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On the Quaternionic Curves in the Semi-Euclidean Space $\mathbf{E}_2^4$

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Abstract. In this study, we investigate the semi-real quaternionic curves in the semi-Euclidean space $\mathbb{E}_2^4$. Firstly, we introduce algebraic properties of semi-real quaternions. Then, we give some characterizations of semi-real quaternionic involute-evolute curves in the semi-Euclidean space $\mathbb{E}_2^4$. Finally, we give an example illustrated with Mathematica Programme.

Keywords: semi-real quaternionic involute-evolute curve, semi-real quaternion, semi-quaternionic space.

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1. Introduction

The quaternions were discovered by the Irish mathematician Sir William R. Hamilton for the first time in 1843. Hamilton generalized the complex numbers to use geometric optics. Thus, the quaternions which is a more general form of a complex number were found by him in [4]. The quaternions, as well as having a normal vector algebra of finite rotations for mathematical calculations of physical problems, provides a simple and elegant representation for describing finite rotations in space. Furthermore, they represent many useful methods, such as Euler angles or the quaternionic formulation of the equation of motion in the theory of

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relativity.

The set of quaternions $\mathbb{Q}$ coincides with the four-dimensional vector space $\mathbb{R}^4$ over the real numbers. The Serret-Frenet formulae for a quaternionic curve in $\mathbb{R}^3$ were introduced by Bharathi and Nagaraj. Moreover, they obtained the Serret-Frenet formulae for the quaternionic curve in $\mathbb{R}^4$ by the formulae in $\mathbb{R}^3$, [1]. A lot of studies have been published by considering their studies. One of them is Çöken and Tuna's study [3] which gives Serret-Frenet formulae of inclined curves, harmonic curvatures and some characterizations for a quaternionic curve in the semi-Euclidean spaces $\mathbb{E}_1^3$ and $\mathbb{E}_2^4$. Kahraman et al. defined a new kind of slant helix, which they called $\mathbf{B}_2-$slant helix and they gave some characterizations of this slant helix in $\mathbb{E}_2^4$, [5].

This study deals with special types of curves called involutes and evolutes. A pair of curves are said to be involute-evolute curves if there exists a one to one correspondence between their points such that one's tangent and the other's principal normal are linear dependent at their corresponding points. Bükçü and Karacan generalized the involute and evolute curves of a spacelike curve with a spacelike binormal in Minkowski 3-Space, [2].

The main purpose of this paper is to obtain some characterizations of semi-real quaternionic involute-evolute curves in semi-quaternionic space $\mathbb{Q}_v$. To do this, by considering the Serret-Frenet formulae of a curve in $\mathbb{E}_2^4$, firstly some characterizations of semi-real quaternionic involute-evolute curves are obtained in $\mathbb{E}_2^4$. In addition, some results for semi-real quaternionic $w-curves$ which have constant curvatures are given. Then, it is seen that the semi-real spatial quaternionic curves in $\mathbb{E}_1^3$ associated with semi-real quaternionic involute-evolute curves in $\mathbb{E}_2^4$ aren't semi-real spatial quaternionic involute-evolute curves. Finally, we give an example and draw related figures by using Mathematica Programme.

2. Preliminaries

A semi-real quaternion is defined with $q = q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3 + q_4$ (or $q = S_q + \boldsymbol{V}_q$ where the symbols $S_q = q_4$ and $\boldsymbol{V}_q = q_1\boldsymbol{e}_1 + q_2\boldsymbol{e}_2 + q_3\boldsymbol{e}_3$ denote scalar and vector parts of q, respectively) such that

$$\begin{array}{lll} i) & \boldsymbol{e}_i \times \boldsymbol{e}_i = -\varepsilon_{\boldsymbol{e}_i}, & (1 \leq i \leq 3) \\ ii) & \boldsymbol{e}_i \times \boldsymbol{e}_j = \varepsilon_{\boldsymbol{e}_i}\varepsilon_{\boldsymbol{e}_j}\boldsymbol{e}_k & in\ \mathbb{R}_1^3 \\ & \boldsymbol{e}_i \times \boldsymbol{e}_j = -\varepsilon_{\boldsymbol{e}_i}\varepsilon_{\boldsymbol{e}_j}\boldsymbol{e}_k & in\ \mathbb{R}_2^4 \end{array} \tag{2.1}$$

where (ijk) is an even permutation of (123). For every $p, q \in \mathbb{Q}_v$, using these basic products one can expand the product of two semi-real quaternions as

$$p \times q = S_p S_q + g\left(\boldsymbol{V}_p, \boldsymbol{V}_q\right) + S_p \boldsymbol{V}_q + S_q \boldsymbol{V}_p + \boldsymbol{V}_p \wedge_{\mathbb{L}} \boldsymbol{V}_q \tag{2.2}$$

where we have used the usual inner and cross products in semi-Euclidean space $\mathbb{R}_1^3$, [3]. A feature of semi-real quaternions is that the product of two semi-real quaternions is non-commutative. The conjugate of the semi-real quaternion q is denoted by $\widehat{q}$ and defined as $\widehat{q} = S_q - \boldsymbol{V}_q$. Thus, we define symmetric, non-degenerate valued bilinear form h as follows:

$$\begin{array}{ll} h(p,q) = \frac{1}{2}\left[\varepsilon_p \varepsilon_{\widehat{q}}\left(p \times \widehat{q}\right) + \varepsilon_q \varepsilon_{\widehat{p}}\left(q \times \widehat{p}\right)\right] & in\ \mathbb{R}_1^3 \\ h(p,q) = \frac{1}{2}\left[-\varepsilon_p \varepsilon_{\widehat{q}}\left(p \times \widehat{q}\right) - \varepsilon_q \varepsilon_{\widehat{p}}\left(q \times \widehat{p}\right)\right] & in\ \mathbb{R}_2^4 \end{array} \tag{2.3}$$

and it is called semi-real quaternion inner product, [3]. The norm of a semi-real quaternion $q = (q_1, q_2, q_3, q_4) \in \mathbb{Q}_v$ is

$$N(q) = \sqrt{|{q_1}^2 + {q_2}^2 - {q_3}^2 - {q_4}^2|}. \tag{2.4}$$

If $N(q) = 1$, then q is called a semi-real unit quaternion, [6]. q is called a semi-real spatial quaternion whenever $q + \widehat{q} = 0$, [1]. Moreover, the quaternionic product of two spatial quaternions is $p \times q = g\left(p, q\right) + p \wedge_{\mathbb{L}} q$. q is a semi-real temporal quaternion whenever $q - \widehat{q} = 0$. Any q can be written as $q = \frac{1}{2}\left(q + \widehat{q}\right) + \frac{1}{2}\left(q - \widehat{q}\right)$, [3].

The 4−dimensional semi-Euclidean space $\mathbb{E}_2^4$ is identified with the space of unit semi-real quaternions which is denoted by $\mathbb{Q}_v$. Let

$$\xi : I \subset \mathbb{R} \to \mathbb{Q}_v, \qquad s \to \xi(s) = \sum_{i=1}^{4} \xi_i(s) \boldsymbol{e}_i, \quad (1 \leq i \leq 4), \qquad \boldsymbol{e}_4 = 1$$

be a smooth curve defined over the interval $I = [0, 1]$. Let the arc-length parameter s be chosen such that the tangent $\boldsymbol{T} = \xi'(s)$ has unit magnitude, [3]. The Serret-Frenet apparatus of the semi-real quaternionic curve ξ are given by

$$\begin{array}{l} \boldsymbol{T}(s) = \xi', \\ \boldsymbol{N}(s) = \frac{\xi''}{N(\xi'')}, \\ \boldsymbol{B}(s) = \eta \varepsilon_{\boldsymbol{n}} \varepsilon_{\boldsymbol{T}} \left(\boldsymbol{E} \wedge_{\mathbb{L}} \boldsymbol{T} \wedge_{\mathbb{L}} \boldsymbol{N}\right), \\ \boldsymbol{E}(s) = -\eta \varepsilon_{\boldsymbol{n}} \varepsilon_{\boldsymbol{b}} \varepsilon_{\boldsymbol{T}} \varepsilon_{\boldsymbol{N}} \frac{\boldsymbol{T} \wedge_{\mathbb{L}} \boldsymbol{N} \wedge_{\mathbb{L}} \xi'''}{N(\boldsymbol{T} \wedge_{\mathbb{L}} \boldsymbol{N} \wedge_{\mathbb{L}} \xi''')}, \quad (\eta = \pm 1) \end{array} \tag{2.5}$$

and

$$\begin{array}{l} \kappa(s) = \varepsilon_{\boldsymbol{N}} N(\xi''), \\ k(s) = \varepsilon_{\boldsymbol{N}} \frac{N(\boldsymbol{T} \wedge_{\mathbb{L}} \mathbf{N} \wedge_{\mathbb{L}} \xi''')}{N(\xi'')}, \\ \left(r - \varepsilon_{\boldsymbol{t}} \varepsilon_{\boldsymbol{T}} \varepsilon_{\boldsymbol{N}} \kappa\right)(s) = \varepsilon_{\boldsymbol{b}} \varepsilon_{\boldsymbol{T}} \varepsilon_{\boldsymbol{N}} \frac{h\left(\xi^{(iv)}, \boldsymbol{E}\right)}{N(\boldsymbol{T} \wedge_{\mathbb{L}} \boldsymbol{N} \wedge_{\mathbb{L}} \xi''')}. \end{array} \tag{2.6}$$

Theorem 2.1. *Let $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}, \boldsymbol{E}\}$ be the Serret-Frenet frame of the semi-real quaternionic curve ξ at the point $\xi(s)$ and s is the arc-length parameter of the semi-real quaternionic curve ξ. Then the Serret-Frenet equations are*

$$\begin{aligned} \boldsymbol{T}' &= \varepsilon_{\boldsymbol{N}}\kappa\boldsymbol{N}, \\ \boldsymbol{N}' &= -\varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{N}}\kappa\boldsymbol{T} + \varepsilon_{\boldsymbol{n}}k\boldsymbol{B}, \\ \boldsymbol{B}' &= -\varepsilon_{\boldsymbol{t}}k\boldsymbol{N} + \varepsilon_{\boldsymbol{n}}\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}}\varepsilon_{\boldsymbol{N}}\kappa\right)\boldsymbol{E}, \\ \boldsymbol{E}' &= -\varepsilon_{\boldsymbol{b}}\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}}\varepsilon_{\boldsymbol{N}}\kappa\right)\boldsymbol{B} \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} &\kappa = \varepsilon_{\boldsymbol{N}}\|\boldsymbol{T}'\|, \ \ \boldsymbol{N} = \varepsilon_{\boldsymbol{T}}(\boldsymbol{t}\times\boldsymbol{T}), \ \ \boldsymbol{B} = \varepsilon_{\boldsymbol{T}}(n\times\boldsymbol{T}), \ \ \boldsymbol{E} = \varepsilon_{\boldsymbol{T}}(\boldsymbol{b}\times\boldsymbol{T}) \\ &h(\boldsymbol{T},\boldsymbol{T}) = \varepsilon_{\boldsymbol{T}}, \ \ h(\boldsymbol{N},\boldsymbol{N}) = \varepsilon_{\boldsymbol{N}}, \ \ h(\boldsymbol{B},\boldsymbol{B}) = \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}}, \ \ h(\boldsymbol{E},\boldsymbol{E}) = \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}}, \end{aligned} \tag{2.8}$$

[3].

The Serret-Frenet formula of the semi-real quaternionic curve $\xi = \xi(s)$ is obtained by making use of the Serret-Frenet formula of a semi-real spatial quaternionic curve $\alpha = \alpha(s)$ where α is a semi-real spatial quaternionic curve associated with the semi-real quaternionic curve ξ. In addition, $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ is the Frenet frame of the semi-real spatial quaternionic curve α in $\mathbb{R}_1^3$. Furthermore, there are relationships between the curvatures of the curves ξ and α as follows: The torsion of ξ is the principal curvature of the curve α and the third curvature of ξ is $(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}}\varepsilon_{\boldsymbol{N}}\kappa)$, where r is the torsion of α and κ is the principal curvature of ξ, [3].

3. Some Characterizations of Semi-Real Quaternionic Involute-Evolute Curves

In this section, some characterizations of semi-real quaternionic involute-evolute curves are obtained in $\mathbb{E}_2^4$. In addition some results for semi-real quaternionic $w-curves$ which have constant curvatures are given. Finally, we give an example and draw related figures by using Mathematica Programme.

Definition 3.1. Let $\phi, \xi : I \subset \mathbb{R} \to \mathbb{Q}_v$ be two semi-real quaternionic curves with parameter s^* and s, respectively. Moreover, $\{\boldsymbol{T}_\phi, \boldsymbol{N}_\phi, \boldsymbol{B}_\phi, \boldsymbol{E}_\phi\}$ and $\{\boldsymbol{T}_\xi, \boldsymbol{N}_\xi, \boldsymbol{B}_\xi, \boldsymbol{E}_\xi\}$ denote the Serret-Frenet frame of the curves ϕ and ξ, respectively. If

$$h(\boldsymbol{T}_\phi(s^*), \boldsymbol{T}_\xi(s)) = 0.$$

then, the curves (ϕ, ξ) are called as semi-real quaternionic involute-evolute curves in $\mathbb{Q}_v$.

Theorem 3.2. *Let $\xi, \phi : I \to \mathbb{Q}_v$ be two unit speed semi-real quaternionic curves. If the semi-real quaternionic curve $\phi : I \to \mathbb{Q}_v$ is a*

quaternionic involute of the curve ξ, then $d_{\mathbb{L}}(\xi(s),\phi(s^*)) = |c-s|$ where c is real number.

Proof. Let $\xi,\phi : I \to \mathbb{Q}_v$ be two unit speed semi-real quaternionic involute-evolute curves. From the Definition 3.1., we know that

$$\phi(s^*) = \xi(s) + \lambda(s)\,\boldsymbol{T}_\xi(s). \tag{3.1}$$

Then differentiating the equation (3.1) with respect to s, we get

$$\frac{d\phi}{ds^*}\frac{ds^*}{ds} = \left(1+\lambda'\right)\boldsymbol{T}_\xi + \varepsilon_{\boldsymbol{N}_\xi}\lambda\kappa_\xi\boldsymbol{N}_\xi.$$

If the semi-real quaternionic product is made with $\boldsymbol{T}_\xi$ in the last equation, we can find easily that

$$h\left(\frac{d\phi}{ds^*}\frac{ds^*}{ds},\boldsymbol{T}_\xi(s)\right) = \left(1+\lambda'\right)h\left(\boldsymbol{T}_\xi,\boldsymbol{T}_\xi\right)+\varepsilon_{\boldsymbol{N}_\xi}\lambda\kappa_\xi h\left(\boldsymbol{N}_\xi,\boldsymbol{T}_\xi\right). \tag{3.2}$$

Considering the equation (2.8), we obtain

$$\lambda(s) = c-s. \tag{3.3}$$

By substituting the equation (3.3) into (3.1) we get

$$\phi(s^*) = \xi(s) + (c-s)\,\boldsymbol{T}_\xi(s). \tag{3.4}$$

Considering

$$\begin{aligned} N^2\left((c-s)\boldsymbol{T}_\xi\right) &= \left|h\left((c-s)\boldsymbol{T}_\xi,(c-s)\boldsymbol{T}_\xi\right)\right| \\ &= (c-s)^2\left|h\left(\boldsymbol{T}_\xi,\boldsymbol{T}_\xi\right)\right| = (c-s)^2\left|\varepsilon_{\boldsymbol{T}_\xi}\right| = (c-s)^2, \end{aligned}$$

we obtain that

$$d_{\mathbb{L}}\left(\xi(s),\phi(s^*)\right) = N\left((c-s)\mathbf{T}_\xi\right) = |c-s|.$$

□

Theorem 3.3. *Let $\phi,\xi : I \to \mathbb{Q}_v$ be unit speed semi-real quaternionic involute-evolute curves. The Serret-Frenet frame of semi-real quaternionic curve ϕ, $\{\boldsymbol{T}_\phi,\boldsymbol{N}_\phi,\boldsymbol{B}_\phi,\boldsymbol{E}_\phi\}$, can be formed by frame of ξ, $\{\boldsymbol{T}_\xi,\boldsymbol{N}_\xi,\boldsymbol{B}_\xi,\boldsymbol{E}_\xi\}$.*

Proof. Let ξ be a unit speed semi-real quaternionic curve. Without loss of generality, suppose that ϕ is the involute of ξ. By using the equation (2.7), if we differentiate (3.4) with respect to s, we get

$$\boldsymbol{T}_\phi(s^*)\frac{ds^*}{ds} = \varepsilon_{\boldsymbol{N}_\xi}(c-s)\kappa_\xi\boldsymbol{N}_\xi(s).$$

So, it is seen that

$$\boldsymbol{T}_\phi(s^*) = \varepsilon_{\boldsymbol{N}_\xi}\boldsymbol{N}_\xi(s) \tag{3.5}$$

where $\frac{ds^*}{ds} = (c-s)\kappa_\xi$.
Let us investigate $\boldsymbol{N}_\phi(s^*)$. If we take differentiation of (3.5) with respect to s, we get

$$\boldsymbol{T}_\phi{}' = \phi'' = \varepsilon_{\boldsymbol{N}_\xi}\boldsymbol{N}_\xi{}'$$

i.e.,

$$\phi'' = -\varepsilon_{\boldsymbol{t}}\kappa_\xi \boldsymbol{T}_\xi + \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi} k \boldsymbol{B}_\xi. \tag{3.6}$$

Thus by using the equations (2.4) and (3.6), and considering

$$\begin{aligned} N^2(\phi'') = |h(\phi'',\phi'')| &= \left|k^2 h(\boldsymbol{B}_\xi,\boldsymbol{B}_\xi) + {\kappa_\xi}^2 h(\boldsymbol{T}_\xi,\boldsymbol{T}_\xi)\right| \\ &= \left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}{\kappa_\xi}^2\right|, \end{aligned}$$

we obtain that

$$N(\phi'') = \sqrt{\left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}{\kappa_\xi}^2\right|}. \tag{3.7}$$

Using the equations (2.5), (3.6) and (3.7), we can write

$$\boldsymbol{N}_\phi(s^*) = \frac{-\varepsilon_{\boldsymbol{t}}\kappa_\xi \boldsymbol{T}_\xi + \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi} k \boldsymbol{B}_\xi}{\sqrt{\left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}{\kappa_\xi}^2\right|}}. \tag{3.8}$$

Now we obtain that the binormal vector $\boldsymbol{E}_\phi(s^*)$. If we differentiate the equation (3.6), we get

$$\phi''' = -\varepsilon_{\boldsymbol{t}}{\kappa_\xi}'\boldsymbol{T}_\xi - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{N}_\xi}\left(\varepsilon_{\boldsymbol{n}}k^2 + {\kappa_\xi}^2\right)\boldsymbol{N}_\xi + \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}k'\boldsymbol{B}_\xi - \varepsilon_{\boldsymbol{N}_\xi}k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)\boldsymbol{E}_\xi. \tag{3.9}$$

Therefore, from the equations (3.5), (3.8) and (3.9) we find that

$$\boldsymbol{T}_\phi \wedge \boldsymbol{N}_\phi \wedge \phi''' = \frac{-\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)\left(\varepsilon_{\boldsymbol{N}_\xi}k\boldsymbol{T}_\xi + \varepsilon_{\boldsymbol{t}}\kappa_\xi\boldsymbol{B}_\xi\right) + \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left({\kappa_\xi}'k - \kappa_\xi k'\right)\boldsymbol{E}_\xi}{\sqrt{\left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}{\kappa_\xi}^2\right|}}. \tag{3.10}$$

The semi-real quaternionic norm of the vector field in the equation (3.10) is

$$N\left(\boldsymbol{T}_\phi \wedge \boldsymbol{N}_\phi \wedge \phi'''\right) = \sqrt{\left|\frac{\varepsilon_{\boldsymbol{T}_\xi}k^2\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^2\left(k^2 + \varepsilon_{\boldsymbol{n}}{\kappa_\xi}^2\right) + \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left({\kappa_\xi}'k - \kappa_\xi k'\right)^2}{\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}{\kappa_\xi}^2}\right|}. \tag{3.11}$$

Moreover, using the equations (3.10), (3.11) and considering $\varepsilon_{\boldsymbol{n}} = \varepsilon_{\boldsymbol{n}^*}\varepsilon_{\boldsymbol{b}^*}$ and $\varepsilon_{\boldsymbol{N}_\phi} = \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}$, we obtain that

$$\boldsymbol{E}_\phi = \eta\frac{\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)\left(\varepsilon_{\boldsymbol{N}_\xi}k\boldsymbol{T}_\xi + \varepsilon_{\boldsymbol{t}}\kappa_\xi\boldsymbol{B}_\xi\right) - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left({\kappa_\xi}'k - \kappa_\xi k'\right)\boldsymbol{E}_\xi}{\sqrt{\left|\varepsilon_{\boldsymbol{T}_\xi}k^2\left(r - \varepsilon_t\varepsilon_{T_\xi}\varepsilon_{N_\xi}\right)^2\left(k^2 + \varepsilon_{\boldsymbol{n}}{\kappa_\xi}^2\right) + \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left({\kappa_\xi}'k - \kappa_\xi k'\right)^2\right|}} \tag{3.12}$$

where $\eta = \pm 1$ providing that $\det\left(\boldsymbol{T}_\xi(s), \boldsymbol{N}_\xi(s), \boldsymbol{B}_\xi(s), \boldsymbol{E}_\xi(s)\right) = +1$. Similarly, considering the equations (2.5), (3.5), (3.8) and (3.12), if we make necessary arrangements, the binormal vector $\boldsymbol{B}_\phi(s^*)$ is obtained as

$$\boldsymbol{B}_\phi = \eta\varepsilon_{\boldsymbol{b}^*}\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{N}_\xi}\frac{\left(\kappa_\xi k' - \kappa_\xi' k\right)\left(-\varepsilon_{\boldsymbol{t}}k\boldsymbol{T}_\xi + \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\boldsymbol{B}_\xi\right) - \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{N}_\xi}k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)\left(k^2 + \varepsilon_{\boldsymbol{n}}{\kappa_\xi}^2\right)\boldsymbol{E}_\xi}{\sqrt{\left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}{\kappa_\xi}^2\right|}\sqrt{\left|\varepsilon_{\boldsymbol{T}_\xi}k^2\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^2\left(k^2 + \varepsilon_{\boldsymbol{n}}{\kappa_\xi}^2\right) + \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left({\kappa_\xi}'k - \kappa_\xi k'\right)^2\right|}}. \tag{3.13}$$

So, the proof is completed. □

Theorem 3.4. *Let $\phi, \xi : I \to \mathbb{Q}_v$ be unit speed semi-real quaternionic involute-evolute curves. The Serret-Frenet curvatures of semi-real quaternionic curve ϕ, $\left\{\kappa_\phi, k^*, \left(r^* - \varepsilon_{\boldsymbol{t}^*}\varepsilon_{\boldsymbol{T}_\phi}\varepsilon_{\boldsymbol{N}_\phi}\kappa_\phi\right)\right\}$ can be formed by curvatures of ξ, $\left\{\kappa_\xi, k, \left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)\right\}$.*

Proof. Let $\phi, \xi : I \to \mathbb{Q}_v$ be unit speed semi-real quaternionic involute-evolute curves. Using the equations (2.6), (3.7) and considering $\varepsilon_{\boldsymbol{N}_\phi} = \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}$, we obtain that

$$\kappa_\phi = \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}\sqrt{\left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}\kappa_\xi{}^2\right|}. \tag{3.14}$$

Similarly, using the equations (2.6), (3.7) and (3.11), we have

$$k^* = \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}\frac{\sqrt{\left|\varepsilon_{\boldsymbol{T}_\xi}k^2\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^2\left(k^2 + \varepsilon_{\boldsymbol{n}}\kappa_\xi{}^2\right) + \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left(\kappa_\xi{}'k - \kappa_\xi k'\right)^2\right|}}{\left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}\kappa_\xi{}^2\right|}. \tag{3.15}$$

Lastly, let us investigate the third curvature, $\left(r^* - \varepsilon_{\boldsymbol{t}^*}\varepsilon_{\boldsymbol{T}_\phi}\varepsilon_{\boldsymbol{N}_\phi}\kappa_\phi\right)$. For this purpose if we calculate $\phi^{(iv)}$, we find that

$$\begin{aligned}\phi^{(iv)} = {} & \left[-\varepsilon_{\boldsymbol{t}}\kappa_\xi{}'' + \kappa_\xi\left(\varepsilon_{\boldsymbol{n}}k^2 + \kappa_\xi{}^2\right)\right]\boldsymbol{T}_\xi + \left[-3\varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\kappa_\xi{}' - 3\varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}kk'\right]\boldsymbol{N}_\xi \\ & + \left[-\varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}k\left(\varepsilon_{\boldsymbol{n}}k^2 + \kappa_\xi{}^2\right) + \varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}k'' - \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{N}_\xi}k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^2\right]\boldsymbol{B}_\xi \\ & + \varepsilon_{\boldsymbol{N}_\xi}\left[2k'\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right) + k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)'\right]\boldsymbol{E}_\xi.\end{aligned} \tag{3.16}$$

Considering the equations (3.12) and (3.16) we get

$$h\left(\phi^{(iv)}, \boldsymbol{E}_\phi\right) = \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{N}_\xi}\frac{\begin{aligned}&\varepsilon_{\boldsymbol{n}}k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)\left(\kappa_\xi{}''k - \kappa_\xi k''\right) - \varepsilon_{\boldsymbol{b}}\kappa_\xi k^2\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^3 \\ &+ \left(2k'\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right) + k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)'\right)\left(\kappa_\xi{}'k - \kappa_\xi k'\right)\end{aligned}}{\left[\left|\varepsilon_{\boldsymbol{T}_\xi}k^2\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^2\left(k^2 + \varepsilon_{\boldsymbol{n}}\kappa_\xi{}^2\right) + \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left(\kappa_\xi{}'k - \kappa_\xi k'\right)^2\right|\right]}. \tag{3.17}$$

Finally, using the equations (2.6), (3.11) and (3.17), we get the third curvature of the semi-real quaternionic curve ϕ as follows

$$r^* - \varepsilon_{\boldsymbol{t}^*}\varepsilon_{\boldsymbol{T}_\phi}\varepsilon_{\boldsymbol{N}_\phi}\kappa_\phi = \varepsilon_{\boldsymbol{b}^*}\varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{N}_\xi}\frac{\begin{aligned}&\varepsilon_{\boldsymbol{n}}k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)\left(\kappa_\xi{}''k - \kappa_\xi k''\right) - \varepsilon_{\boldsymbol{b}}\kappa_\xi k^2\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^3 \\ &+ \left(2k'\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right) + k\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)'\right)\left(\kappa_\xi{}'k - \kappa_\xi k'\right)\end{aligned}}{\frac{1}{\sqrt{\left|\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2 + \varepsilon_{\boldsymbol{T}_\xi}\kappa_\xi{}^2\right|}}\left[\left|\varepsilon_{\boldsymbol{T}_\xi}k^2\left(r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}\kappa_\xi\right)^2\left(k^2 + \varepsilon_{\boldsymbol{n}}\kappa_\xi{}^2\right) + \varepsilon_{\boldsymbol{b}}\varepsilon_{\boldsymbol{T}_\xi}\left(\kappa_\xi{}'k - \kappa_\xi k'\right)^2\right|\right]}. \tag{3.18}$$

This completed the proof. □

From the Theorem 3.4, we give following corollary.

Corollary 3.5. *Let $\phi, \xi : I \to \mathbb{Q}_v$ be unit speed semi-real quaternionic involute-evolute curves. If quaternionic curve ξ is a quaternionic $w-$ curve, then the Serret-Frenet vectors of ξ are*

$$\begin{aligned} \mathbf{T}_\phi(s^*) &= \varepsilon_{\mathbf{N}_\xi}\mathbf{N}_\xi, & \mathbf{N}_\phi(s^*) &= \frac{-\varepsilon_{\mathbf{t}}\kappa_\xi\mathbf{T}_\xi+\varepsilon_{\mathbf{n}}\varepsilon_{\mathbf{N}_\xi}k\mathbf{B}_\xi}{\sqrt{\left|\varepsilon_{\mathbf{n}}\varepsilon_{\mathbf{T}_\xi}k^2+\varepsilon_{\mathbf{T}_\xi}\kappa_\xi{}^2\right|}}, \\ \mathbf{B}_\phi(s^*) &= \eta\varepsilon_{\mathbf{n}}\varepsilon_{\mathbf{b}}\varepsilon_{\mathbf{b}^*}\mathbf{E}_\xi, & \mathbf{E}_\phi(s^*) &= -\eta\varepsilon_{\mathbf{n}}\varepsilon_{\mathbf{T}_\xi}\frac{\varepsilon_{\mathbf{N}_\xi}k\mathbf{T}_\xi+\varepsilon_{\mathbf{t}}\kappa_\xi\mathbf{B}_\xi}{\sqrt{\left|\varepsilon_{\mathbf{T}_\xi}k^2+\varepsilon_{\mathbf{n}}\varepsilon_{\boldsymbol{T}_\xi}\kappa_\xi{}^2\right|}}. \end{aligned} \tag{3.19}$$

Theorem 3.6. *Let $\phi, \xi : I \subset \mathbb{R} \to \mathbb{Q}_v$ be semi-real quaternionic curves with parameter s^* and s, respectively. If (ϕ, ξ) are semi-real quaternionic involute-evolute curves, then the semi-real spatial quaternionic curves (β, α), associated with ϕ and ξ, respectively, aren't the semi-real spatial quaternionic involute-evolute curves.*

Proof. Let (ϕ, ξ) be semi-real quaternionic involute-evolute curves with parameter s^* and s, respectively. So, from the equation (2.8) we write $\boldsymbol{t}^* = \varepsilon_{\boldsymbol{T}_\phi}\boldsymbol{N}_\phi \times \widehat{\boldsymbol{T}_\phi}$ for the semi-real quaternionic curve $\phi = \phi(s^*)$. Here, if we use the equations (3,5) and (3.8), we can write $\boldsymbol{t}^* = (x\boldsymbol{T}_\xi + y\boldsymbol{B}_\xi)\times \varepsilon_{\boldsymbol{N}_\xi}\widehat{\boldsymbol{N}_\xi}$ where $x = \frac{-\varepsilon_t\kappa_\xi}{\sqrt{\left|\varepsilon_{\boldsymbol{T}_\xi}\kappa_\xi{}^2+\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2\right|}}$, $y = \frac{\varepsilon_{\boldsymbol{n}}k}{\sqrt{\left|\varepsilon_{\boldsymbol{T}_\xi}\kappa_\xi{}^2+\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{T}_\xi}k^2\right|}}$. From the last equation, we obtain that $\boldsymbol{t}^* = \varepsilon_{\boldsymbol{N}_\xi}x\left(\boldsymbol{T}_\xi \times \widehat{\boldsymbol{N}_\xi}\right)+\varepsilon_{\boldsymbol{N}_\xi}y\left(\boldsymbol{B}_\xi \times \widehat{\boldsymbol{N}_\xi}\right)$. By using the equation (2.8), we find that

$$\begin{aligned} \boldsymbol{t}^* &= \varepsilon_{\boldsymbol{N}_\xi}x\left(\boldsymbol{T}_\xi \times \varepsilon_{\boldsymbol{T}_\xi}\widehat{\boldsymbol{T}_\xi} \times \hat{\boldsymbol{t}}\right) + \varepsilon_{\boldsymbol{N}_\xi}y\left(\boldsymbol{B}_\xi \times \varepsilon_{\boldsymbol{T}_\xi}\widehat{\boldsymbol{T}_\xi} \times \hat{\boldsymbol{t}}\right), \\ \boldsymbol{t}^* &= \varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}x\hat{\boldsymbol{t}} + \varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}y\left(\varepsilon_{\boldsymbol{T}_\xi}\boldsymbol{n} \times \hat{\boldsymbol{t}}\right), \\ \boldsymbol{t}^* &= -\varepsilon_{\boldsymbol{T}_\xi}\varepsilon_{\boldsymbol{N}_\xi}x\boldsymbol{t} + \varepsilon_{\boldsymbol{N}_\xi}y\boldsymbol{b}. \end{aligned}$$

From the last equation, we see that $\boldsymbol{t}^*$ is orthogonal to $\boldsymbol{n}$. However, if $\boldsymbol{t}^*$ is linearly dependent with $\boldsymbol{n}$ then (β, α) are semi-real spatial quaternionic involute-evolute curves. So (β, α) aren't semi -real spatial quaternionic involute-evolute curves. □

Now, we will give an example for the above theorem.

Example 3.7. We consider a quaternionic curve with the arc-length parameter s, $\xi : I \subset \mathbb{R} \to \mathbb{E}_2^4$ given by

$$\xi(s) = \left(\cosh(s), \sqrt{2}s, \sinh(s), \sqrt{2}\right)$$

for all $s \in I$. By considering the equations (2.5) and (2.6) we find that the Frenet apparatus of the quaternionic curve $\xi = \xi(s)$ as follows

$$\begin{aligned}
&\boldsymbol{T}_{\xi}(s) = \left(\sinh(s), \sqrt{2}, \cosh(s), 0\right) \\
&\boldsymbol{N}_{\xi}(s) = \left(\cosh(s), 0, \sinh(s), 0\right) \\
&\boldsymbol{B}_{\xi}(s) = -\varepsilon_{\boldsymbol{b}}\left(\sqrt{2}\sinh(s), 1, \sqrt{2}\cosh(s), 0\right) \\
&\boldsymbol{E}_{\xi}(s) = -\varepsilon_{\boldsymbol{n}}\varepsilon_{\boldsymbol{b}}\left(0, 0, 0, 1\right).
\end{aligned}$$

The curvatures of the quaternionic curve $\xi = \xi(s)$ are as follows

$$\kappa_{\xi} = -1, \quad k = \sqrt{2}, \quad r - \varepsilon_{\boldsymbol{t}}\varepsilon_{\boldsymbol{T}_{\xi}}\varepsilon_{\boldsymbol{N}_{\xi}}\kappa_{\xi} = 0.$$

By using the equation (3.4), if we make necessary arrangement, we can easily find the quaternionic involute curve of the quaternionic curve $\xi = \xi(s)$, as follows

$$\phi(s) = \left((c-s)\sinh(s) + \cosh(s), \sqrt{2}c, (c-s)\cosh(s) + \sinh(s), \sqrt{2}\right)$$

which c is a real number.
By using the equation (3.1), the spatial quaternionic curve $\alpha = \alpha(s)$ in $\mathbb{R}_1^3$ associated with quaternionic curve $\xi = \xi(s)$ in $\mathbb{R}_2^4$ is given by

$$\alpha(s) = \left(\sqrt{2}\sinh(s), s, -\sqrt{2}\cosh(s)\right)$$

where s is the arc-length parameter of α and its curvature functions are as follows

$$k = \sqrt{2}, \quad r = -1.$$

Similarly, the spatial quaternionic curve β in $\mathbb{R}_1^3$ associated with quaternionic curve $\phi = \phi(s)$ in $\mathbb{R}_2^4$ is given by

$$\beta(s) = (c, -s + c, c)$$

which c is a real number. Now, if we calculate the quaternionic inner product $h(\boldsymbol{t}, \boldsymbol{t}^*)$, then we obtain that

$$h(\boldsymbol{t}, \boldsymbol{t}^*) = -1 \neq 0$$

where $\boldsymbol{t}$ and $\boldsymbol{t}^*$ are the tangent vector fields of the semi-real spatial quaternionic curves α and β, respectively.

So, we can easily see that the spatial quaternionic curves (β, α) aren't spatial quaternionic involute-evolute curves.
The figures of the some projections of this quaternionic curve $\xi = \xi(s)$, the quaternionic involute curves of ξ and their associated spatial quaternionic curves are as follows;

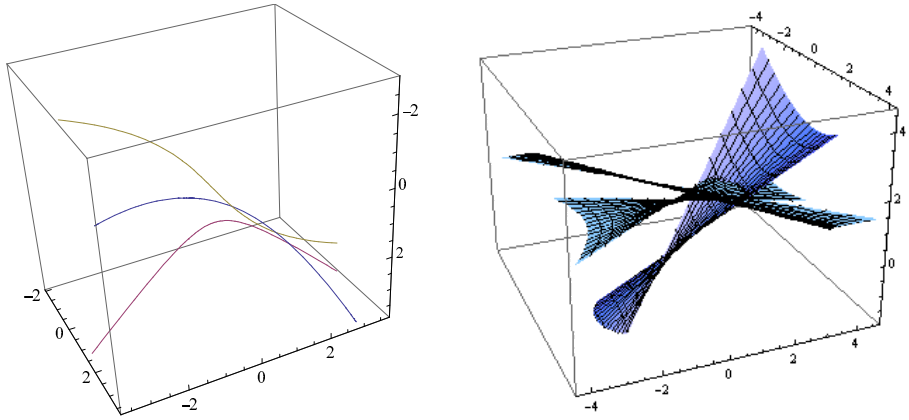

Figure 4.1. Some projections of the quaternionic curve $\xi = \xi(s)$ (on the left) and the quaternionic involutes of ξ (on the right).

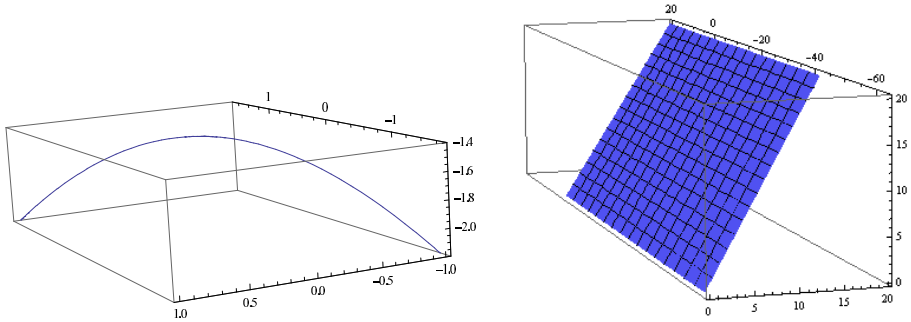

Figure 4.2. The spatial quaternionic curve associated with the quaternionic curve $\xi = \xi(s)$ (on the left) and the spatial quaternionic curves associated with the quaternionic involutes of ξ (on the right).

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