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An effective method for approximating the solution of singular integral equations with Cauchy kernel type

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Abstract. In present paper, a numerical approach for solving Cauchy type singular integral equations is discussed. Lagrange interpolation with Gauss Legendre quadrature nodes and Taylor series expansion are utilized to reduce the computation of integral equations into some algebraic equations. Finally, five examples with exact solution are given to show efficiency and applicability of the method. Also, we give the maximum of computed absolute errors for some examples.

Keywords: Singular integral equation, Cauchy kernel, Lagrange interpolation, Taylor series expansion, Gauss Legendre quadrature nodes.

2000 Mathematics subject classification: 45E05; 45B05; 45E10.

1. Introduction

Singular integral equation has enormous applications in applied problems including fluid mechanics, biomechanics, electromagnetic theory and chemistry applications such as heat conduction, crystal growth and electrochemistry. An integral equation is called a singular integral equation if one or both limits of integration become infinite, or if the kernel

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of the equation becomes infinite at one or more points in the interval of integration. Such equations can not be solved exactly, that is why various approximate methods have been developed and applied(see [1, 2, 4-6, 8-10]). The present paper deals with approximate approach to solve the above mentioned singular integral equation. Our study is based on approximating the solution of the Cauchy singular integral equation by Lagrange interpolation. As will be seen, we may overcome the singularity using truncated Taylor series expansion. We first present the Lagrange interpolation and Taylor series expansion of a function and then we introduce singular integral equations with Cauchy kernel type.

2. Gauss Legendre nodes and weights

Let $p_n(x)$ be the Legendre polynomial of degree n on the interval $[-1, 1]$. The nodes for quadrature order n are given by the roots of the Legendre polynomial $p_n(x)$ which occur symmetrically about 0. The Gauss Legendre nodes are

$$-1 < x_0 < x_1 < ... < x_{n-1} < x_n < 1.$$

No explicit formulas are known for the points x_i, and so they are computed numerically. Also, the weights w_i are given by

$$\begin{aligned} w_i &= \frac{2}{(1-x_i^2)\left(p_n'(x_i)\right)^2} \\ &= \frac{2(1-x_i^2)}{(n+1)^2\left(p_{n+1}(x_i)\right)^2}, \quad i = 0, 1, ..., n. \end{aligned}$$

For detail see [7]. Beyer (1987) gave a table of nodes and weights up to $n = 16$. (see [3])

3. Lagrange interpolation and Taylor series expansion

In this paper, we use the idea of the interpolation by Lagrange functions to approximate the solution of the singular integral equations. The collocation points for the interpolation formula are Gauss Legendre quadrature nodes. A function $\psi(x)$ may be approximated by using Lagrange interpolation as

$$\psi(x) \simeq \sum_{i=0}^{n} \psi_i l_i(x), \tag{3.1}$$

where,

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \left(\frac{x - x_j}{x_i - x_j}\right).$$

Also, $l_i(x_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta, which simply implies that, $\psi_i = \psi(x_i)$. Now with the Taylor series expansion of $\psi(y)$ based on expanding about the given point $x \in (-1, 1)$, we have the Taylor series approximation of $\psi(y)$ in the following form

$$\begin{aligned}\psi(y) &= \sum_{r=0}^{p} \frac{\psi^{(r)}(x)}{r!}(y-x)^r + r_p(x) \\ &= \psi(x) + \sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{r!}(y-x)^r + r_p(x) \\ &\simeq \psi(x) + \sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{r!}(y-x)^r, \qquad (3.2)\end{aligned}$$

where,

$$r_p(x) = \frac{\psi^{(p+1)}(\xi_x)}{(p+1)!}(y-x)^{p+1},$$

is called the remainder term or truncation error, and ξ_x is between y and x. Consequently, from (3.2) we obtain

$$\psi(y) - \psi(x) \simeq \sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{r!}(y-x)^r. \qquad (3.3)$$

Now, we consider the following singular integral equation

$$\alpha(x)\psi(x) + \beta(x)\int_{-1}^{1} \frac{\psi(y)}{y-x}dy = f(x), \quad |x| < 1, \qquad (3.4)$$

where, $\alpha(x) \neq 0, \beta(x) \neq 0$ and $\alpha(x), \beta(x), f(x) \in L^2[-1,1]$ are given functions and $\psi(x)$ is the unknown function to be determined.

4. Method of the solution

To approximate the integral part of the Eq. (3.4), we may proceed as follows

$$\begin{aligned}\int_{-1}^{1} \frac{\psi(y)}{y-x}dy &= \int_{-1}^{1} \frac{\psi(y)-\psi(x)+\psi(x)}{y-x}dy \\ &= \int_{-1}^{1} \frac{\psi(y)-\psi(x)}{y-x}dy + \int_{-1}^{1} \frac{\psi(x)}{y-x}dy \\ &= \int_{-1}^{1} \frac{\psi(y)-\psi(x)}{y-x}dy + \psi(x)\int_{-1}^{1} \frac{1}{y-x}dy \\ &= I_1(x) + \psi(x) I_2(x), \qquad (4.1)\end{aligned}$$

By using (3.3), we can approximate $I_1(x)$ as follows

$$\begin{aligned} I_1(x) &= \int_{-1}^{1} \frac{\psi(y)-\psi(x)}{y-x} dy \\ &\simeq \int_{-1}^{1} \frac{\sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{r!}(y-x)^r}{y-x} dy \\ &= \int_{-1}^{1} \sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{r!}(y-x)^{r-1} dy \\ &= \sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{r!} \int_{-1}^{1} (y-x)^{r-1} dy \\ &= \sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{rr!} \left[(1-x)^r - (-1-x)^r\right], \quad |x|<1, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} I_2(x) &= \int_{-1}^{1} \frac{1}{y-x} dy \\ &= \ln\left(\frac{1-x}{1+x}\right), \quad |x|<1, \end{aligned} \tag{4.3}$$

by substituting (4.2) and (4.3) into (4.1), we obtain

$$\int_{-1}^{1} \frac{\psi(y)}{y-x} dy \simeq \sum_{r=1}^{p} \frac{\psi^{(r)}(x)}{rr!} \left[(1-x)^r - (-1-x)^r\right] + \psi(x) \ln\left(\frac{1-x}{1+x}\right), \quad |x|<1. \tag{4.4}$$

Now, we return back to main problem that is approximating the solution for the following singular integral equation

$$\alpha(x)\psi(x) + \beta(x) \int_{-1}^{1} \frac{\psi(y)}{y-x} dy = f(x), \quad |x|<1, \tag{4.5}$$

and for simplicity of notation we set

$$\gamma(x) = \int_{-1}^{1} \frac{\psi(y)}{y-x} dy. \tag{4.6}$$

Representation of the functions $\alpha(x), \beta(x), f(x)$ and $\psi(x)$ in terms of Lagrange function yields

$$\alpha(x) \simeq \sum_{i=0}^{n} \alpha_i l_i(x), \quad \alpha_i = \alpha(x_i), \tag{4.7}$$

$$\beta(x) \simeq \sum_{i=0}^{n} \beta_i l_i(x), \quad \beta_i = \beta(x_i), \tag{4.8}$$

$$f(x) \simeq \sum_{i=0}^{n} f_i l_i(x), \quad f_i = f(x_i), \tag{4.9}$$

$$\psi(x) \simeq \sum_{i=0}^{n} \psi_i l_i(x), \quad \psi_i = \psi(x_i), \tag{4.10}$$

by substituting the approximations (4.7)-(4.10) and (4.6) into (4.5), we obtain

$$\left(\sum_{i=0}^{n} \alpha_i l_i(x)\right)\left(\sum_{i=0}^{n} \psi_i l_i(x)\right) + \left(\sum_{i=0}^{n} \beta_i l_i(x)\right)\gamma(x) = \sum_{i=0}^{n} f_i l_i(x), \tag{4.11}$$

collocating (4.11) at the same Lagrange interpolation points $x_j, j = 0, 1, ..., n$ and using the fact that $l_i(x_j) = \delta_{ij}$, we obtain

$$\left(\sum_{i=0}^{n} \alpha_i \delta_{ij}\right)\left(\sum_{i=0}^{n} \psi_i \delta_{ij}\right) + \left(\sum_{i=0}^{n} \beta_i \delta_{ij}\right)\gamma(x_j) = \sum_{i=0}^{n} f_i \delta_{ij}, \tag{4.12}$$

which simply yields

$$\alpha_j \psi_j + \beta_j \gamma(x_j) = f_j, \quad j = 0, 1, ..., n \tag{4.13}$$

where

$$\gamma(x_j) = \sum_{r=1}^{p} \frac{\psi^{(r)}(x_j)}{r r!}\left[(1-x_j)^r - (-1-x_j)^r\right] + \psi_j \ln\left(\frac{1-x_j}{1+x_j}\right).$$

Equation (4.13) is a system of $n+1$ linear equations which can be solved for $\psi_j, j = 0, 1, ..., n$; therefore, desired approximation for $\psi(x)$ may be obtained by (3.1).

5. Illustrative Examples

In this section, five examples with the exact solutions are given. To implement the method numerically, the points for the Lagrange interpolation are chosen to be the Gauss Legendre quadrature nodes which lead to the more accurate approximations. The computations associated with the examples were performed using Maple 9 on a Personal Computer. In the following examples n is the number of Lagrange functions and p is the number of the terms of the Taylor series expansion.

Example 1:

$$\alpha(x)\psi(x) + \beta(x) \int_{-1}^{1} \frac{\psi(y)}{y-x} dy = f(x), \quad |x| < 1,$$

where

$$\alpha(x) = 1+x^2, \quad \beta(x) = 1+x^2, \quad f(x) = x^3 \ln(\frac{1-x}{1+x}) + x^3 + 2x^2 + \frac{4-\pi}{2},$$

and the exact solution $\psi(x) = \frac{x^3}{1+x^2}$.
Table 1 shows the computed error $|\frac{x^3}{1+x^2} - \psi_n(x)|$ for example 1 with $n = 4, p = 3$ and $n = 7, p = 7$.

Table 1
Estimated errors for example 1

t	example 1(n=4, p=3)	example 1(n=7, p=7)
0.0	9×10^{-2}	3×10^{-5}
0.1	8×10^{-2}	5×10^{-4}
0.2	8×10^{-2}	5×10^{-4}
0.3	7×10^{-2}	8×10^{-6}
0.4	7×10^{-2}	6×10^{-4}
0.5	7×10^{-2}	9×10^{-4}
0.6	6×10^{-2}	5×10^{-4}
0.7	5×10^{-2}	1×10^{-4}
0.8	3×10^{-2}	2×10^{-4}
0.9	1×10^{-2}	5×10^{-4}

Example 2:

$$\alpha(x)\psi(x) + \beta(x) \int_{-1}^{1} \frac{\psi(y)}{y-x} dy = f(x), \quad |x| < 1,$$

where

$$\alpha(x) = 1, \quad \beta(x) = 1, \quad f(x) = x + 2 + x \ln(\frac{1-x}{1+x}),$$

and the exact solution $\psi(x) = x$. In this example, for $n = 4$ and $p = 3$ we have

$$\begin{aligned}
\psi_4(x) &= \sum_{i=0}^{4} \psi_i l_i(x) \\
&= .9999999913(-2.719530422x - 1.464383670)x(-.6922095881x + .3727336193) \\
&(-.5517668485x + .5000000000) + .9999999913(2.719530422x + 2.464383670)x \\
&(-.9285580268x + .5000000000)(-.6922095881x + .6272663807) + 0.24 \times 10^{-8} \\
&(1.103533697x + 1.000000000)(1.857116054x + 1.000000000)(-1.857116054x + 1.000000000) \\
&(-1.103533697x + 1.000000000) + 1.000000006(.6922095881x + .6272663807) \\
&(.9285580268x + .5000000000)x(-2.719530422x + 2.464383670) + 1.000000006 \\
&(.5517668485x + .5000000000)(.6922095881x + .3727336193)x(2.719530422x - 1.464383670) \\
&= 0.2400000000 \times 10^{-8} + 0.5 \times 10^{-8}x^2 + .9999999989x \\
&\simeq x. \qquad (\textit{exact solution})
\end{aligned}$$

So, maximum error is obtained as

$$|x - \psi_4(x)| < .84 \times 10^{-8}, \quad |x| < 1.$$

Example 3:

$$\alpha(x)\psi(x) + \beta(x)\int_{-1}^{1} \frac{\psi(y)}{y - x}dy = f(x), \quad |x| < 1,$$

where

$$\alpha(x) = x^2+x+1, \quad \beta(x) = x^2+x+1, \quad f(x) = x^5+x^4+x^3+(x^2+x+1)\left(\frac{2}{3} + 2x^2 + x^3 \ln\left(\frac{1-x}{1+x}\right)\right),$$

and the exact solution $\psi(x) = x^3$. In this example, for $n = 4$ and $p = 3$ we have

$$\begin{aligned}
\psi_4(x) &= \sum_{i=0}^{4} \psi_i l_i(x) \\
&= .8211619121(-2.719530422x - 1.464383670)x(-.6922095881x + .3727336193) \\
&(-.5517668485x + .5000000000) + .2899491901(2.719530422x + 2.464383670)x \\
&(-.9285580268x + .5000000000)(-.6922095881x + .6272663807) + 0.25 \times 10^{-8} \\
&(1.103533697x + 1.000000000)(1.857116054x + 1.000000000)(-1.857116054x + 1.000000000) \\
&(-1.103533697x + 1.000000000) + .2899492033(.6922095881x + .6272663807) \\
&(.9285580268x + .5000000000)x(-2.719530422x + 2.464383670) + .8211619249 \\
&(.5517668485x + .5000000000)(.6922095881x + .3727336193)x(2.719530422x - 1.464383670) \\
&= 0.6 \times 10^{-9}x^4 + .9999999976x^3 + 0.31 \times 10^{-8}x^2 - 0.6 \times 10^{-9}x + 0.2500000000 \times 10^{-8} \\
&\simeq x^3. \qquad (\textit{exact solution})
\end{aligned}$$

So, maximum error is obtained as

$$|x^3 - \psi_4(x)| < .92 \times 10^{-8}, \quad |x| < 1.$$

Example 4:

$$\alpha(x)\psi(x) + \beta(x)\int_{-1}^{1} \frac{\psi(y)}{y - x}dy = f(x), \quad |x| < 1,$$

where

$$\alpha(x) = 2, \quad \beta(x) = 1, \quad f(x) = 2x^4 + 2x^3 + \frac{2}{3}x + x^4 \ln\left(\frac{1 - x}{1 + x}\right),$$

and the exact solution $\psi(x) = x^4$. In this example, for $n = 6$ and $p = 5$ we have

$$\begin{aligned}
\psi_6(x) &= \sum_{i=0}^{6} \psi_i l_i(x) \\
&= -.8549618710(-4.817495912x - 3.572323476)(-1.840729889x - .7470512979) \\
&\quad x(-.7380329471x + .2995270921)(-.5914922942x + .4386099848) \\
&\quad (-.5268104867x + .5000000000) - .4077446059(4.817495912x + 4.572323476) \\
&\quad (-2.978974044x - 1.209002168)x(-.8715536177x + .3537158087)(-.6742804709x \\
&\quad + .5000000000)(-.5914922942x + .5613900151) - 0.6684683406 \times 10^{-1}(1.840729889x \\
&\quad + 1.747051298)(2.978974044x + 2.209002168)x(-1.231996982x + .5000000000) \\
&\quad (-.8715536177x + .6462841913)(-.7380329471x + .7004729079) - 0.1195 \times 10^{-7} \\
&\quad (1.053620973x + .9999999996)(1.348560942x + 1.000000000)(2.463993964x \\
&\quad + .9999999999)(-2.463993964x + .9999999999)(-1.348560942x + 1.000000000) \\
&\quad (-1.05362097x + .999999999) + 0.668468474 \times 10^{-1}(.738032947x + .7004729079) \\
&\quad (.871553617x + .6462841913)(1.231996982x + .5000000000)x(-2.978974044x \\
&\quad + 2.209002168)(-1.840729889x + 1.747051298) + .4077446403(.5914922942x + .5613900151) \\
&\quad (.6742804709x + .5000000000)(.8715536177x + .3537158087)x(2.978974044x \\
&\quad - 1.209002168)(-4.817495912x + 4.572323476) + .8549619302(.5268104867x + .5000000000) \\
&\quad (.5914922942x + .4386099848)(.7380329471x + .2995270921)x(1.840729889x \\
&\quad - .7470512979)(4.817495912x - 3.572323476) \\
&= 0.32 \times 10^{-8}x + 0.24 \times 10^{-8}x^2 + 0.18 \times 10^{-8}x^3 + 0.9 \times 10^{-8}x^5 + .999999981x^4 \\
&\quad - 0.51 \times 10^{-8}x^6 - 0.1194999999 \times 10^{-8} \\
&\simeq x^4. \qquad (\textit{exact solution})
\end{aligned}$$

So, maximum error is obtained as

$$|x^4 - \psi_6(x)| < .41694999999 \times 10^{-7}, \quad |x| < 1.$$

Example 5:

$$\alpha(x)\psi(x) + \beta(x)\int_{-1}^{1} \frac{\psi(y)}{y - x}dy = f(x), \quad |x| < 1,$$

where

$$\alpha(x) = 2, \quad \beta(x) = 1, \quad f(x) = 2x^6 + 2x^5 + \frac{2}{3}x^3 + \frac{2}{5}x + x^6 \ln\left(\frac{1 - x}{1 + x}\right),$$

and the exact solution $\psi(x) = x^6$. In this example, for $n = 7$ and $p = 7$ we have

$$\begin{aligned}\psi_7(x) &= \sum_{i=0}^{7} \psi_i l_i(x)\\ &= 0.392807 \times 10^{-7}x - 0.903043 \times 10^{-7}x^2 - 0.4851 \times 10^{-6}x^3 + 0.1453 \times 10^{-5}x^5\\ &\quad + 0.3636 \times 10^{-6}x^4 + .99999971x^6 - 0.11577436 \times 10^{-5}x^7 + 0.683 \times 10^{-8}\\ &\simeq x^6. \qquad (\textit{exact solution})\end{aligned}$$

So, maximum error is obtained as

$$|x^6 - \psi_7(x)| < .38858649 \times 10^{-5}, \quad |x| < 1.$$

6. Conclusion

In the present paper, Lagrange functions and truncated Taylor series expansion are applied to solve the Cauchy singular integral equation. Gauss Legendre quadrature nodes were used as the Lagrange interpolation points to obtain high accuracy of the approximated solutions. This approach, transforms a singular integral equation to a set of algebraic equations. As was seen, the maximum absolute error was estimated on some examples that show efficiency and applicability of the proposed method. Finally, by some modifications, this method can be extended and applied to integral equations of the form

$$\alpha(x)\psi(x) + \beta(x)\int_{-1}^{1} \frac{k(x,y)\psi(y)}{(y-x)^{\alpha}}dy = f(x), \quad |x| < 1, \quad 0 < \alpha \le 1.$$

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