

## Growth Properties of the Cherednik-Opdam Transform in the Space $L^p_{\alpha,\beta}(\mathbb{R})$

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ABSTRACT. In this paper, using a generalized translation operator, we obtain a generalization of Younis Theorem 5.2 in [3] for the Cherednik-Opdam transform for functions satisfying the  $(\delta, \gamma, p)$ -Cherednik-Opdam Lipschitz condition in the space  $L^p_{\alpha,\beta}(\mathbb{R})$ .

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### 1. INTRODUCTION AND PRELIMINARIES

Various investigators such as Mittal and Mishra [6], Mishra et al. [7]-[11] and Mishra and Mishra [12] have determined the degree of approximation of  $2\pi$ -periodic signals (functions) belonging to various classes  $Lip\alpha$ ,  $Lip(\alpha, r)$ ,  $Lip(\xi(t), r)$  and  $W(L_r, \xi(t))$ , ( $r \geq 1$ ), of functions through trigonometric Fourier approximation using different summability matrices with monotone rows. In this direction, Theorem 5.2 of Younis [3] characterized the set of functions in  $L^2(\mathbb{R})$  satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

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**Theorem 1.1.** [3] *Let  $f \in L^2(\mathbb{R})$ . Then the following are equivalents*

- (i)  $\|f(x+h) - f(x)\| = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right)$ , as  $h \rightarrow 0, 0 < \delta < 1, \gamma \geq 0$ ,
- (ii)  $\int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right)$ , as  $r \rightarrow \infty$ ,

where  $\widehat{f}$  stands for the Fourier transform of  $f$ .

In this paper, we prove the generalization of Theorem 1.1 for the Cherednik-Opdam transform for functions satisfying the Cherednik-Opdam Lipschitz condition in the space  $L^p_{\alpha,\beta}(\mathbb{R})$ . For this purpose, we use the generalized translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator  $T^{(\alpha,\beta)}$ . Further details can be found in [1] and [2]. In the following we fix parameters  $\alpha, \beta$  subject to the constraints  $\alpha \geq \beta \geq -\frac{1}{2}$  and  $\alpha > \frac{-1}{2}$ .

Let  $\rho = \alpha + \beta + 1$  and  $\lambda \in \mathbb{C}$ . The Opdam hypergeometric functions  $G_\lambda^{(\alpha,\beta)}$  on  $\mathbb{R}$  are eigenfunctions  $T^{(\alpha,\beta)}G_\lambda^{(\alpha,\beta)}(x) = i\lambda G_\lambda^{(\alpha,\beta)}(x)$  of the differential-difference operator

$$T^{(\alpha,\beta)}f(x) = f'(x) + [(2\alpha+1) \coth x + (2\beta+1) \tanh x] \frac{f(x) - f(-x)}{2} - \rho f(-x),$$

that are normalized such that  $G_\lambda^{(\alpha,\beta)}(0) = 1$ . In the notation of Cherednik one would write  $T^{(\alpha,\beta)}$  as

$$T(k_1+k_2)f(x) = f'(x) + \left\{ \frac{2k_1}{1+e^{-2x}} + \frac{4k_2}{1-e^{-4x}} \right\} (f(x) - f(-x)) - (k_1+2k_2)f(x),$$

with  $\alpha = k_1 + k_2 - \frac{1}{2}$  and  $\beta = k_2 - \frac{1}{2}$ . Here  $k_1$  is the multiplicity of a simply positive root and  $k_2$  the (possibly vanishing) multiplicity of a multiple of this root. By [1] or [2], the eigenfunction  $G_\lambda^{(\alpha,\beta)}$  is given by

$$G_\lambda^{(\alpha,\beta)}(x) = \varphi_\lambda^{\alpha,\beta}(x) - \frac{1}{\rho - i\lambda} \frac{\partial}{\partial x} \varphi_\lambda^{\alpha,\beta}(x) = \varphi_\lambda^{\alpha,\beta}(x) + \frac{\rho}{4(\alpha+1)} \sinh(2x) \varphi_\lambda^{\alpha+1,\beta+1}(x),$$

where  $\varphi_\lambda^{\alpha,\beta}(x) = {}_2F_1\left(\frac{\rho+i\lambda}{2}; \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x\right)$  is the classical Jacobi function.

**Lemma 1.2.** *The following inequalities are valid for Jacobi functions*

- (i)  $|\varphi_\lambda^{\alpha,\beta}(x)| \leq 1$ .
- (ii)  $|1 - \varphi_\lambda^{\alpha,\beta}(x)| \leq x^2(\lambda^2 + \rho^2)$ .
- (iii) *there is a constant  $c > 0$  such that*

$$1 - \varphi_\lambda^{\alpha,\beta}(x) \geq c,$$

for  $|\lambda x| \geq 1$ .

*Proof.* (See [4], Lemma 3.1, Lemma 3.2).  $\square$

Denote  $L_{\alpha,\beta}^p(\mathbb{R})$ , the space of measurable functions  $f$  on  $\mathbb{R}$  such that

$$\|f\|_{p,\alpha,\beta} = \left( \int_{\mathbb{R}} |f(x)|^p A_{\alpha,\beta}(x) dx \right)^{1/p} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$

$$\|f\|_{\infty,\alpha,\beta} = \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)| < +\infty,$$

and  $L_{\sigma}^p(\mathbb{R})$ ,  $p \geq 1$ , the space of measurable functions  $f$  on  $\mathbb{R}$  such that

$$\|f\|_{p,\sigma} = \left( \int_{\mathbb{R}} |f(\lambda)|^p d\sigma(\lambda) \right)^{1/p} < +\infty,$$

where  $A_{\alpha,\beta}(x) = (\sinh |x|)^{2\alpha+1} (\cosh |x|)^{2\beta+1}$  and  $d\sigma$  is the measure given by

$$d\sigma(\lambda) = \left(1 - \frac{\rho}{i\lambda}\right) \frac{d\lambda}{8\pi |c_{\alpha,\beta}(\lambda)|^2}.$$

here

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda} \Gamma(\alpha+1) \Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda)) \Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}.$$

The Cherednik-Opdam transform of  $f \in C_c(\mathbb{R})$  is defined by

$$\mathcal{H}f(\lambda) = \int_{\mathbb{R}} f(x) G_{\lambda}^{(\alpha,\beta)}(-x) A_{\alpha,\beta}(x) dx \quad \text{for all } \lambda \in \mathbb{C}.$$

The inverse transform is given as

$$\mathcal{H}^{-1}g(x) = \int_{\mathbb{R}} g(\lambda) G_{\lambda}^{(\alpha,\beta)}(x) d\sigma(\lambda).$$

The corresponding Plancherel formula was established in [1], to the effect that

$$\begin{aligned} \int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x) dx &= \int_0^{+\infty} (|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2) \frac{d\lambda}{16\pi |c_{\alpha,\beta}(\lambda)|^2} \\ &= \int_{\mathbb{R}} \mathcal{H}f(\lambda) \overline{\mathcal{H}\check{f}(-\lambda)} d\sigma(\lambda), \end{aligned}$$

where  $\check{f}(x) := f(-x)$ .

**Lemma 1.3.** *Let  $\alpha \geq \beta \geq -\frac{1}{2}$  with  $\alpha \neq -\frac{1}{2}$  and let  $p \in [1, 2)$ ,  $q = \frac{p}{p-1}$ . There exists a constant  $c_p < \infty$  such that*

$$\|\mathcal{H}f\|_{q,\sigma} \leq c_p \|f\|_{p,\alpha,\beta},$$

for every  $f \in L_{\alpha,\beta}^p(\mathbb{R})$ .

*Proof.* (See [5], Lemma 3.1).  $\square$

According to [2] there exists a family of signed measures  $\mu_{x,y}^{(\alpha,\beta)}$  such that the product formula

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y) = \int_{\mathbb{R}} G_\lambda^{(\alpha,\beta)}(z)d\mu_{x,y}^{(\alpha,\beta)}(z),$$

holds for all  $x, y \in \mathbb{R}$  and  $\lambda \in \mathbb{C}$ , where

$$d\mu_{x,y}^{(\alpha,\beta)}(z) = \begin{cases} \mathcal{K}_{\alpha,\beta}(x, y, z)A_{\alpha,\beta}(z)dz & \text{if } xy \neq 0, \\ d\delta_x(z) & \text{if } y = 0, \\ d\delta_y(z) & \text{if } x = 0, \end{cases}$$

and

$$\begin{aligned} \mathcal{K}_{\alpha,\beta}(x, y, z) &= M_{\alpha,\beta} |\sinh x \cdot \sinh y \cdot \sinh z|^{-2\alpha} \int_0^\pi g(x, y, z, \chi)_+^{\alpha-\beta-1} \\ &\times [1 - \sigma_{x,y,z}^\chi + \sigma_{x,z,y}^\chi + \sigma_{z,y,x}^\chi + \frac{\rho}{\beta + \frac{1}{2}} \coth x \cdot \coth y \cdot \coth z (\sin \chi)^2] \times (\sin \chi)^{2\beta} d\chi \end{aligned}$$

if  $x, y, z \in \mathbb{R} \setminus \{0\}$  satisfy the triangular inequality  $\|x - y\| < |z| < |x| + |y|$ , and  $\mathcal{K}_{\alpha,\beta}(x, y, z) = 0$  otherwise. Here

$$\forall x, y, z \in \mathbb{R}, \chi \in [0, 1], \sigma_{x,y,z}^\chi = \begin{cases} \frac{\cosh x + \cosh y - \cosh z \cos \chi}{\sinh x \sinh y} & \text{if } xy \neq 0, \\ 0 & \text{if } xy = 0, \end{cases}$$

and  $g(x, y, z, \chi) = 1 - \cosh^2 x - \cosh^2 y \cdot \cosh^2 z + 2 \cosh x \cdot \cosh y \cdot \cosh z \cdot \cos \chi$ . The product formula is used to obtain explicit estimates for the generalized translation operators

$$\tau_x^{(\alpha,\beta)} f(y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^{(\alpha,\beta)}(z).$$

It is known from [2] that

$$\mathcal{H}\tau_x^{(\alpha,\beta)} f(\lambda) = G_\lambda^{(\alpha,\beta)}(x)\mathcal{H}f(\lambda), \quad (1.1)$$

for  $f \in C_c(\mathbb{R})$ .

## 2. MAIN RESULT

In this section we give the main result of this paper. We need first to define  $(\delta, \gamma, p)$ -Cherednik-Opdam Lipschitz class.

**Definition 2.1.** Let  $\delta, \gamma > 0$ . A function  $f \in L_{\alpha,\beta}^p(\mathbb{R})$  is said to be in the  $(\delta, \gamma, p)$ -Cherednik-Opdam Lipschitz class, denoted by  $Lip(\delta, \gamma, p)$ , if

$$\|\tau_h^{(\alpha,\beta)} f(x) + \tau_{-h}^{(\alpha,\beta)} f(x) - 2f(x)\|_{p,\alpha,\beta} = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right) \quad \text{as } h \rightarrow 0.$$

**Lemma 2.2.** For  $f \in L^p_{\alpha,\beta}(\mathbb{R})$ , then

$$\int_{\mathbb{R}} |\varphi_{\lambda}^{\alpha,\beta}(h) - 1|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \leq \left(\frac{c_p}{2}\right)^q \|\tau_h^{(\alpha,\beta)} f(x) + \tau_{-h}^{(\alpha,\beta)} f(x) - 2f(x)\|_{p,\alpha,\beta}^q,$$

where  $p \in [1, 2)$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* From formula 1.1, we have

$$\mathcal{H}(\tau_h^{(\alpha,\beta)} f + \tau_{-h}^{(\alpha,\beta)} f - 2f)(\lambda) = (G_{\lambda}^{(\alpha,\beta)}(h) + G_{\lambda}^{(\alpha,\beta)}(-h) - 2)\mathcal{H}(f)(\lambda),$$

Since

$$G_{\lambda}^{(\alpha,\beta)}(h) = \varphi_{\lambda}^{\alpha,\beta}(h) + \frac{\rho}{4(\alpha+1)} \sinh(2h) \varphi_{\lambda}^{\alpha+1,\beta+1}(h),$$

and  $\varphi_{\lambda}^{\alpha,\beta}$  is even, then

$$\mathcal{H}(\tau_h^{(\alpha,\beta)} f + \tau_{-h}^{(\alpha,\beta)} f - 2f)(\lambda) = 2(\varphi_{\lambda}^{\alpha,\beta}(h) - 1)\mathcal{H}(f)(\lambda).$$

By Lemma 1.3, we have the result.  $\square$

**Theorem 2.3.** Let  $f(x)$  belong to  $Lip(\delta, \gamma, p)$ . Then

$$\int_{|\lambda| \geq r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{r^{-q\delta}}{(\log r)^{q\gamma}}\right), \quad \text{as } r \rightarrow \infty,$$

where  $p \in [1, 2)$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Let  $f \in Lip(\delta, \gamma, p)$ . Then we have

$$\|\tau_h^{(\alpha,\beta)} f(x) + \tau_{-h}^{(\alpha,\beta)} f(x) - 2f(x)\|_{p,\alpha,\beta} = O\left(\frac{h^{\delta}}{(\log \frac{1}{h})^{\gamma}}\right) \quad \text{as } h \rightarrow 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}} |\varphi_{\lambda}^{\alpha,\beta}(h) - 1|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \leq \left(\frac{c_p}{2}\right)^q \|\tau_h^{(\alpha,\beta)} f(x) + \tau_{-h}^{(\alpha,\beta)} f(x) - 2f(x)\|_{p,\alpha,\beta}^q.$$

If  $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$ , then  $|\lambda h| \geq 1$  and (iii) of Lemma 1.2 implies that

$$1 \leq \frac{1}{c^q} |1 - \varphi_{\lambda}^{\alpha,\beta}(h)|^q.$$

Then

$$\begin{aligned} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &\leq \frac{1}{c^q} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |1 - \varphi_{\lambda}^{\alpha,\beta}(h)|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\ &\leq \frac{1}{c^q} \int_{\mathbb{R}} |1 - \varphi_{\lambda}^{\alpha,\beta}(h)|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\ &\leq \left(\frac{c_p}{2c}\right)^q \|\tau_h^{(\alpha,\beta)} f(x) + \tau_{-h}^{(\alpha,\beta)} f(x) - 2f(x)\|_{p,\alpha,\beta}^q \\ &= O\left(\frac{h^{q\delta}}{(\log \frac{1}{h})^{q\gamma}}\right). \end{aligned}$$

We obtain

$$\int_{r \leq |\lambda| \leq 2r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \leq C \frac{r^{-q\delta}}{(\log r)^{q\gamma}}, \quad r \rightarrow \infty.$$

where  $C$  is a positive constant. Now,

$$\begin{aligned} \int_{|\lambda| \geq r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\ &\leq C \left( \frac{r^{-q\delta}}{(\log r)^{q\gamma}} + \frac{(2r)^{-q\delta}}{(\log 2r)^{q\gamma}} + \frac{(4r)^{-2\delta}}{(\log 4r)^{2\gamma}} + \dots \right) \\ &\leq C \frac{r^{-q\delta}}{(\log r)^{q\gamma}} \left( 1 + 2^{-q\delta} + (2^{-q\delta})^2 + (2^{-q\delta})^3 + \dots \right) \\ &\leq K_\delta \frac{r^{-q\delta}}{(\log r)^{q\gamma}}, \end{aligned}$$

where  $K_\delta = C(1 - 2^{-q\delta})^{-1}$  since  $2^{-q\delta} < 1$ .

Consequently

$$\int_{|\lambda| \geq r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) = O \left( \frac{r^{-q\delta}}{(\log r)^{q\gamma}} \right), \quad \text{as } r \rightarrow \infty.$$

□

**Definition 2.4.** Let  $\gamma > 0$ . A function  $f \in L_{\alpha, \beta}^p(\mathbb{R})$  is said to be in the  $(\psi, \gamma, p)$ -Cherednik-Opdam Lipschitz class, denoted by  $Lip(\psi, \gamma, p)$ , if

$$\|\tau_h^{(\alpha, \beta)} f(x) + \tau_{-h}^{(\alpha, \beta)} f(x) - 2f(x)\|_{p, \alpha, \beta} = O \left( \frac{\psi(h)}{(\log \frac{1}{h})^\gamma} \right) \quad \text{as } h \rightarrow 0,$$

where  $\psi$  is a continuous increasing function on  $[0, \infty)$ ,  $\psi(0) = 0$  and  $\psi(ts) = \psi(t)\psi(s)$  for all  $t, s \in [0, \infty)$ .

**Theorem 2.5.** Let  $f(x)$  belong to  $Lip(\psi, \gamma, p)$ . Then

$$\int_{|\lambda| \geq r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) = O \left( \frac{\psi(r^{-q})}{(\log r)^{q\gamma}} \right), \quad \text{as } r \rightarrow \infty,$$

where  $p \in [1, 2)$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

*Proof.* Let  $f \in Lip(\psi, p)$ . Then we have

$$\|\tau_h^{(\alpha, \beta)} f(x) + \tau_{-h}^{(\alpha, \beta)} f(x) - 2f(x)\|_{p, \alpha, \beta} = O \left( \frac{\psi(h)}{(\log \frac{1}{h})^\gamma} \right) \quad \text{as } h \rightarrow 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}} |\varphi_\lambda^{\alpha, \beta}(h) - 1|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \leq \left( \frac{c_p}{2} \right)^q \|\tau_h^{(\alpha, \beta)} f(x) + \tau_{-h}^{(\alpha, \beta)} f(x) - 2f(x)\|_{p, \alpha, \beta}^q.$$

If  $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$ , then  $|\lambda h| \geq 1$  and (iii) of Lemma 1.2 implies that

$$1 \leq \frac{1}{c^q} |1 - \varphi_\lambda^{\alpha, \beta}(h)|^q.$$

Then

$$\begin{aligned} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &\leq \frac{1}{c^q} \int_{\frac{1}{h} \leq |\lambda| \leq \frac{2}{h}} |1 - \varphi_\lambda^{\alpha, \beta}(h)|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\ &\leq \frac{1}{c^q} \int_{\mathbb{R}} |1 - \varphi_\lambda^{\alpha, \beta}(h)|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\ &\leq \left(\frac{c_p}{2c}\right)^q \|\tau_h^{(\alpha, \beta)} f(x) + \tau_{-h}^{(\alpha, \beta)} f(x) - 2f(x)\|_{p, \alpha, \beta}^q \\ &= O\left(\frac{\psi(h)^q}{(\log \frac{1}{h})^{q\gamma}}\right) = O\left(\frac{\psi(h^q)}{(\log \frac{1}{h})^{q\gamma}}\right). \end{aligned}$$

We obtain

$$\int_{r \leq |\lambda| \leq 2r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \leq C \frac{\psi(r^{-q})}{(\log r)^{q\gamma}}, \quad r \rightarrow \infty.$$

where  $C$  is a positive constant. Now,

$$\begin{aligned} \int_{|\lambda| \geq r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &= \sum_{i=0}^{\infty} \int_{2^i r \leq |\lambda| \leq 2^{i+1} r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\ &\leq C \left( \frac{\psi(r^{-q})}{(\log r)^{q\gamma}} + \frac{\psi((2r)^{-q})}{(\log 2r)^{q\gamma}} + \frac{\psi((4r)^{-q})}{(\log 4r)^{q\gamma}} + \dots \right) \\ &\leq C \frac{\psi(r^{-q})}{(\log r)^{q\gamma}} (1 + \psi(2^{-q}) + (\psi(2^{-q}))^2 + (\psi(2^{-q}))^3 + \dots) \\ &\leq K_\delta \frac{\psi(r^{-q})}{(\log r)^{q\gamma}}, \end{aligned}$$

where  $K_\delta = C(1 - \psi(2^{-q}))^{-1}$  since  $\psi(2^{-q}) < 1$ .

Consequently

$$\int_{|\lambda| \geq r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right), \quad \text{as } r \rightarrow \infty.$$

□

### 3. Conclusions

In this work we have succeeded to generalise the theorem in [3] for the Cherednik-Opdam transform in the space  $L_{\alpha, \beta}^p(\mathbb{R})$ . We proved that  $f(x)$  belong to  $Lip(\psi, \gamma, p)$ . Then

$$\int_{|\lambda| \geq r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right), \quad \text{as } r \rightarrow \infty,$$

where  $p \in [1, 2)$  and  $q$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

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