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# Bounds on First Reformulated Zagreb Index of Graph

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Abstract. The first reformulated Zagreb index $EM_1(G)$ of a simple graph $G$ is defined as the sum of the terms $(d_u + d_v - 2)^2$ over all edges $uv$ of $G$. In this paper, the various upper and lower bounds for the first reformulated Zagreb index of a connected graph interms of other topological indices are obtained.

Keywords: Zagreb index, reformulated Zagreb index.

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## 1. Introduction

A *chemical graph* is a graph whose vertices denote atoms and edges denote bonds between those atoms of any underlying chemical structure. A *topological index* for a (chemical) graph G is a numerical quantity invariant under automorphisms of G and it does not depend on the labeling or pictorial representation of the graph. In the current chemical literature, a large number of graph-based structure descriptors (topological indices) have been put forward, that all depend only on the degrees of the vertices of the molecular graph. More details on vertex-degree-based topological indices and on their comparative study can be found in [5, 6, 9, 11]. The topological indices are graph invariants which have been used for examing quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships

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(QSAR) extensively in which the biological activity or other properties of molecules are correlated with their chemical structures, in [3].

For a (molecular) graph $G$, The *first Zagreb index* $M_1(G)$ is equal to the sum of the squares of the degrees of the vertices, and the *second Zagreb index* $M_2(G)$ is equal to the sum of the products of the degrees of pairs of adjacent vertices, that is, $M_1(G) = \sum_{u \in V(G)} d_G^2(u) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$, $M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v)$, where $d_G(u)$ is the degree of a vertex $u$ in $G$.

Milicević et al. [14] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees $EM_1(G) = \sum_{e \in E(G)} d(e)^2$, where $d(e)$ denotes the degree of the edge $e$ in $G$, which is defined by $d(e) = d(u) + d(v) - 2$ with $e = uv$. The use of these descriptors in QSPR study was also discussed in their report [14]. Reformulated Zagreb index, particularly its upper and lower bounds has attracted recently threat tension of many mathematicians and computer scientists, in [2, 10, 12, 13, 14, 19, 21]. In this sequence, we obtain here, the various upper and lower bounds for the first reformulated Zagreb index of a connected graph in terms of other topological indices.

## 2. Preliminaries

In this section, we recall some definitions and notations which will be used throughout the paper.

Let $G$ be a connected graph. We denote by $\delta$ and $\Delta$ the minimum and maximum vertex degrees of $G$, respectively. The distance $d_G(u, v)$ between the vertices $u$ and $v$ of $G$ is defined as the length of any shortest path in $G$ connecting $u$ and $v$. Let $D(G)$ be the diameter of $G$. For an edge $uv = e \in E(G)$, the number of vertices of $G$ whose distance to the vertex $u$ is smaller than the distance to the vertex $v$ in $G$ is denoted by $n_u(e)$; analogously, $n_v(e)$ is the number of vertices of $G$ whose distance to the vertex $v$ in $G$ is smaller than the distance to the vertex $u$. The *vertex PI index* of $G$, denoted by $PI(G)$, is defined as $PI(G) = \sum_{e = uv \in E(G)} \Big( n_u(e) + n_v(e) \Big)$.

A multiplicative version of the first Zagreb index called multiplicative sum Zagreb index was proposed by Eliasi et al. [7] in 2010. The *multiplicative sum Zagreb index* $\Pi_1^*(G)$ of $G$ is defined as $\Pi_1^*(G) = \prod_{uv \in E(G)} (du + dv)$.

In 1975, Milan Randić [16] proposed a structural descriptor, based on the end-vertex degrees of edges in a graph, called the branching index that later became the well-known Randić connectivity index. The *Randić index* $R(G)$ of $G$ is defined as $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}$. The Randić index is one of the most successful molecular descriptors in QSPR and QSAR studies, suitable for measuring the extent of branching of the carbon-atom skeleton of saturated hydrocarbons.

Another variant of the Randić connectivity index named the harmonic index was introduced by Fajtlowicz [8] in 1987. The *harmonic index* $H(G)$ of $G$ is defined as $H(G) = \sum\limits_{uv\in E(G)} \frac{1}{d(u)+d(v)}$. Motivated by definition of the Randić connectivity index, Vukicević and Furtula [20] proposed another vertex-degree-based topological index, named the geometric-arithmetic index. The *geometric-arithmetic index* of a graph $G$ is denoted by $GA(G)$ and defined as $GA(G) = \sum\limits_{uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}$.

The eccentric connectivity index was introduced by Sharma et al. [17] in 1997. The *eccentric connectivity index* $\xi^c(G)$ of $G$ is defined as $\xi^c(G) = \sum\limits_{uv\in E(G)} d(u)\epsilon(u)$. The eccentric connectivity index can also be expressed as a sum over edges of $G$, $\xi^c(G) = \sum\limits_{uv\in E(G)} (\epsilon(u)+\epsilon(v))$.

Recently, Shirdel et al. [18] introduced a variant of the first Zagreb index called hyper-Zagreb index. The *hyper-Zagreb index* of $G$ is denoted by $HM(G)$ and defined as $HM(G) = \sum\limits_{uv\in E(G)} (d(u)+d(v))^2$.

## 3. Bounds for $EM_1$

In this section, we obtain the different kinds of upper and lower bonuds for first reformulated Zagreb index of a given connceted graph.

**Theorem 3.1.** *Let $G$ be a triangle free graph. Then $EM_1(G) \leq HM(G) - 4PI(G) + 4m$ with equality if and only if diameter of $G$ is 2.*

*Proof.* Since $n_u(e) + n_v(e) \geq d(u) + d(v)$ for each edge $e = uv \in E(G)$. Thus

$$\begin{aligned} EM_1(G) &= \sum_{uv\in E(G)} (d(u)+d(v))^2 - 4\sum_{uv\in E(G)} (d(u)+d(v)) + 4m \\ &\leq HM(G) - 4\sum_{uv\in E(G)} (n_u(e)+n_v(e)) + 4m \\ &= HM(G) - 4PI(G) + 4m. \end{aligned}$$

Assume that $D(G) = 2$ and there exists an edge $e = uv$ such that $n_u(e)+n_v(e) > d_G(u)+d_G(v)$. Then there exists a vertex $r$ such that $d_G(u,r) \neq 1$ and $d_G(v,r) \neq 1$. Suppose $x$ is closer to $u$ than $v$. Hence $d_G(x,v) > d_G(x,u) \geq 2$, which contradicts by our assumption that $D(G) = 2$. Therefore, for each edge $e = uv, n_u(e) + n_v(e) = d_G(u) + d_G(v)$ and so $EM_1(G) = HM(G) - 4PI(G) + 4m$. □

**Theorem 3.2.** *Let $G$ be a graph with $m$ edges. Then $4m(\delta-1)^2 \leq EM_1(G) \leq 4m(\Delta-1)^2$ with equality if and only if $G$ is regular.*

*Proof.* One can observe that $2\delta - 2 \leq d(u) + d(v) - 2 \leq 2\Delta - 2$ for every edge $uv$ in $G$. Thus

$$4m(\delta-1)^2 = \sum_{uv\in E(G)} (2\delta-2)^2 \leq \sum_{uv\in E(G)} (d(u)+d(v)-2)^2 = EM_1(G) \leq \sum_{uv\in E(G)} (2\Delta-2)^2 = 4m(\Delta-1)^2.$$

The equalities hold if and only if $2\delta - 2 = d(u) + d(v) - 2 = 2\Delta - 2$, for each edge $uv \in E(G)$. This implies that $G$ is regular. □

For positive integer $r \neq s$,a graph $G$ is said to be $(r, s)$-*semiregular* if its vertex degrees assume only the value $r$ and $s$, and if there is atleast one vertex of degree $r$ and atleast one of degree $s$

**Theorem 3.3.** *Let G be a graph with m edges and p pendent vertices. If* $\delta_1$ *is the minimal nonpendent vertex of* $G$, *then* $4\delta_1^2(m-p) + (1+\delta_1)^2 p + 4m - 4M_2(G) \leq EM_1(G) \leq 4\Delta^2(m-p) + (1+\Delta)^2 p + 4m - 4M_2(G)$ *with equality if and only if G is regular (or)* $(1, \Delta)$-*semiregular.*

*Proof.* From the definition of $EM_1$,

$$\begin{aligned} EM_1(G) &= \sum_{uv\in E(G),\ d(u),d(v)\neq 1} (d(u)+d(v))^2 + \sum_{uv\in E(G),\ d(u)=1} (1+d(v))^2 + 4m - 4M_2(G) \\ &\leq \sum_{uv\in E(G),\ d(u),d(v)\neq 1} (2\Delta)^2 + \sum_{uv\in E(G),\ d(u)=1} (1+\Delta)^2 + 4m - 4M_2(G) \\ &= 4\Delta^2(m-p) + (1+\Delta)^2 p + 4m - 4M_2(G). \end{aligned}$$

Similarly, We can prove $EM_1(G) \geq 4\delta_1^2(m-p) + (1+\delta_1)^2 p + 4m - 4M_2(G)$. The equalities hold if and only if $d(u) = d(v) = \delta_1 = \Delta$, for each $uv \in E(G)$, with $d(u)$, $d(v) \neq 1$ and $d(u) = \delta_1 = \Delta$, for each $uv \in E(G)$ with $d(u) = 1$. Thus $G$ is regular if $p = 0$ and $G$ is $(1, \Delta)$-semiregular if $p > 0$. □

**Theorem 3.4.** *Let G be a tree on n vertices. Then* $EM_1(G) \leq (n-1)(n-2)^2$ *with equality if and only if G is a star.*

*Proof.* Since $G$ is a tree, $d(u) + d(v) - 2 \leq n - 2$ for all $uv \in E(G)$. Thus

$$EM_1(G) = \sum_{uv\in E(G)} (d(u)+d(v)-2)^2 \leq m(n-2)^2 = (n-1)(n-2)^2.$$

Equality holds for $d(u)+d(v)-2 = n-2$ for all $uv \in E(G)$. Thus $G$ is a star. □

**Lemma 3.5.** *(Cauchy-Schwarz inequality) Let* $X = (x_1, x_2 \ldots x_n)$ *and* $Y = (y_1, y_2, \ldots, y_n)$ *be two sequences of real numbers. Then* $\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2$ *with equality if and only if the sequences X and Y are proportional, i.e., there exists a constant c such that* $x_i = cy_i$, *for each* $1 \leq i \leq n$.

*As a special case of the Cauchy-Schwarz inequality, when* $y_1 = y_2 = \ldots = y_n$, *we get the following result.*

**Corollary 3.6.** *Let $x_1, x_2, \ldots, x_n$ be real numbers. Then* $\Big(\sum_{i=1}^{n} x_i\Big)^2 \le n \sum_{i=1}^{n} x_i^2$ *with equality if and only if* $x_1 = x_2 = \ldots = x_n$.

**Theorem 3.7.** *Let G be a graph on m edges. Then* $EM_1(G) \ge \frac{(M_1(G)-2m)^2}{m}$ *with equality if and only if G is regular.*

*Proof.* From the definition of $EM_1$, $EM_1(G) = \sum_{uv\in E(G)} (d(u) + d(v) - 2)^2$.

By Corollary 3.6, we have

$$EM_1(G) \ge \frac{\Big(\sum_{uv\in E(G)} (d(u) + d(v) - 2)\Big)^2}{m} = \frac{(M_1(G) - 2m)^2}{m}.$$

The equality is hold if and only if $G$ is regular. □

**Lemma 3.8.** *(Polya-Szego inequality* [15]*) Let* $0 < m_1 \le x_i \le M_1$ *and* $0 < m_2 \le y_i \le M_2$, *for* $1 \le i \le n$. *Then* $\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 \le \frac{1}{4}\Big(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\Big)^2 (\sum_{i=1}^{n} x_iy_i)^2$.

**Theorem 3.9.** *Let G be a graph on m edges. Then* $EM_1(G) \le \frac{1}{4m}\frac{(\Delta+\delta-2)^2}{(\delta-1)(\Delta-1)}(M_1(G) - 2m)^2$.

*Proof.* One can see that $2\delta - 2 \le d(u) + d(v) - 2 \le 2\Delta - 2$ for all $uv \in E(G)$. Setting $m_1 = 2\delta - 2, x_i = d(u) + d(v) - 2, 1 \le i \le m$ $M_1 = 2\Delta - 2, m_2 = y_i = M_2 = 1, 1 \le i \le m$, in Polya-Szego inequality, we have

$$\begin{aligned}\sum_{uv\in E(G)} (d(u) + d(v) - 2)^2 \sum_{uv\in E(G)} (1)^2 &\le \frac{1}{4}\Big(\sqrt{\frac{2\Delta - 2}{2\delta - 2}} + \sqrt{\frac{2\delta - 2}{2\Delta - 2}}\Big)^2 \Big(\sum_{uv\in E(G)} (d(u) + d(v) - 2)\Big)^2 \\ &= \frac{1}{4}\Big(\sqrt{\frac{\Delta - 1}{\delta - 1}} + \sqrt{\frac{\delta - 1}{\Delta - 1}}\Big)^2 (M_1(G) - 2m)^2.\end{aligned}$$

Hence $EM_1(G) \le \frac{1}{4m}\frac{(\Delta+\delta-2)^2}{(\delta-1)(\Delta-1)}(M_1(G) - 2m)^2$.

**Lemma 3.10.** *(Diaz-Metcalf inequality* [4]*) Let* $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ *be real numbers such that* $px_i \le y_i \le Px_i$, *for* $1 \le i \le n$. *Then* $\sum_{i=1}^{n} y_i^2 + pP \sum_{i=1}^{n} x_i^2 \le (p + P) \sum_{i=1}^{n} x_iy_i$ *with equality if and only if* $y_i = Px_i$ *or* $y_i = px_i$, *for* $1 \le i \le n$.

**Theorem 3.11.** *Let G be a graph on m edges. Then* $EM_1(G) \le 2(\delta + \Delta - 2)(M_1(G) - 2m) - 4(\delta - 1)(\Delta - 1)m$ *with equality if and only if G is regular.*

*Proof.* Setting $p = 2\delta - 2$, $P = 2\Delta - 2$, $x_i = 1$ and $y_i = d(u) + d(v) - 2$, $1 \leq i \leq m$, in Diaz-Metcalf inequality, we have

$$\begin{aligned} \sum_{uv \in E(G)} (d(u)+d(v)-2)^2 + (2\delta-2)(2\Delta-2) \sum_{uv \in E(G)} (1)^2 &\leq (2\delta - 2 + 2\Delta - 2) \sum_{uv \in E(G)} (d(u)+d(v)-2) \\ &= (2\delta + 2\Delta - 4)(M_1(G) - 2m). \end{aligned}$$

This implies that $EM_1(G) \leq 2(\delta + \Delta - 2)(M_1(G) - 2m) - 4(\delta - 1)(\Delta - 1)m$.

Equality holds if and only if $d(u)+d(v)-2 = 2\delta-2$ or $d(u)+d(v)-2 = 2\Delta-2$ for all $uv \in E(G)$.

$\Rightarrow G$ is regular. □

Let $x_1, x_2, \ldots x_n$ be positive real numbers.

The *arithmetic mean* of $x_1, x_2, \ldots x_n$ is equal to $AM(x_1, x_2, \ldots x_n) = \frac{x_1+x_2+\ldots+x_n}{n}$.

The *geometric mean* of $x_1, x_2, \ldots x_n$ is equal to $GM(x_1, x_2, \ldots x_n) =^n \sqrt{x_1 x_2 \ldots x_n}$.

The *harmonic mean* of $x_1, x_2, \ldots x_n$ is equal to $HM(x_1, x_2, \ldots x_n) = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\ldots+\frac{1}{x_n}}$.

**Lemma 3.12.** *(AM-GM-HM inequality). Let $x_1, x_2, \ldots x_n$ be positive real numbers. Then $AM(x_1, x_2, \ldots x_n) \geq GM(x_1, x2, \ldots x_n) \geq HM(x_1, x_2, \ldots x_n)$,*

*with equality if and only if $x_1 = x_2 = \ldots = x_n$.*

**Theorem 3.13.** *Let G be a graph on m edges. Then $EM_1(G) \geq 4M_2(G) - 4M_1(G) + 4m$ with equality if and only if G is regular.*

*Proof.* By using AM-GM inequality, we have

$$\begin{aligned} EM_1(G) &= \sum_{uv \in E(G)} (d(u)+d(v)-2)^2 \\ &= \sum_{uv \in E(G)} [(d(u)+d(v))^2 + 4 - 4(d(u)+d(v))] \\ &\geq \sum_{uv \in E(G)} (2\sqrt{d(u)d(v)})^2 + 4m - 4M_1(G) \\ &= 4M_2(G) - 4M_1(G) + 4m \end{aligned}$$

with equality if and only if $d(u) = d(v)$ for each $uv \in E(G)$. Thus $G$ is regular. □

**Theorem 3.14.** *Let G be a graph on m edges. Then $(\delta - 4)M_1(G) + 2M_2(G) + 4m \leq EM_1(G)(\Delta - 4)M_1(G) + 2M_2(G) + 4m$ with equality if and only if G is regular.*

*Proof.* Since $\delta \leq d(u) \leq \Delta$, for each $u \in V(G)$. Thus

$$\begin{aligned} EM_1(G) &= \sum_{uv\in E(G)} (d(u)+d(v)-2)^2 \\ &= \sum_{uv\in E(G)} [d^2(u)+d^2(v)+4+2d(u)d(v)-4d(u)-4d(v)] \\ &= \sum_{u\in V(G)} d(u)d^2(u)+4m+2M_2(G)-4M_1(G) \\ &\leq \Delta M_1(G)+4m+2M_2(G)-4M_1(G) \\ &= (\Delta-4)M_1(G)+2M_2(G)+4m. \end{aligned}$$

Similarly we obtain $EM_1(G) \geq (\delta-4)M_1(G)+2M_2(G)+4m$.

Equality hold if and only if $\delta = d(u) = \Delta$, for each $u \in V(G)$. Thus $G$ is regular. □

**Theorem 3.15.** *Let G be a graph on m edges. Then* $EM_1 \geq m(\sqrt{\pi_1^*(G)^2}) + 4m - 4M_1(G)$ *with equality if and only if G is regular.*

*Proof.* Using AM-GM inequality, we have

$$\begin{aligned} EM_1(G) &= \sum_{uv\in E(G)} (d(u)+d(v))^2+4m-4M_1(G) \\ &\geq m\Big(\sqrt[m]{\prod_{uv\in E(G)} (d(u)+d(v))^2}\Big)+4m-4M_1(G) \\ &= m\Big(\sqrt[m]{\pi_1^*(G)^2}\Big)+4m-4M_1(G). \end{aligned}$$

*Proof.*

**Theorem 3.16.** *For any graph G,* $4(\delta-1)^2\delta R(G) \leq EM_1(G) \leq 4(\Delta-1)^2\delta R(G)$ *with equality if and only if G is regular.*

*Proof.* Clearly $d(u)+d(v) \leq 2\Delta$ for each edge $uv \in E(G)$.

$4(\delta-1)^2\delta = (2\delta-2)^2\sqrt{\delta^2} \leq (d(u)+d(v)-2)^2\sqrt{d(u)d(v)} \leq (2\Delta-2)^2\sqrt{\Delta^2} = 4(\Delta-1)^2\delta.$

By the definition of Randić index, we have

$$4(\delta-1)^2\delta R(G) \leq EM_1(G) = \sum_{uv\in E(G)} (d(u)+d(v)-2)^2\frac{\sqrt{d(u)d(v)}}{\sqrt{d(u)d(v)}} = 4(\Delta-1)^2\Delta R(G).$$

Equality hold if and only if $d(u) = d(v) = \Delta = \delta$. This implies $G$ is regular. □

**Theorem 3.17.** *Let G be a graph on m edges. Then*
$G$, $EM_1(G) \geq \frac{4m^3}{R(G)} - 4M_1(G) + 4m$.

*Proof.*

$$\begin{aligned}
\frac{EM_1(G)}{4m} &= \frac{1}{4m}\Big[\sum_{uv\in E(G)}(d(u)+d(v))^2+4m-4M_1(G)\Big]\\
&= \frac{m}{m^2}\sum_{uv\in E(G)}\Big(\frac{(d(u)+d(v))}{2}\Big)^2-\frac{M_1(G)}{m}+1\\
&\geq \frac{1}{m^2}\Big(\sum_{uv\in E(G)}\frac{d(u)+d(v)}{2}\Big)^2-\frac{M_1(G)}{m}+1\\
&\geq \Big(\frac{\sum\limits_{uv\in E(G)}\sqrt{d(u)d(v)}}{m}\Big)^2-\frac{M_1(G)}{m}+1\\
&\geq \Big(\frac{m}{\sum\limits_{uv\in E(G)}\frac{1}{\sqrt{d(u)d(v)}}}\Big)^2-\frac{M_1(G)}{m}+1\\
&= (\frac{m}{R(G)})^2-\frac{M_1(G)}{m}+1\\
EM_1(G) &\geq \frac{4m^3}{R(G)}-4M_1(G)+4m,
\end{aligned}$$

where $R(G)$ is the Randic index of $G$.

**Theorem 3.18.** *Let $G$ be a graph with $m$ edges. Then*
$4\delta^3H(G)+4m-4M_1(G)\leq EM_1(G)\leq 4\Delta^3H(G)+4m-4M_1(G)$.

*Proof.* It is easy to see that, for each $uv\in E(G)$,

$$4\delta^3=\frac{(2\delta)^3}{2}\leq\frac{(d(u)+d(v))^3}{2}\leq\frac{(2\Delta)^3}{2}=4\Delta^3.$$

By the definition of first reformulated Zagreb index,

$$\begin{aligned}
EM_1(G) &= \sum_{uv\in E(G)}(d(u)+d(v))^2+4m-4M_1(G)\\
&= \sum_{uv\in E(G)}\Big(\frac{2}{d(u)+d(v)}\Big)\Big(\frac{(d(u)+d(v))^3}{2}\Big)+4m-4M_1(G).
\end{aligned}$$

$$EM_1(G) \leq 4\Delta^3H(G)+4m-4M_1(G),$$

where $H(G)$ is the harmonic index of $G$. □

Similarly,we obtain $4\delta^3H(G)+4m-4M_1(G)\leq EM_1(G)$.

**Theorem 3.19.** *For any graph $G$, $EM_1(G)\geq\frac{4\delta^2}{m}(GA(G))^2+4m-4M_1(G)$.*

*Proof.*

$$
\begin{aligned}
EM_1(G) &= \sum_{uv\in E(G)} (d(u)+d(v))^2 + 4m - 4M_1(G)\\
&= 4\sum_{uv\in E(G)} (\frac{d(u)+d(v)}{2})^2 + 4m - 4M_1(G)\\
&\geq 4\sum_{uv\in E(G)} \Big(\frac{2}{\frac{1}{d(u)}+\frac{1}{d(v)}}\Big)^2 + 4m - 4M_1(G)\\
&= 4\sum_{uv\in E(G)} \Big(\frac{2d(u)d(v)}{d(u)+d(v)}\Big)^2 + 4m - 4M_1(G)\\
&\geq \frac{4}{m}\Big(\sum_{uv\in E(G)} \frac{2d(u)d(v)}{d(u)+d(v)}\Big)^2 + 4m - 4M_1(G)\\
&= \frac{4}{m}\Big(\sum_{uv\in E(G)} \frac{2\sqrt{d(u)d(v)}\sqrt{d(u)d(v)}}{d(u)+d(v)}\Big)^2 + 4m - 4M_1(G)\\
&\geq \frac{4\delta^2}{m}\Big(\sum_{uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}\Big)^2 + 4m - 4M_1(G)\\
&= \frac{4\delta^2}{m}(GA(G))^2 + 4m - 4M_1(G),
\end{aligned}
$$

where $GA(G)$ is the geometric arithmetic index of $G$. □

**Lemma 3.20.** *Let $G$ be a nontrivial connected graph of order $n$. For each $u \in V(G), \xi_u \leq n-d(u)$, with equality if and only if $G \cong P_4$ or $G \cong K_n - iK_2, 0 \leq i \leq, \left\lfloor \frac{n}{2} \right\rfloor$, where $K_n - iK_2$ denotes the graph obtained from the complete graph $K_n$ by removing $i$ independent edges.*

**Theorem 3.21.** *For any graph $G, EM_1(G) \leq 4n^2m + \xi_3(G) + 2\xi_2(G) - 4n\xi^c(G) + 4m - 4M_1(G)$, where $\xi_3(G) = \sum_{u\in V(G)} (\xi_u^2 + \xi_v^2)$ and the equality holds if and only if $G \cong P_4$ or $G \cong K_n - iK_2, 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.*

*Proof.* By the definition of $EM_1$

$$
\begin{aligned}
EM_1(G) &= \sum_{uv\in E(G)} (d(u)+d(v))^2 + 4m - 4M_1(G)\\
&\leq \sum_{uv\in E(G)} (n-\xi_u+n-\xi_v)^2 + 4m - 4M_1(G)\\
&= \sum_{uv\in E(G)} [4n^2 + (\xi_u^2+\xi_v^2) + 2\xi_u\xi_v - 4n(\xi_u+\xi_v)] + 4m - 4M_1(G), \text{ by Lemma 3.20}\\
&= 4n^2m + \xi_3(G) + 2\xi_2(G) - 4n\xi^c(G) + 4m - 4M_1(G),
\end{aligned}
$$

where $\xi_3(G) = \sum_{u\in V(G)} (\xi_u^2 + \xi_v^2)$. By Lemma 3.20, the equality holds if and only if $d(u) = n - \xi_u$ for each $u \in V(G)$, which by Lemma 3.20 implies that $G \cong P_4$ or $G \cong K_n - iK_2, 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$. □

**Lemma 3.22.** *(Zhou and Wu 2005) Let G be a graph of order* $n \geq 4$ *such that both G and* $\overline{G}$ *are connected . Then* $\frac{n(n-1)^2}{2} \leq M_1(G) + M_1(\overline{G}) \leq n(n-1)^2$, *the right equality holds if and onlt if G or* $\overline{G}$ *is the complete graph* $K_n$, *the left equlaity holds if and only if G or* $\overline{G}$ *is* $2k$*-reqular graph on* $n = 4k + 1$ *vertices.*

**Theorem 3.23.** *Let G be a simple graph with* $n \geq 4$ *vertices and* $\overline{G}$ *its complement. If both G and* $\overline{G}$ *are connected ,then* $EM_1(G) + EM_1(\overline{G}) \leq HM(G) + HM(\overline{G}) - 2n(n-1)(2n-3)$ *and* $EM_1(G) + EM_1(\overline{G}) \geq HM(G) + HM(\overline{G}) - 2n(n-1)(n-2)$.

*Proof.*

$$\begin{aligned} EM_1(G) + EM_1(\overline{G}) &= \sum_{uv\in E(G)} (d_G(u) + d_G(v) - 2)^2 + \sum_{uv\in E(\overline{G})} (d_{\overline{G}}(u) + d_{\overline{G}}(v) - 2)^2 \\ &= HM(G) - 4M_1(G) + 4m + HM(\overline{G}) - 4M_1(\overline{G}) + 4\left|E(\overline{G})\right| \\ &= HM(G) + HM(\overline{G}) - 4\Big(M_1(G) + M_1(\overline{G})\Big) + 2n(2n-1). \end{aligned}$$

By Lemma 3.22, we get the desired result. □

## References


[1] A.R. Ashrafi, T. Doslić, A. Hamzeh, The Zagreb coindices of graph operations, Discrete Appl. Math. 158(2010)1571–1578.
[2] N. De, Some bounds of reformulated Zagreb indices, Appl. Math. Sci. 101 (2012)5005-5012.
[3] M. Dehmer , K. Varmuza , D. Bonchev, editors (2012) Statistical Modelling of Molecular Descriptors in QSAR/QSPR. Quantitative and Network Biology. Wiley-FBlackwell.
[4] J.B. Diaz, F.T. Metcalf, Stronger forms of a class of inequalities of G. Polya-G Szego and L.V. kantorovich, Bull. Amer. Math Soc 69(1963 ), 415–418
[5] T. Doslić, B. Furtula, A. Graovać, I. Gutman, S. Moradi, Z. Yarahmadi, On vertexdegreebased molecular structure descriptors, MATCH Commun. Math. Comput. Chem. 66 (2011) 613–626.
[6] T. Doslić, T. Reti, D. Vukicević, On the vertex degree indices of connected graphs, Chem. Phys. Lett. 512 (2011) 283–286.
[7] M. Eliasi, A. Iranmanish, I. Gutman, Multiplicative versions of first Zagerb index, MATCH Commun. Math. Comput.Chem. 68 (2012) 217–230.
[8] S. Fajtlowicz, On conjectures on Graffiti-*II*, Congr. Numer. 60 (1987) 187-197.
[9] B. Furtula, I. Gutman, M. Dehmer, On structure-sensitivity of degreebased topological indices, Appl. Math. Comput. 219 (2013) 8973–8978.
[10] B. Furtula, I. Gutman, A forgotten topological index, J. Math. Chem. 53(2015)1184-1190.
[11] I. Gutman, Degree-based topological indices, Croat. Chem. Acta 86 (2013) 351–361
[12] A. Ilić, D.Stevanović, On comparing Zagreb indices, MATCH Commun. Math. Comput. Chem. 62(2009)681–687.

[13] A. Ilić, B. Zhou, On reformulated Zagreb indices,Discrete Appl. Math. 160(2012)204-209.
[14] A. Milićević, S.Nikolić, N.Trinajstić, On reformulated Zagreb indices, Mol. Divers. 8(2004) 93-399.
[15] G. Polya and G. Szego, Problems and theorems in analysis, series. Integral Calculus, Theory of Functions , Springer-Verlag, Berlin, 1972.
[16] M. Randić, On characterization of molecular branching, J. Am. Chem. Soc. 97 (1975) 6609-6615.
[17] V. Sharma, R. Goswami and A.K. Madan, Eccentric connectivity index: A novel highly discriminating topological descriptor for structure-property and structure-activity studies, J. Chem. Inf. Comput. Sci. 37 (1997) 273-282.
[18] G.H. Shirdel, H. Rezapour and A.M. Sayadi, The hyper-Zagreb index of graph operations, Iranian J. Math. Chem. 4 (2013) 213-220.
[19] G. Su, L. Xiong, L. Xu, B. Ma, On the maximum and minimum first reformulated Zagreb index with connectivity of at most *k*, Filomat 25(2011)75-83.
[20] D. Vukicević, B. Furtula, Topological index Based on the ration of geometrical and arithmetical means of end vertex degree of edges, J. Math. Chem 46 (2009) 1369-1376.
[21] B. Zhou, N. Trinajstić, Some properties of the reformulated Zagreb indices,J. Math. Chem. 48(2010)714-719.