

Some Results for the Jacobi-Dunkl Transform in the Space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$

S. El Ouadih¹ and R. Daher²

¹ Department of Mathematics, Faculty of Sciences Ain Chock,
University Hassan II, Casablanca, Morocco

² Department of Mathematics, Faculty of Sciences Ain Chock,
University Hassan II, Casablanca, Morocco

ABSTRACT. In this paper, using a generalized Jacobi-Dunkl translation operator, we obtain a generalization of Titchmarsh's theorem for the Dunkl transform for functions satisfying the (ϕ, p) -Lipschitz Jacobi-Dunkl condition in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$, $\alpha \geq \beta \geq \frac{-1}{2}$, $\alpha \neq \frac{-1}{2}$.

Keywords: Jacobi-Dunkl operator, Jacobi-Dunkl transform, generalized Jacobi-Dunkl translation.

2000 Mathematics subject classification: 42B37.

1. INTRODUCTION AND PRELIMINARIES

Titchmarsh's ([8], Theorem 85) characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy-Lipschitz Condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have:

Theorem 1.1. [8] *Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalent:*

(a) $\|f(x+h) - f(x)\| = O(h^\alpha)$, as $h \rightarrow 0$,

¹ Corresponding author: salahwadih@gmail.com

Received: 28 September 2015

Revised: 9 July 2016

Accepted: 9 July 2016

(b) $\int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda = O(r^{-2\alpha}), \quad \text{as } r \rightarrow \infty,$
 where \widehat{f} stands for the Fourier transform of f .

A similar result of Theorem 1.1 has been established for the Jacobi transform in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx)$ (see [11]). In this paper, we prove a generalization of Theorem 1.1 for the Jacobi-Dunkl transform for functions satisfying the (ϕ, p) -Lipschitz Jacobi-Dunkl condition in the space $L^p(\mathbb{R}, A_{\alpha,\beta}(x)dx), 1 < p \leq 2$. For this purpose, we use the generalized Jacobi-Dunkl translation operator.

In this section, we recapitulate from ([1]-[6]) some results related to the harmonic analysis associated with Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$. The Jacobi-Dunkl function with parameters $(\alpha, \beta), \alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$, is defined by the formula:

$$\forall x \in \mathbb{R}, \psi_{\lambda}^{\alpha,\beta}(x) = \begin{cases} \varphi_{\mu}^{\alpha,\beta}(x) - \frac{i}{\lambda} \frac{d}{dx} \varphi_{\mu}^{\alpha,\beta}(x), & \text{if } \lambda \in \mathbb{C} \setminus \{0\}, \\ 1, & \text{if } \lambda = 0, \end{cases}$$

with $\lambda^2 = \mu^2 + \rho^2, \rho = \alpha + \beta + 1$ and $\varphi_{\mu}^{\alpha,\beta}$ is the Jacobi function given by:

$$\varphi_{\mu}^{\alpha,\beta}(x) = F\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}, \alpha + 1, -(\sinh(x))^2\right),$$

F is the Gauss hypergeometric function (see [1],[7]).

$\psi_{\lambda}^{\alpha,\beta}$ is the unique C^{∞} -solution on \mathbb{R} of the differential-difference equation

$$\begin{cases} \Lambda_{\alpha,\beta} \mathcal{U} = i\lambda \mathcal{U}, & \lambda \in \mathbb{C}, \\ \mathcal{U}(0) = 1, \end{cases}$$

(see [5]), where $\Lambda_{\alpha,\beta}$ is the Jacobi-Dunkl operator given by:

$$\Lambda_{\alpha,\beta} \mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + \frac{A'_{\alpha,\beta}(x)}{A_{\alpha,\beta}(x)} \times \left(\frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}\right),$$

with

$$A_{\alpha,\beta}(x) = 2^{\rho} (\sinh |x|)^{2\alpha+1} (\cosh |x|)^{2\beta+1},$$

i.e.,

$$\Lambda_{\alpha,\beta} \mathcal{U}(x) = \frac{d\mathcal{U}(x)}{dx} + [(2\alpha + 1) \coth x + (2\beta + 1) \tanh x] \times \frac{\mathcal{U}(x) - \mathcal{U}(-x)}{2}.$$

Using the relation

$$\frac{d}{dx} \varphi_{\mu}^{\alpha,\beta}(x) = -\frac{\mu^2 + \rho^2}{4(\alpha + 1)} \sinh(2x) \varphi_{\mu}^{\alpha+1,\beta+1}(x),$$

the function $\psi_\lambda^{\alpha,\beta}$ can be written in the form above (see [2])

$$\psi_\lambda^{\alpha,\beta}(x) = \varphi_\mu^{\alpha,\beta}(x) + i \frac{\lambda}{4(\alpha+1)} \sinh(2x) \varphi_\mu^{\alpha+1,\beta+1}(x), \quad x \in \mathbb{R},$$

where $\lambda^2 = \mu^2 + \rho^2$, $\rho = \alpha + \beta + 1$.

Denote $L_{\alpha,\beta}^p(\mathbb{R}) = L_{\alpha,\beta}^p(\mathbb{R}, A_{\alpha,\beta}(x)dx)$, $1 < p \leq 2$ the space of measurable functions f on \mathbb{R} such that

$$\|f\|_{p,\alpha,\beta} = \left(\int_{\mathbb{R}} |f(x)|^p A_{\alpha,\beta}(x) dx \right)^{1/p} < +\infty.$$

Using the eigenfunctions $\psi_\lambda^{\alpha,\beta}$ of the operator $\Lambda_{\alpha,\beta}$ called the Jacobi-Dunkl kernels, we define the Jacobi-Dunkl transform by

$$\mathcal{F}_{\alpha,\beta} f(\lambda) = \int_{\mathbb{R}} f(x) \psi_\lambda^{\alpha,\beta}(x) A_{\alpha,\beta}(x) dx, \quad \lambda \in \mathbb{R},$$

and the inversion formula by

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{\alpha,\beta} f(\lambda) \psi_{-\lambda}^{\alpha,\beta}(t) d\sigma(\lambda),$$

where

$$d\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2 - \rho^2} |C_{\alpha,\beta}(\sqrt{\lambda^2 - \rho^2})|} \mathbb{I}_{\mathbb{R} \setminus]-\rho, \rho[}(\lambda) d\lambda.$$

Here,

$$C_{\alpha,\beta}(\mu) = \frac{2^{\rho-i\mu} \Gamma(\alpha+1) \Gamma(i\mu)}{\Gamma(\frac{1}{2}(\rho+i\mu)) \Gamma(\frac{1}{2}(\alpha-\beta+1+i\mu))}, \quad \mu \in \mathbb{C} \setminus (i\mathbb{N}),$$

and $\mathbb{I}_{\mathbb{R} \setminus]-\rho, \rho[}$ is the characteristic function of $\mathbb{R} \setminus]-\rho, \rho[$.

The Jacobi-Dunkl transform is a unitary isomorphism from $L_{\alpha,\beta}^2(\mathbb{R})$ onto $L^2(\mathbb{R}, d\sigma(\lambda))$, i.e.,

$$\|f\|_{2,\alpha,\beta} = \|\mathcal{F}_{\alpha,\beta}(f)\|_{L^2(\mathbb{R}, d\sigma(\lambda))}. \quad (1.1)$$

Plancherel's theorem (1.1) and the Marcinkiewics interpolation theorem (see [8]) we get for $f \in L_{\alpha,\beta}^p(\mathbb{R})$ with $1 < p \leq 2$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathcal{F}_{\alpha,\beta}(f)\|_{L^q(\mathbb{R}, d\sigma(\lambda))} \leq K \|f\|_{p,\alpha,\beta}, \quad (1.2)$$

where K is a positive constant (see [6]).

The operator of Jacobi-Dunkl translation is defined by :

$$T_x f(y) = \int_{\mathbb{R}} f(z) d\nu_{x,y}^{\alpha,\beta}(z), \quad \forall x, y \in \mathbb{R},$$

where $\nu_{x,y}^{\alpha,\beta}(z)$, $x, y \in \mathbb{R}$ are the signed measures given by

$$d\nu_{x,y}^{\alpha,\beta}(z) = \begin{cases} K_{\alpha,\beta}(x, y, z)A_{\alpha,\beta}(z)dz, & \text{if } x, y \in \mathbb{R}^*, \\ \delta_x, & \text{if } y = 0, \\ \delta_y, & \text{if } x = 0. \end{cases}$$

Here, δ_x is the Dirac measure at x . And,

$$K_{\alpha,\beta}(x, y, z) = M_{\alpha,\beta}(\sinh(|x|) \sinh(|y|) \sinh(|z|))^{-2\alpha} \mathbb{I}_{I_{x,y}} \times \int_0^\pi \rho_\theta(x, y, z) \\ \times (g_\theta(x, y, z))_+^{\alpha-\beta-1} \sin^{2\beta} \theta d\theta,$$

where

$$I_{x,y} = [-|x| - |y|, -||x| - |y||] \cup [||x| - |y||, |x| + |y|] \\ \rho_\theta(x, y, z) = 1 - \sigma_{x,y,z}^\theta + \sigma_{z,x,y}^\theta + \sigma_{z,y,x}^\theta, \\ \forall z \in \mathbb{R}, \theta \in [0, \pi], \sigma_{x,y,z}^\theta = \begin{cases} \frac{\cosh(x) + \cosh(y) - \cosh(z) \cos(\theta)}{\sinh(x) \sinh(y)}, & \text{if } xy \neq 0, \\ 0, & \text{if } xy = 0, \end{cases} \\ g_\theta(x, y, z) = 1 - \cosh^2(x) - \cosh^2(y) - \cosh^2(z) + 2 \cosh(x) \cosh(y) \cosh(z) \cos \theta, \\ t_+ = \begin{cases} t, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

and,

$$M_{\alpha,\beta} = \begin{cases} \frac{2^{-2\rho} \Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha-\beta) \Gamma(\beta+\frac{1}{2})}, & \text{if } \alpha > \beta, \\ 0, & \text{if } \alpha = \beta. \end{cases}$$

In [2], we have

$$\mathcal{F}_{\alpha,\beta}(T_h f)(\lambda) = \psi_\lambda^{\alpha,\beta}(h) \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad (1.3)$$

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta} f)(\lambda) = i\lambda \mathcal{F}_{\alpha,\beta}(f)(\lambda). \quad (1.4)$$

For $\alpha \geq \frac{-1}{2}$, we introduce the normalized Bessel function of first kind and order α [10] defined by:

$$j_\alpha(x) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n+\alpha+1)}, \quad x \in \mathbb{R}.$$

Moreover, we see that

$$\lim_{x \rightarrow 0} \frac{j_\alpha(x) - 1}{x^2} \neq 0,$$

by consequence, there exists $C_1 > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Rightarrow |j_\alpha(x) - 1| \geq C_1 |x|^2. \quad (1.5)$$

Lemma 1.2. *Let $\alpha \geq \beta \geq \frac{-1}{2}, \alpha \neq \frac{-1}{2}$. Then for $|\nu| \leq \rho$, there exists a positive constant C_2 such that*

$$|1 - \varphi_{\mu+i\nu}^{\alpha,\beta}(x)| \geq C_2|1 - j_\alpha(\mu x)|.$$

Proof. (See [4], Lemma 9). □

Denote by $L_p^m(\Lambda_{\alpha,\beta}), 1 < p \leq 2, m = 0, 1, 2, \dots$, the class of functions $f \in L_{\alpha,\beta}^p(\mathbb{R})$ that have on \mathbb{R} generalized derivatives $f'(x), f''(x), \dots, f^{(2m)}(x)$ in the sense of Levi (see [9]) and belong to $L_{\alpha,\beta}^p(\mathbb{R})$ with $\Lambda_{\alpha,\beta}^m f \in L_{\alpha,\beta}^p(\mathbb{R})$. i.e.,

$$L_p^m(\Lambda_{\alpha,\beta}) = \left\{ f \in L_{\alpha,\beta}^p(\mathbb{R}) / \Lambda_{\alpha,\beta}^m f \in L_{\alpha,\beta}^p(\mathbb{R}) \right\},$$

where $\Lambda_{\alpha,\beta}^0 f = f, \Lambda_{\alpha,\beta}^m f = \Lambda_{\alpha,\beta}(\Lambda_{\alpha,\beta}^{m-1} f), m = 0, 1, 2, \dots$

2. MAIN RESULT

In this section we give the main result of this paper. We need first to define (ϕ, p) -Lipschitz Jacobi-Dunkl class.

Denote N_h by

$$N_h = T_h + T_{-h} - 2I,$$

where I is the unit operator in the space $L_{\alpha,\beta}^p(\mathbb{R})$.

Definition 2.1. A function $f \in L_p^m(\Lambda_{\alpha,\beta})$ is said to be in (ϕ, p) -Lipschitz Jacobi-Dunkl class, denoted by $Lip(\phi, p, \alpha, \beta)$, if

$$\|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)), \quad \text{as } h \rightarrow 0,$$

where $m = 0, 1, 2, \dots$ and ϕ is a continuous increasing function on $[0, \infty)$, satisfying $\phi(0) = 0$ and $\phi(ts) = \phi(t)\phi(s)$ for all $t, s \in [0, \infty)$.

Lemma 2.2. *For $f \in L_p^m(\Lambda_{\alpha,\beta})$, then*

$$\left(\int_{\mathbb{R}} 2^q \lambda^{qm} |\varphi_{\mu}^{\alpha,\beta}(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \right)^{\frac{1}{q}} \leq K \|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta},$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, \dots$

Proof. From (1.4), we have

$$\mathcal{F}_{\alpha,\beta}(\Lambda_{\alpha,\beta}^m f)(\lambda) = i^m \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda), \quad m = 0, 1, 2, \dots \quad (2.1)$$

We use formulas (1.3) and (2.1), we conclude that

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = i^m (\psi_{\lambda}^{(\alpha,\beta)}(h) + \psi_{\lambda}^{(\alpha,\beta)}(-h) - 2) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda),$$

Since

$$\psi_{\lambda}^{(\alpha,\beta)}(h) = \varphi_{\mu}^{\alpha,\beta}(h) + i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi_{\mu}^{\alpha+1,\beta+1}(h),$$

$$\psi_{\lambda}^{(\alpha,\beta)}(-h) = \varphi_{\mu}^{\alpha,\beta}(-h) - i \frac{\lambda}{4(\alpha+1)} \sinh(2h) \varphi_{\mu}^{\alpha+1,\beta+1}(-h),$$

and $\varphi_{\mu}^{\alpha,\beta}$ is even, then

$$\mathcal{F}_{\alpha,\beta}(N_h \Lambda_{\alpha,\beta}^m f)(\lambda) = 2i^m (\varphi_{\mu}^{\alpha,\beta}(h) - 1) \lambda^m \mathcal{F}_{\alpha,\beta}(f)(\lambda).$$

By formula 1.2, we have the result. \square

Theorem 2.3. *Let f belong to $Lip(\phi, p, \alpha, \beta)$. Then*

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})), \quad \text{as } r \rightarrow \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, \dots$

Proof. Assume that $f \in Lip(\phi, p, \alpha, \beta)$, then we have

$$\|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)), \quad \text{as } h \rightarrow 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}} \lambda^{qm} |\varphi_{\mu}^{\alpha,\beta}(h) - 1|^q |\mathcal{F}_{\alpha,\beta} f(\lambda)|^q d\sigma(\lambda) \leq \frac{K^q}{2^q} \|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta}^q.$$

By (1.5) and Lemma 1.2, we get:

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |1 - \varphi_{\mu}^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \geq C_1^q C_2^q \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} |\mu h|^{2q} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

From $\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}$ we have

$$\begin{aligned} \left(\frac{\eta}{2h}\right)^2 - \rho^2 &\leq \mu^2 \leq \left(\frac{\eta}{h}\right)^2 - \rho^2 \\ \Rightarrow \mu^2 h^2 &\geq \frac{\eta^2}{4} - \rho^2 h^2. \end{aligned}$$

Take $h \leq \frac{\eta}{3\rho}$, then we have $\mu^2 h^2 \geq C_3 = C_3(\eta)$.

So,

$$\int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |1 - \varphi_{\mu}^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \geq C_1^q C_2^q C_3^q \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda).$$

There exists then a positives constants C and K_1 such that

$$\begin{aligned} \int_{\frac{\eta}{2h} \leq |\lambda| \leq \frac{\eta}{h}} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) &\leq C \int_{\mathbb{R}} \lambda^{qm} |1 - \varphi_{\mu}^{\alpha,\beta}(h)|^q |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &\leq K_1 \phi^q(h) = K_1 \phi(h^q). \end{aligned}$$

For all $0 < h < \frac{\eta}{3\rho}$. Then we have,

$$\int_{r \leq |\lambda| \leq 2r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \leq K_2 \phi(r^{-q}), \quad r \rightarrow \infty.$$

where $K_2 = K_1\phi(\eta^q 2^{-q})$.

Furthermore, we obtain

$$\begin{aligned} \int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) &= \left(\int_{r \leq |\lambda| \leq 2r} + \int_{2r \leq |\lambda| \leq 4r} + \int_{4r \leq |\lambda| \leq 8r} + \dots \right) \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \\ &\leq K_2\phi(r^{-q}) + K_2\phi((2r)^{-q}) + K_2\phi((4r)^{-q}) + \dots \\ &\leq K_2\phi(r^{-q}) + K_2\phi(2^{-q})\phi(r^{-q}) + K_2\phi((2^{-q})^2)\phi(r^{-q}) + \dots \\ &\leq K_2\phi(r^{-q})(1 + \phi(2^{-q}) + \phi((2^{-q})^2) + \dots). \end{aligned}$$

We have $\phi(2^{-q}) < 1$, then

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) \leq K_3\phi(r^{-q}),$$

where $K_3 = K_2(1 - \phi(2^{-q}))^{-1}$.

Finally, we get

$$\int_{|\lambda| \geq r} \lambda^{qm} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(\phi(r^{-q})), \quad \text{as } r \rightarrow \infty.$$

Thus, the proof is finished. \square

Corollary 2.4. *Let $f \in L_p^m(\Lambda_{\alpha,\beta})$, and let*

$$\|N_h \Lambda_{\alpha,\beta}^m f\|_{p,\alpha,\beta} = O(\phi(h)), \quad \text{as } h \rightarrow 0.$$

Then

$$\int_{|\lambda| \geq r} |\mathcal{F}_{\alpha,\beta}(f)(\lambda)|^q d\sigma(\lambda) = O(r^{-qm}\phi(r^{-q})), \quad \text{as } r \rightarrow \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m = 0, 1, 2, \dots$

Acknowledgements

The authors would like to thank the referee for his valuable comments and suggestions.

REFERENCES

- [1] H. Ben Mohamed and H. Mejjaoli, Distributional Jacobi-Dunkl transform and applications, *Afr.Diaspora J.Math.* **1**(2004), 24-46.
- [2] H. Ben Mohamed, The Jacobi-Dunkl transform on \mathbb{R} and the convolution product on new space of distributions, *Ramanujan J.* **21**(2010), 145-175.
- [3] N. Ben Salem and A. Ahmed Salem, Convolution structure associated with the Jacobi-Dunkl operator on \mathbb{R} , *Ramanujan J.* **12**(3) (2006), 359-378.
- [4] O. W. Bray and M. A.Pinsky, Growth properties of Fourier transforms via module of continuity, *Journal of Functional Analysis.* **255**(2008), 2256-2285.
- [5] F. Chouchane, M. Mili and K. Trimche, Positivity of the intertwining operator and harmonic analysis associated with the Jacobi-Dunkl operator on \mathbb{R} , *J.Anal. Appl.* **1**(4)(2003), 387-412.

- [6] T. R. Johansen, Remarks on the inverse cherednik-opdam transform on the real line, arxiv: 1502.01293v1 (2015).
- [7] T. H. Koornwinder, A new proof of a Paley-Wiener type theorems for the Jacobi transform , Ark.Math. **13**(1975),145-159.
- [8] E. C. Titchmarsh, Introduction to the theory of Fourier Integrals , Clarendon, Oxford. (1948), Komkniga, Moscow, (2005).
- [9] S. M. Nikol'skii, Aproximation of functions of several variables and embedding theorems, (Nauka,Moscow.1977)[in Russian].
- [10] B. M. Levitan, Expansion in Fourier series and integrals over Bessel functions, Uspekhi Math.Nauk. **6**(2) (1951), 102-143.
- [11] A. Abouelaz, A. Belkhadir And R. Daher, An Analog of Titchmarsh's theorem for the Jacobi-Dunkl transform in the space $L^2(\mathbb{R}, A_{\alpha,\beta}(x)dx)$, Int. J. of Anal. Appl., V. **8**(1) (2015), 15-21.