F(x,\lambda) + z_1(x,\lambda)\widetilde{F}(x,\lambda) \\ \qquad\qquad +(\widetilde{m}(\lambda) - m(\lambda))z_1(x,\lambda)\widetilde{z}_1(x,\lambda). \end{cases} \tag{4.5}$$

Therefore, since $m(\lambda) = \widetilde{m}(\lambda)$, $\mathbf{M}_{11}(x,\lambda)$ and $\mathbf{M}_{12}(x,\lambda)$ are entire in λ for fixed x. Moreover, from (2.9), (3.5) and (4.2), for $\rho \in B_r := \{\lambda : |\lambda - \lambda_n| > r,\ n = 1,2,3,\dots\}$ we have

$$|\Phi^{(m)}(x,\lambda)| \le \alpha_r |\rho|^{m-2} \exp(-|\Im\rho|(x-a)), \quad m = 0,1,$$

and for $\rho \in \widetilde{B}_r := \{\lambda : |\lambda - \widetilde{\lambda}_n| > r,\ n = 1, 2, 3, \dots\}$,

$$|\widetilde{\Phi}^{(m)}(x,\lambda)| \le \widetilde{\alpha}_r |\rho|^{m-2} \exp(-|\Im\rho|(x-a)), \quad m = 0, 1,$$

where α_r and $\widetilde{\alpha}_r$ are positive constants. Thus, for $\rho \in B_r \cup \widetilde{B}_r$, as $|\rho| \to \infty$ we obtain the following approximations:

$$|\mathbf{M}_{11}(x,\lambda)| \le \alpha_r, \quad |\mathbf{M}_{12}(x,\lambda)| \le \alpha_r |\rho|^{-2}.$$

These together with (4.5) yield $\mathbf{M}_{11}(x,\lambda) \equiv p(x)$ and $\mathbf{M}_{12}(x,\lambda) \equiv 0$. So,

$$z_1(x,\lambda) = p(x)\widetilde{z}_1(x,\lambda), \quad \Phi(x,\lambda) = p(x)\widetilde{\Phi}(x,\lambda).$$

Since $W(z_1, \Phi) = W(\widetilde{z}_1, \widetilde{\Phi}) \equiv 1$, $p(x) \equiv 1$. Hence, $z_1(x,\lambda) = \widetilde{z}_1(x,\lambda)$ and $\Phi(x,\lambda) = \widetilde{\Phi}(x,\lambda)$, and consequently, $\mathcal{L} = \widetilde{\mathcal{L}}$, i.e. $\chi(x) = \widetilde{\chi}(x)$ and $A_{ij} = \widetilde{A}_{ij}$. □

Remark 4.2. If $a = -b$. Then, the nodal points $\{x_n^j\}$, $n = 1, 2, 3, \dots$, $j = 1, 2, \dots, n-1$, of the boundary value problem $\mathcal{L}$ have the form

$$x_n^{j,1} = \begin{cases} -b + \frac{(2j-1)\pi}{n-1} + O(\frac{1}{n^2}), & x \in [-b, 0), \\ \frac{(2j-1)\pi}{n-1} + O(\frac{1}{n^2}), & x \in (0, b]. \end{cases}$$

Using Remark 4.2 and by the method used in the proof of Theorem 3 in [17], we can prove the following theorem.

Theorem 4.3. *(Uniqueness Theorem 2) Let $\chi(x)$ and $\widetilde{\chi}(x)$ be the potentials in (1.1) for $\mathcal{L}$ and $\widetilde{\mathcal{L}}$, respectively, $a = -b$, and $\{x_n^j\}, \{\widetilde{x}_n^j\}$ be the nodal points of $\mathcal{L}, \widetilde{\mathcal{L}}$, respectively. If $x_n^j = \widetilde{x}_n^j$, then $\chi(x) = \widetilde{\chi}(x)$ a.e. on $(-b, 0) \cup (0, b)$.*

Now, we denote

$$D(x,\lambda,\zeta) := \frac{W(z_1(x,\lambda), z_1(x,\zeta))}{\lambda - \zeta} = \int_a^x z_1(t,\lambda), z_1(t,\zeta)dt. \tag{4.6}$$

Let $\lambda_n = \rho_n^2$ and $\zeta_n = \sigma_n^2$. It follows from Lemma 2.1 that

$$D(x,\lambda_n,\zeta_n) = \begin{cases} \int_a^x z_1(t,\rho_n) z_1(t,\sigma_n)dt, & x \in [a, 0), \\ \int_a^0 z_1(t,\rho_n) z_1(t,\sigma_n)dt + \\ \qquad \int_0^x z_1(t,\rho_n) z_1(t,\sigma_n)dt, & x \in (0, b]. \end{cases}$$

Similarly, we can obtain for $\widetilde{D}(x,\lambda,\zeta)$. Further, using the method used in the proof of Theorem 3 in [8], it can be shown that for $\lambda = \rho^2$ and

$\zeta=\sigma^2\geq 0$, the following relations hold:

$$z_1(x,\lambda)=\widetilde{z}_1(x,\lambda)-\frac{1}{2\pi i}\int_\Lambda \widetilde{T}(x,\lambda,\zeta)z_1(x,\zeta)d\zeta, \tag{4.7}$$

$$T(x,\lambda,\zeta)=\widetilde{T}(x,\lambda,\zeta)-\frac{1}{2\pi i}\int_\Lambda \widetilde{T}(x,\lambda,\theta)T(x,\zeta,\theta)d\theta, \tag{4.8}$$

where

$$\begin{cases} T(x,\lambda,\zeta)=D(x,\lambda,\zeta)(m(\lambda)-\widetilde{m}(\lambda)),\\ \widetilde{T}(x,\lambda,\zeta)=\widetilde{D}(x,\lambda,\zeta)(m(\lambda)-\widetilde{m}(\lambda)),\end{cases} \tag{4.9}$$

$\Lambda=\Lambda_1\cup\Lambda_2$, Λ_1 is a bounded closed contour encircling the set $\{\lambda_n=\rho_n^2:\rho_n\neq 0,\Im\rho_n\geq 0\}$, and Λ_2 is the two-sided cutting along the $\{\lambda:\lambda>0,\lambda\in\mathbb{C}\backslash\Lambda_1\}$. We denote by $\mathbf{B}(\Lambda)$ the Banach space containing bounded functions $f(\lambda)$, $\lambda\in\Lambda$, with the norm $||f||:=\sup_{\lambda\in\Lambda}|f(\lambda)|$.

Now, for constructing the solution of the inverse problem, we need to prove the following uniqueness theorem for the solution of the equation (4.7) which playes the central role in studing the inverse problem.

Theorem 4.4. *For each fixed $x\in(a,0)\cup(0,b)$, the equation (4.7) has a unique solution $z_1\in\boldsymbol{B}(\Lambda)$.*

Proof. For fixed $x\in(a,0)$, consider the linear bounded operators H and $\widetilde{H}$ in $\mathbf{B}(\Lambda)$ as follows:

$$Hf(\lambda)=f(\lambda)-\frac{1}{2\pi i}\int_\Lambda T(x,\lambda,\zeta)f(\zeta)d\zeta,$$
$$\widetilde{H}f(\lambda)=f(\lambda)+\frac{1}{2\pi i}\int_\Lambda \widetilde{T}(x,\lambda,\zeta)f(\zeta)d\zeta.$$

Therefore,

$$\widetilde{H}Hf(\lambda)=f(\lambda)-\frac{1}{2\pi i}\int_\Lambda\{T(x,\lambda,\zeta)-\widetilde{T}(x,\lambda,\zeta) +\frac{1}{2\pi i}\int_\Lambda \widetilde{T}(x,\lambda,\zeta)T(x,\theta,\zeta)d\theta\}f(\zeta)d\zeta.$$

This together with (4.8) yields that for $f(\zeta)\in\mathbf{B}(\Lambda)$, $\widetilde{H}Hf(\lambda)=f(\lambda)$. Similarly, we can obtain $H\widetilde{H}f(\lambda)=f(\lambda)$. Thus, $\widetilde{H}H=H\widetilde{H}=I$, where I is the identity operator. So, $\widetilde{H}$ has a bounded inverse operator. Consequently, for each fixed $x\in(a,0)$, the equation (4.7) has a unique solution. For $x\in(0,b)$, the claim is proved, analogously. □

In the next theorem, we construct the solution of the inverse problem.

Theorem 4.5. *The following relations hold:*

$$\chi(x)=\widetilde{\chi}(x)-2\varepsilon'(x), \tag{4.10}$$

$$\begin{cases} A_{11} = \widetilde{A}_{11} + \varepsilon(a)\widetilde{A}_{12}, & A_{12} = \widetilde{A}_{12} - \varepsilon(a), \\ A_{21} = \widetilde{A}_{21} + \varepsilon(b)\widetilde{A}_{22}, & A_{22} = \widetilde{A}_{12}, \end{cases} \tag{4.11}$$

where

$$\varepsilon(x) = \frac{1}{2\pi i}\int_{\Lambda} z_1(x,\zeta)\widetilde{z}_1(x,\zeta)m(\zeta)d\zeta. \tag{4.12}$$

Proof. According to (4.6), (4.9) and (4.12) we have

$$\widetilde{z}_1'(x,\lambda) = z_1(x,\lambda) + \widetilde{z}_1(x,\lambda)\varepsilon(x) + \frac{1}{2\pi i}\int_{\Lambda}\widetilde{T}(x,\lambda,\zeta)\widetilde{z}_1'(x,\zeta)d\zeta, \tag{4.13}$$

$$\begin{aligned}\widetilde{z}_1''(x,\lambda) = z_1(x,\lambda) &+ \frac{1}{2\pi i}\int_{\Lambda}\widetilde{T}(x,\lambda,\zeta)\widetilde{z}_1''(x,\zeta)d\zeta \\ &+\frac{1}{2\pi i}\int_{\Lambda} 2\widetilde{z}_1(x,\lambda)\widetilde{z}_1(x,\zeta)z_1'(x,\zeta)m(\zeta)d\zeta \\ &+\frac{1}{2\pi i}\int_{\Lambda} 2(\widetilde{z}_1(x,\lambda)\widetilde{z}_1(x,\zeta))'z_1(x,\zeta)m(\zeta)d\zeta.\end{aligned} \tag{4.14}$$

Since

$$\begin{aligned}\widetilde{z}_1''(x,\lambda) &= \widetilde{\chi}(x)\widetilde{z}_1(x,\lambda) - \lambda\widetilde{z}_1(x,\lambda), \\ z_1''(x,\zeta) &= \chi(x)z_1(x,\zeta) - \zeta z_1(x,\zeta),\end{aligned}$$

substituting these and the relation (4.7) into (4.14) give us

$$\begin{aligned}(\chi(x) - \widetilde{\chi}(x))\widetilde{z}_1(x,\lambda) = &-\frac{1}{2\pi i}\int_{\Lambda} W(z_1(x,\lambda), z_1(x,\zeta))z_1(x,\zeta)m(\zeta)d\zeta \\ &-\frac{z_1(x,\lambda)}{2\pi i}\int_{\Lambda} 2z_1(x,\zeta)z_1'(x,\zeta)m(\zeta)d\zeta \\ &-\frac{1}{2\pi i}\int_{\Lambda}(\widetilde{z}_1(x,\lambda)\widetilde{z}_1(x,\zeta))'z_1(x,\zeta)m(\zeta)d\zeta,\end{aligned}$$

and hence (4.10) is obtained after cancelling the terms with $z_1'(x,\lambda)$. Also, taking $x = a$ and $x = b$ in (4.7) and (4.13), we arrive at (4.11). The proof is complete. □

5. Conclusions

In this paper, we investigated a boundary value problem of Sturm-Liouville type on two disjoint bounded intervals $[a,0)\cup(0,b]$, with transmission conditions at the interface point $x = 0$. Using two methods, first by the Weyl-Titchmarsh m-function, and second by the nodal points (the zeros of the eigenfunctions) of the Sturm-Liouville problem, we proved that the coefficients of the boundary value problem can be uniquely determined by spectral data. Then, by the spectral mappings method, we determined the continuous potential function $\chi(x)$ and the nonzero real coefficients A_{ij}, $i, j = 1, 2$, of the boundary conditions (1.2)-(1.3).

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