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A New Method for Computing Determinants By Reducing The Orders By Two

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Abstract. In this paper we will present a new method to calculate determinants of square matrices. The method is based on the Chio-Dodgson's condensation formula and our approach automatically affects in reducing the order of determinants by two. Also, using the Chio's condensation method we present an inductive proof of Dodgson's determinantal identity.

Keywords: Chio's condensation method, Dodgson's condensation method, determinants, determinantal identity, Laplace expansion.

2000 Mathematics subject classification: 15A15; 11C20; 65F40.

1. Introduction and Motivation

Determinants are frequently appeared in many areas of mathematics, science and engineering. For matrices of small sizes, the calculation of their determinants is an easy task based on *Laplace expansion* by rows or columns and the difficulties only arise when someone needs to work with matrices of very large sizes. There are only a few interesting untraditional methods of computing the determinant of a square matrix in the old literature. Fortunately, these methods are based on what is

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called the *condensation* method; that is, reducing the order of the original determinant.
The two most famous of these classic methods are called *Chio* and *Dodgson*'s condensation. In this paper while reviewing the above-mentioned condensation methods, we will present a new method for calculating the determinant of a square matrix by successively reducing it's order based on Chio and Dodgson's determinantal identities to ultimately obtain a determinant of order two which can be easily computed.

2. Chio's Condensation Method

One of the interesting determinantal formulas for computing the determinant of a square matrix is the so-called Chio's Condensation Method which was originally due to Chio[1].

Lemma 2.1 (Chio's Condensation Method)**.** *Let* $\mathbf{A} = (a_{i,j})$ *be an* $n \times n$ *matrix, where without loss of generality we will assume that* $a_{p,q} \neq 0$*. If* $\mathbf{B}$ *is an* $(n-1) \times (n-1)$ *matrix constructed from* $\mathbf{A}$ *by defining*

$$b_{i,j} = \begin{cases} \begin{vmatrix} a_{i,j} & a_{i,q} \\ a_{p,j} & \boxed{a_{p,q}} \end{vmatrix} & \text{if } \; 1 \leq i \leq p-1, \quad 1 \leq j \leq q-1, \\ \\ \begin{vmatrix} a_{i,q} & a_{i,j} \\ \boxed{a_{p,q}} & a_{p,j} \end{vmatrix} & \text{if } \; 1 \leq i \leq p-1, \quad q+1 \leq j \leq n, \\ \\ \begin{vmatrix} a_{p,j} & \boxed{a_{p,q}} \\ a_{i,j} & a_{i,q} \end{vmatrix} & \text{if } \; p+1 \leq i \leq n, \quad 1 \leq j \leq q-1, \\ \\ \begin{vmatrix} \boxed{a_{p,q}} & a_{p,j} \\ a_{i,q} & a_{i,j} \end{vmatrix} & \text{if } \; p+1 \leq i \leq n, \quad q+1 \leq j \leq n, \end{cases}$$

then

$$\det(\mathbf{A}) = \frac{1}{a_{p,q}^{n-2}} \det(\mathbf{B}).$$

Proof. Dividing the p-th row of $\mathbf{A}$ by $a_{p,q}$ yields a matrix $\mathbf{X}$ for which we have $x_{p,q} = 1$. Now subtract suitable multiples of the q-th column of $\mathbf{X}$ from other columns of this matrix in order to make the elements of the q-th column equal to zero except $x_{p,q}$ which is equal to unity, and thereby obtain the matrix Y. The Laplace expansion of Y via the q-th column leads to $\mathbf{B}$. □

In Lemma 2.1, $a_{p,q}$ is often called a *pivot*. Other proofs of the above lemma can be found in [2, 3]. In order to apply the Chio's condensation method to a given square matrix, we start with the original $n\times n$ matrix and compute the $(n-1)\times(n-1)$ matrix, then the $(n-2)\times(n-2)$ matrix, and so on, until we arrive at a 1×1 matrix whose only entry is the determinant of the original $n\times n$ matrix. Clearly, at each stage a nonzero entry of the matrix should be used as a pivot.

Corollary 2.2. *For $p=1$ and $q=1$, if $a_{1,1}\neq 0$, then we have*

$$\begin{vmatrix} \boxed{a_{1,1}} & a_{1,2} & \cdots & a_{1,n}\\ a_{2,1} & a_{2,2} & \cdots & a_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n,1} & a_{n,2} & \cdots & a_{n,n}\end{vmatrix} = \frac{1}{a_{1,1}^{n-2}} \begin{vmatrix} \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{2,1} & a_{2,3}\end{vmatrix} & \cdots & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,n}\\ a_{2,1} & a_{2,n}\end{vmatrix}\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{3,1} & a_{3,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{3,1} & a_{3,3}\end{vmatrix} & \cdots & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,n}\\ a_{3,1} & a_{3,n}\end{vmatrix}\\ \vdots & \vdots & \ddots & \vdots\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{n,1} & a_{n,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{n,1} & a_{n,3}\end{vmatrix} & \cdots & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,n}\\ a_{n,1} & a_{n,n}\end{vmatrix}\end{vmatrix}.$$

Example 2.3. If $a_{11}\neq 0$ and $a_{11}a_{12}-a_{12}a_{21}\neq 0$, we have

$$\det(A) = \begin{vmatrix} \boxed{a_{1,1}} & a_{1,2} & a_{1,3} & a_{1,4}\\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4}\\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4}\\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4}\end{vmatrix} = \frac{1}{a_{1,1}^{4-2}} \begin{vmatrix} \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{2,1} & a_{2,3}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,4}\\ a_{2,1} & a_{2,4}\end{vmatrix}\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{3,1} & a_{3,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{3,1} & a_{3,3}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,4}\\ a_{3,1} & a_{3,4}\end{vmatrix}\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{4,1} & a_{4,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{4,1} & a_{4,3}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,4}\\ a_{4,1} & a_{4,4}\end{vmatrix}\end{vmatrix}$$

$$= \frac{1}{a_{1,1}^{4-2}\begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix}^{3-2}} \begin{vmatrix} \begin{vmatrix} \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{2,1} & a_{2,3}\end{vmatrix}\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{3,1} & a_{3,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{3,1} & a_{3,3}\end{vmatrix}\end{vmatrix} & \begin{vmatrix} \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,4}\\ a_{2,1} & a_{2,4}\end{vmatrix}\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{3,1} & a_{3,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,4}\\ a_{3,1} & a_{3,4}\end{vmatrix}\end{vmatrix}\\ \begin{vmatrix} \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{2,1} & a_{2,3}\end{vmatrix}\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{4,1} & a_{4,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,3}\\ a_{4,1} & a_{4,3}\end{vmatrix}\end{vmatrix} & \begin{vmatrix} \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,4}\\ a_{2,1} & a_{2,4}\end{vmatrix}\\ \begin{vmatrix}\boxed{a_{1,1}} & a_{1,2}\\ a_{4,1} & a_{4,2}\end{vmatrix} & \begin{vmatrix}\boxed{a_{1,1}} & a_{1,4}\\ a_{4,1} & a_{4,4}\end{vmatrix}\end{vmatrix}\end{vmatrix}$$

$$= \frac{1}{a_{1,1}^{4-2}\begin{vmatrix}a_{1,1} & a_{1,2}\\ a_{2,1} & a_{2,2}\end{vmatrix}^{3-2}} \begin{vmatrix} a_{1,1}\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3}\\ a_{2,1} & a_{2,2} & a_{2,3}\\ a_{3,1} & a_{3,2} & a_{3,3}\end{vmatrix} & a_{1,1}\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,4}\\ a_{2,1} & a_{2,2} & a_{2,4}\\ a_{3,1} & a_{3,2} & a_{3,4}\end{vmatrix}\\ a_{11}\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3}\\ a_{2,1} & a_{2,2} & a_{2,3}\\ a_{4,1} & a_{4,2} & a_{4,3}\end{vmatrix} & a_{1,1}\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,4}\\ a_{2,1} & a_{2,2} & a_{2,4}\\ a_{4,1} & a_{4,2} & a_{4,4}\end{vmatrix}\end{vmatrix}.$$

Therefore, we have

$$\det(A) = \frac{1}{\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} & \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,4} \end{vmatrix} \\ \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{vmatrix} & \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,4} \\ a_{4,1} & a_{4,2} & a_{4,4} \end{vmatrix} \end{vmatrix}.$$

Now we generalize the above formula for a matrix whose entries are the determinants of some 3×3 submatrices of A. For each matrix A, the notation $\left|A_{i=i_1,i_2}^{j=j_1,j_2}\right|$ stands for the determinant of the matrix obtained from A by choosing the i-th rows ($i = i_1, i_2$) and j-th columns ($j = j_1, j_2$). Other notations like $|A_{i=i_1,i_2,i_3}^{j=j_1,j_2,j_3}|$ are defined in a similar way.

Theorem 2.4. *Let A be an $n\times n$ and $\left|A_{i=1,2}^{j=1,2}\right| \neq 0$, then we have*

$$\det(A) = \frac{1}{\left|A_{i=1,2}^{j=1,2}\right|} \begin{vmatrix} |A_{i=1,2,3}^{j=1,2,3}| & |A_{i=1,2,3}^{j=1,2,4}| & \cdots & |A_{i=1,2,3}^{j=1,2,n}| \\ |A_{i=1,2,4}^{j=1,2,3}| & |A_{i=1,2,4}^{j=1,2,4}| & \cdots & |A_{i=1,2,4}^{j=1,2,n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{i=1,2,n}^{j=1,2,3}| & |A_{i=1,2,n}^{j=1,2,4}| & \cdots & |A_{i=1,2,n}^{j=1,2,n}| \end{vmatrix}.$$

Theorem 2.5. *If A is an $n\times n$ and $\left|A_{i=1,2,\cdots,k}^{j=1,2,\cdots,k}\right| \neq 0$, then we have*

$$\det(A) = \frac{1}{\left|A_{i=1,2,\cdots,k}^{j=1,2,\cdots,k}\right|} \begin{vmatrix} \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k,k+1}\right| & \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,k+1}\right| & \cdots & \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,n}\right| \\ \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,n}\right| & \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,n}\right| & \cdots & \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,n}\right| \\ \vdots & \vdots & \ddots & \vdots \\ \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,n}\right| & \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,n}\right| & \cdots & \left|A_{i=1,2,\cdots,k,k+1}^{j=1,2,\cdots,k-1,n}\right| \end{vmatrix}.$$

Proof. The proof can be easily obtained using mathematical induction on k and Chio's Condensation Method. □

3. Dodgson's Condensation Method

The Dodgsons condensation method was introduced in a paper by Charels Ludwidge Dodgson in 1866 [4]. The method is based on a recursive algorithm that computes any determinant of order n by reducing it to calculation of determinants of smaller size $n-1$. By iterating this process, one can finally come up with a determinant of order 2, which can be easily computed.
The following theorem can be proved by using Lemma 2.1.

Theorem 3.1. *If the determinant of the matrix* $(a_{i,j})_{\substack{i\neq n-1,n\\ j\neq n-1,n}}$ *is null, then the following identity holds:*

$$\det\left[(a_{i,j})_{\substack{i\neq n-1\\ j\neq n-1}}\right].\det\left[(a_{i,j})_{\substack{i\neq n\\ j\neq n}}\right]-\det\left[(a_{i,j})_{\substack{i\neq n-1\\ j\neq n}}\right].\det\left[(a_{i,j})_{\substack{i\neq n\\ j\neq n-1}}\right]=0. \tag{3.1}$$

Proof. Assuming $\det\left[(a_{i,j})_{\substack{i\neq n-1,n\\ j\neq n-1,n}}\right]=0$, we prove identity (3.1) by induction. For $n=3$, the proof of (3.1) is trivial. For $n\geq 4$, we suppose that (3.1) holds for $(n-1)$ and prove it for n. Without loss of generality, let us assume that $a_{1,1}\neq 0$. Then it is concluded from Theorem 2.1 that

$$\det_{1\leq i,j\leq n-2}\left[(a_{i,j})_{\substack{i\neq n-1,n\\ j\neq n-1,n}}\right]=\frac{1}{a_{1,1}^{n-4}}\det_{1\leq i,j\leq n-3}\left[(b_{i,j})_{\substack{i\neq n-1,n\\ j\neq n-1,n}}\right], \tag{3.2}$$

where $b_{i,j}=\begin{vmatrix} a_{1,1} & a_{1,j+1}\\ a_{i+1,1} & a_{i+1,j+1}\end{vmatrix}$. Since $\det\left[(a_{i,j})_{\substack{i\neq n-1,n\\ j\neq n-1,n}}\right]=0$, the equation (3) yields

$$\det_{1\leq i,j\leq n-3}\left[(b_{i,j})_{\substack{i\neq n-1,n\\ j\neq n-1,n}}\right]=0.$$

Now, by applying (3.1) to the $(n-1)\times(n-1)$ matrix $(b_{i,j})_{1\leq i,j\leq n-1}$, we arrive at the following formula:

$$\det_{1\leq i,j\leq n-1}\left[(b_{i,j})_{\substack{i\neq n-2\\ j\neq n-2}}\right].\det_{1\leq i,j\leq n-1}\left[(b_{i,j})_{\substack{i\neq n-1\\ j\neq n-1}}\right]-\det_{1\leq i,j\leq n-1}\left[(b_{i,j})_{\substack{i\neq n-2\\ j\neq n-1}}\right].\det_{1\leq i,j\leq n-1}\left[(b_{i,j})_{\substack{i\neq n-1\\ j\neq n-2}}\right]=0.$$

Simplifying the above formula by using Lemma 2.1 leads to (3.1), which completes the proof. □

Now, we are ready to present our proof for Dodgson's condensation theorem which employs Lemma 2.1, and Theorem 3.1. An alternative combinatorial proof for this theorem can be found in [5].

Let $A=(a_{i,j})_{(1\leq i,j\leq n)}$ be an $n\times n$ square matrix and for each $1\leq i_1,j_1\leq n$ denote by $A_{i\neq i_1}^{j\neq j_1}$ the matrix that is resulted from A by deleting the i_1-th row and the j_1-th column. Similarly, for $1\leq i_1,i_2,j_1,j_2\leq n$ denote by $A_{i\neq i_1,i_2}^{j\neq j_1,j_2}$ the matrix that is resulted from A by deleting the i_1-th and i_2-th rows and the j_1-th and j_2-th columns.

Theorem 3.2 (Dodgson's Condensation Theorem). *For any square matrix* $\mathbf{A} = (a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}$, *we have*

$$\det\left[(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}\right]\cdot\det\left[(a_{i,j})_{\substack{i\ne k,k'\\j\ne l,l'}}\right] = \det\begin{bmatrix}\det\left[(a_{i,j})_{\substack{i\ne k\\j\ne l}}\right] & \det\left[(a_{i,j})_{\substack{i\ne k\\j\ne l'}}\right]\\ \det\left[(a_{i,j})_{\substack{i\ne k'\\j\ne l}}\right] & \det\left[(a_{i,j})_{\substack{i\ne k'\\j\ne l'}}\right]\end{bmatrix}, \tag{3.3}$$

for all $k,k',l,l'\in\{1,2,\cdots,n\}$ *with* $k\ne k'$ *and* $l\ne l'$.

Proof. To prove (3.3), it is sufficient to prove the following identity:

$$\det\left[(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}\right]\cdot\det\left[(a_{i,j})_{\substack{i\ne n-1,n\\j\ne n-1,n}}\right] = \det\begin{bmatrix}\det\left[(a_{i,j})_{\substack{i\ne n-1\\j\ne n-1}}\right] & \det\left[(a_{i,j})_{\substack{i\ne n-1\\j\ne n}}\right]\\ \det\left[(a_{i,j})_{\substack{i\ne n\\j\ne n-1}}\right] & \det\left[(a_{i,j})_{\substack{i\ne n\\j\ne n}}\right]\end{bmatrix}, \tag{3.4}$$

since by pair replacing the row k with the row $(n-1)$ and the row k' with the row n, the column l with the column $(n-1)$ and the column l' with the column n, the determinant of the matrix $(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}$ is not changed. In the following, we prove (3.4) in two cases, separately.

Case 1: $\det\left[(a_{i,j})_{\substack{i\ne n-1,n\\j\ne n-1,n}}\right] = 0$. In this case the proof of (3.4) is the same as the proof of Theorem 3.1.

Case 2: $\det\left[(a_{i,j})_{\substack{i\ne n-1,n\\j\ne n-1,n}}\right] \ne 0$. We prove (3.4) by induction. For $n=3$ the proof of (3.4) is trivial. For $n\ge 4$, we suppose that (3.4) is correct for $(n-1)$ and prove it for n. By Lemma 2.1, we have

$$(a_{1,1})^{n-2}\det\left[(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}\right] = \det\left[(b_{i,j})_{\substack{1\le i\le n-1\\1\le j\le n-1}}\right], \tag{3.5}$$

where $b_{i,j} = \begin{vmatrix} a_{1,1} & a_{1,j+1}\\ a_{i+1,1} & a_{i+1,j+1}\end{vmatrix}$.

Multiplying two sides of (3.5) by $(a_{1,1})^{n-4}\det\left[(a_{i,j})_{\substack{1\le i\le n-2\\1\le j\le n-2}}\right]$, we get

$$\left\{(a_{1,1})^{n-3}\right\}^2\det\left[(a_{i,j})_{\substack{1\le i\le n\\1\le j\le n}}\right].\det\left[(a_{i,j})_{\substack{1\le i\le n-2\\1\le j\le n-2}}\right] = \tag{3.6}$$
$$(a_{1,1})^{n-4}\det_{1\le i,j\le n}\left[(a_{i,j})_{\substack{1\le i\le n-2\\1\le j\le n-2}}\right].\det\left[(b_{i,j})_{\substack{1\le i\le n-1\\1\le j\le n-1}}\right].$$

From (3.5), we obtain

$$(a_{1,1})^{n-4}\det\left[(a_{i,j})_{\substack{1\le i\le n-2\\1\le j\le n-2}}\right] = \det\left[(b_{i,j})_{\substack{1\le i\le n-3\\1\le j\le n-3}}\right]. \tag{3.7}$$

Combining (3.6) and (3.7) results in

$$\left\{(a_{1,1})^{n-3}\right\}^2 \det\left[(a_{i,j})_{\substack{1\le i\le n\\ 1\le j\le n}}\right].\det\left[(a_{i,j})_{\substack{1\le i\le n-2\\ 1\le j\le n-2}}\right]$$
$$= \det\left[(b_{i,j})_{\substack{1\le i\le n-3\\ 1\le j\le n-3}}\right].\det\left[(b_{i,j})_{\substack{1\le i\le n-1\\ 1\le j\le n-1}}\right].$$

Since $\det\left[(a_{i,j})_{\substack{i\ne n-1,n\\ j\ne n-1,n}}\right] = \det\left[(a_{i,j})_{\substack{1\le i\le n-2\\ 1\le j\le n-2}}\right]$, we get

$$\left\{(a_{1,1})^{n-3}\right\}^2 \det\left[(a_{i,j})_{\substack{1\le i\le n\\ 1\le j\le n}}\right].\det\left[(a_{i,j})_{\substack{i\ne n-1,n\\ j\ne n-1,n}}\right]$$
$$= \det\left[(b_{i,j})_{\substack{1\le i\le n-3\\ 1\le j\le n-3}}\right].\det\left[(b_{i,j})_{\substack{1\le i\le n-1\\ 1\le j\le n-1}}\right]. \quad (3.8)$$

Applying (3.4) to the $(n-1)\times(n-1)$ matrix $(b_{i,j})_{\substack{1\le i\le n-1\\ 1\le j\le n-1}}$ concludes that

$$\det\left[(b_{i,j})_{\substack{1\le i\le n-3\\ 1\le j\le n-3}}\right]\cdot\det\left[(b_{i,j})_{\substack{1\le i\le n-1\\ 1\le j\le n-1}}\right] = \quad (3.9)$$
$$\det_{1\le i,j\le n-1}\left[(b_{i,j})_{\substack{i\ne n-2\\ j\ne n-2}}\right]\cdot\det_{1\le i,j\le n-1}\left[(b_{i,j})_{\substack{i\ne n-1\\ j\ne n-1}}\right]$$
$$-\det_{1\le i,j\le n-1}\left[(b_{i,j})_{\substack{i\ne n-2\\ j\ne n-1}}\right]\cdot\det_{1\le i,j\le n-1}\left[(b_{i,j})_{\substack{i\ne n-1\\ j\ne n-2}}\right].$$

Substituting (3.9) in (3.8), we get:

$$\{(a_{1,1})^{n-3}\}^2 \det\left[(a_{i,j})_{\substack{1\le i\le n\\ 1\le j\le n}}\right]\cdot\det\left[(a_{i,j})_{\substack{i\ne n-1,n\\ j\ne n-1,n}}\right] = \quad (3.10)$$
$$\det_{1\le i,j\le n-2}[(b_{i,j})]\cdot\det_{1\le i,j\le n-1}\left[(b_{i,j})_{\substack{i\ne n-2\\ j\ne n-2}}\right]$$
$$-\det_{1\le i,j\le n-1}\left[(b_{i,j})_{\substack{i\ne n-2\\ j\ne n-1}}\right]\cdot\det_{1\le i,j\le n-1}\left[(b_{i,j})_{\substack{i\ne n-1\\ j\ne n-2}}\right].$$

According to Lemma 2.1, we have

$$\begin{cases} (a_{1,1})^{n-3} \det \left[(a_{i,j})_{\substack{i\neq n \\ j\neq n}} \right] = \det_{1\leq i\leq n-1} \left[(b_{i,j})_{\substack{1\leq i\leq n-2 \\ 1\leq j\leq n-2}} \right], \\ (a_{1,1})^{n-3} \det \left[(a_{i,j})_{\substack{i\neq n-1 \\ j\neq n-1}} \right] = \det_{1\leq i\leq n-1} \left[(b_{i,j})_{\substack{i\neq n-2 \\ j\neq n-2}} \right], \\ (a_{1,1})^{n-3} \det \left[(a_{i,j})_{\substack{i\neq n \\ j\neq n-1}} \right] = \det_{1\leq i\leq n-1} \left[(b_{i,j})_{\substack{i\neq n-1 \\ j\neq n-2}} \right], \\ (a_{1,1})^{n-3} \det \left[(a_{i,j})_{\substack{i\neq n-1 \\ j\neq n}} \right] = \det_{1\leq i\leq n-1} \left[(b_{i,j})_{\substack{i\neq n-2 \\ j\neq n-1}} \right]. \end{cases} \tag{3.11}$$

Now substituting (3.11) in (3.10), we find

$$\{(a_{1,1})^{n-3}\}^2 \det \left[(a_{i,j})_{\substack{1\leq i\leq n \\ 1\leq j\leq n}} \right] \cdot \det \left[(a_{i,j})_{\substack{i\neq n-1,n \\ j\neq n-1,n}} \right] = \{(a_{1,1})^{n-3}\}^2 . \Bigg(\det \left[(a_{i,j})_{\substack{i\neq n \\ j\neq n}} \right] . \det \left[(a_{i,j})_{\substack{i\neq n-1 \\ j\neq n-1}} \right] - \det \left[(a_{i,j})_{\substack{i\neq n-1 \\ j\neq n}} \right] . \det \left[(a_{i,j})_{\substack{i\neq n \\ j\neq n-1}} \right] \Bigg).$$

This completes the proof. □

Using the Dodgsons condensation method for the determinants of the third order, we obtain:

Example 3.3.

$$\det(A) = \begin{vmatrix} 1 & 0 & 3 & 5 & 1 \\ 0 & 1 & 5 & 1 & 0 \\ 0 & 4 & 0 & 0 & 2 \\ 2 & 3 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{vmatrix} = \frac{1}{\begin{vmatrix} 1 & 5 & 1 \\ 4 & 0 & 0 \\ 3 & 1 & 2 \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & 5 & 1 \\ 0 & 4 & 0 & 0 \\ 2 & 3 & 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 3 & 5 & 1 \\ 1 & 5 & 1 & 0 \\ 4 & 0 & 0 & 2 \\ 3 & 1 & 2 & 0 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 & 5 & 1 \\ 0 & 4 & 0 & 0 \\ 2 & 3 & 1 & 2 \\ 1 & 0 & 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 & 1 & 0 \\ 4 & 0 & 0 & 2 \\ 3 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} \end{vmatrix}$$

$$= \cdots = \frac{1}{-36} \begin{vmatrix} 140 & 170 \\ -4 & -64 \end{vmatrix} = \frac{1}{-36}(-8960 + 680) = \frac{-8280}{-36} = 230.$$

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