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Growth Properties of the Cherednik-Opdam Transform in the Space $L^p_{\alpha,\beta}(\mathbb{R})$

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Abstract. In this paper, using a generalized translation operator, we obtain a generalization of Younis Theorem 5.2 in [3] for the Cherednik-Opdam transform for functions satisfying the (δ, γ, p)-Cherednik-Opdam Lipschitz condition in the space $L^p_{\alpha,\beta}(\mathbb{R})$.

Keywords: Cherednik-Opdam operator, Cherednik-Opdam transform, generalized translation.

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1. Introduction and Preliminaries

Various investigators such as Mittal and Mishra [6], Mishra et al. [7]-[11] and Mishra and Mishra [12] have determined the degree of approximation of 2π-periodic signals (functions) belonging to various classes $Lip\alpha$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L_r, \xi(t))$, $(r \geq 1)$, of functions through trigonometric Fourier approximation using different summability matrices with monotone rows. In this direction, Theorem 5.2 of Younis [3] characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely we have

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Theorem 1.1. [3] *Let $f \in L^2(\mathbb{R})$. Then the following are equivalents*
(i) $\|f(x+h) - f(x)\| = O\left(\frac{h^\delta}{(\log \frac{1}{h})^\gamma}\right)$, *as* $h \to 0, 0 < \delta < 1, \gamma \geq 0$,
(ii) $\displaystyle\int_{|\lambda|\geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\delta}}{(\log r)^{2\gamma}}\right)$, *as* $r \to \infty$,
where $\widehat{f}$ *stands for the Fourier transform of* f.

In this paper, we prove the generalization of Theorem 1.1 for the Cherednik-Opdam transform for functions satisfying the Cherednik-Opdam Lipschitz condition in the space $L^p_{\alpha,\beta}(\mathbb{R})$. For this purpose, we use the generalized translation operator.

In this section, we develop some results from harmonic analysis related to the differential-difference operator $T^{(\alpha,\beta)}$. Further details can be found in [1] and [2]. In the following we fix parameters α, β subject to the constraints $\alpha \geq \beta \geq -\frac{1}{2}$ and $\alpha > \frac{-1}{2}$.
Let $\rho = \alpha + \beta + 1$ and $\lambda \in \mathbb{C}$. The Opdam hypergeometric functions $G^{(\alpha,\beta)}_\lambda$ on $\mathbb{R}$ are eigenfunctions $T^{(\alpha,\beta)}G^{(\alpha,\beta)}_\lambda(x) = i\lambda G^{(\alpha,\beta)}_\lambda(x)$ of the differential-difference operator

$$T^{(\alpha,\beta)}f(x) = f'(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\frac{f(x) - f(-x)}{2} - \rho f(-x),$$

that are normalized such that $G^{(\alpha,\beta)}_\lambda(0) = 1$. In the notation of Cherednik one would write $T^{(\alpha,\beta)}$ as

$$T(k_1+k_2)f(x) = f'(x) + \left\{\frac{2k_1}{1+e^{-2x}} + \frac{4k_2}{1-e^{-4x}}\right\}(f(x) - f(-x)) - (k_1+2k_2)f(x),$$

with $\alpha = k_1 + k_2 - \frac{1}{2}$ and $\beta = k_2 - \frac{1}{2}$. Here k_1 is the multiplicity of a simply positive root and k_2 the (possibly vanishing) multiplicity of a multiple of this root. By [1] or [2], the eigenfunction $G^{(\alpha,\beta)}_\lambda$ is given by

$$G^{(\alpha,\beta)}_\lambda(x) = \varphi^{\alpha,\beta}_\lambda(x) - \frac{1}{\rho - i\lambda}\frac{\partial}{\partial x}\varphi^{\alpha,\beta}_\lambda(x) = \varphi^{\alpha,\beta}_\lambda(x) + \frac{\rho}{4(\alpha+1)}\sinh(2x)\varphi^{\alpha+1,\beta+1}_\lambda(x),$$

where $\varphi^{\alpha,\beta}_\lambda(x) = {}_2F_1(\frac{\rho+i\lambda}{2}; \frac{\rho-i\lambda}{2}; \alpha+1; -\sinh^2 x)$ is the classical Jacobi function.

Lemma 1.2. *The following inequalities are valids for Jacobi functions* $\varphi^{\alpha,\beta}_\lambda(x)$
(i) $|\varphi^{\alpha,\beta}_\lambda(x)| \leq 1$.
(ii) $|1 - \varphi^{\alpha,\beta}_\lambda(x)| \leq x^2(\lambda^2 + \rho^2)$.
(iii) there is a constant $c > 0$ *such that*

$$1 - \varphi^{\alpha,\beta}_\lambda(x) \geq c,$$

for $|\lambda x| \geq 1$.

Proof. (See [4], Lemma 3.1, Lemma 3.2). ☐

Denote $L^p_{\alpha,\beta}(\mathbb{R})$, the space of measurable functions f on $\mathbb{R}$ such that

$$\begin{aligned}\|f\|_{p,\alpha,\beta} &= \left(\int_{\mathbb{R}} |f(x)|^p A_{\alpha,\beta}(x)dx\right)^{1/p} < +\infty, \quad \text{if} \quad 1 \leq p < +\infty,\\ \|f\|_{\infty,\alpha,\beta} &= ess \sup_{x\in\mathbb{R}} |f(x)| < +\infty,\end{aligned}$$

and $L^p_\sigma(\mathbb{R}), p \geq 1$, the space of measurable functions f on $\mathbb{R}$ such that

$$\|f\|_{p,\sigma} = \left(\int_{\mathbb{R}} |f(\lambda)|^p d\sigma(\lambda)\right)^{1/p} < +\infty,$$

where $A_{\alpha,\beta}(x) = (\sinh|x|)^{2\alpha+1}(\cosh|x|)^{2\beta+1}$ and $d\sigma$ is the measure given by

$$d\sigma(\lambda) = \left(1 - \frac{\rho}{i\lambda}\right)\frac{d\lambda}{8\pi|c_{\alpha,\beta}(\lambda)|^2}.$$

here

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma(\frac{1}{2}(\rho+i\lambda))\Gamma(\frac{1}{2}(\alpha-\beta+1+i\lambda))}.$$

The Cherednik-Opdam transform of $f \in C_c(\mathbb{R})$ is defined by

$$\mathcal{H}f(\lambda) = \int_{\mathbb{R}} f(x)G^{(\alpha,\beta)}_\lambda(-x)A_{\alpha,\beta}(x)dx \quad \text{for all} \quad \lambda \in \mathbb{C}.$$

The inverse transform is given as

$$\mathcal{H}^{-1}g(x) = \int_{\mathbb{R}} g(\lambda)G^{(\alpha,\beta)}_\lambda(x)d\sigma(\lambda).$$

The corresponding Plancherel formula was established in [1], to the effect that

$$\begin{aligned}\int_{\mathbb{R}} |f(x)|^2 A_{\alpha,\beta}(x)dx &= \int_0^{+\infty} (|\mathcal{H}f(\lambda)|^2 + |\mathcal{H}\check{f}(\lambda)|^2)\frac{d\lambda}{16\pi|c_{\alpha,\beta}(\lambda)|^2}\\ &= \int_{\mathbb{R}} \mathcal{H}f(\lambda)\overline{\mathcal{H}\check{f}(-\lambda)}d\sigma(\lambda),\end{aligned}$$

where $\check{f}(x) := f(-x)$.

Lemma 1.3. *Let $\alpha \geq \beta \geq -\frac{1}{2}$ with $\alpha \neq -\frac{1}{2}$ and let $p \in [1, 2)$, $q = \frac{p}{p-1}$. There exists a constant $c_p < \infty$ such that*

$$\|\mathcal{H}f\|_{q,\sigma} \leq c_p\|f\|_{p,\alpha,\beta},$$

for every $f \in L^p_{\alpha,\beta}(\mathbb{R})$.

Proof. (See [5], Lemma 3.1). ☐

According to [2] there exists a family of signed measures $\mu_{x,y}^{(\alpha,\beta)}$ such that the product formula

$$G_\lambda^{(\alpha,\beta)}(x)G_\lambda^{(\alpha,\beta)}(y)=\int_{\mathbb{R}}G_\lambda^{(\alpha,\beta)}(z)d\mu_{x,y}^{(\alpha,\beta)}(z),$$

holds for all $x,y\in\mathbb{R}$ and $\lambda\in\mathbb{C}$, where

$$d\mu_{x,y}^{(\alpha,\beta)}(z)=\begin{cases}\mathcal{K}_{\alpha,\beta}(x,y,z)A_{\alpha,\beta}(z)dz & \text{if } xy\neq 0,\\ \\ d\delta_x(z) & \text{if } y=0,\\ d\delta_y(z) & \text{if } x=0,\end{cases}$$

and

$$\begin{aligned}\mathcal{K}_{\alpha,\beta}(x,y,z) &= M_{\alpha,\beta}|\sinh x.\sinh y.\sinh z|^{-2\alpha}\int_0^\pi g(x,y,z,\chi)_+^{\alpha-\beta-1}\\ &\times\quad [1-\sigma_{x,y,z}^\chi+\sigma_{x,z,y}^\chi+\sigma_{z,y,x}^\chi+\frac{\rho}{\beta+\frac{1}{2}}\coth x.\coth y.\coth z(\sin\chi)^2]\times(\sin\chi)^{2\beta}d\chi\end{aligned}$$

if $x,y,z\in\mathbb{R}\backslash\{0\}$ satisfy the triangular inequality $||x|-y||<|z|<|x|+|y|$, and $\mathcal{K}_{\alpha,\beta}(x,y,z)=0$ otherwise. Here

$$\forall x,y,z\in\mathbb{R},\chi\in[0,1],\sigma_{x,y,z}^\chi=\begin{cases}\frac{\cosh x+\cosh y-\cosh z\cos\chi}{\sinh x\sinh y} & \text{if } xy\neq 0,\\ \\ 0 & \text{if } xy=0,\end{cases}$$

and $g(x,y,z,\chi)=1-\cosh^2 x-\cosh^2 y.\cosh^2 z+2\cosh x.\cosh y.\cosh z.\cos\chi$. The product formula is used to obtain explicit estimates for the generalized translation operators

$$\tau_x^{(\alpha,\beta)}f(y)=\int_{\mathbb{R}}f(z)d\mu_{x,y}^{(\alpha,\beta)}(z).$$

It is known from [2] that

$$\mathcal{H}\tau_x^{(\alpha,\beta)}f(\lambda)\quad=\quad G_\lambda^{(\alpha,\beta)}(x)\mathcal{H}f(\lambda),\tag{1.1}$$

for $f\in C_c(\mathbb{R})$.

2. Main result

In this section we give the main result of this paper. We need first to define (δ,γ,p)-Cherednik-Opdam Lipschitz class.

Definition 2.1. Let $\delta,\gamma>0$. A function $f\in L_{\alpha,\beta}^p(\mathbb{R})$ is said to be in the (δ,γ,p)-Cherednik-Opdam Lipschitz class, denoted by $Lip(\delta,\gamma,p)$, if

$$\|\tau_h^{(\alpha,\beta)}f(x)+\tau_{-h}^{(\alpha,\beta)}f(x)-2f(x)\|_{p,\alpha,\beta}=O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right)\quad\text{as}\quad h\to 0.$$

Lemma 2.2. *For* $f \in L^p_{\alpha,\beta}(\mathbb{R})$*, then*

$$\int_{\mathbb{R}} |\varphi_\lambda^{\alpha,\beta}(h)-1|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \le \left(\frac{c_p}{2}\right)^q \|\tau_h^{(\alpha,\beta)} f(x)+\tau_{-h}^{(\alpha,\beta)} f(x)-2f(x)\|^q_{p,\alpha,\beta},$$

where $p \in [1,2)$ *and* q *such that* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From formula 1.1, we have

$$\mathcal{H}(\tau_h^{(\alpha,\beta)} f + \tau_{-h}^{(\alpha,\beta)} f - 2f)(\lambda) = (G_\lambda^{(\alpha,\beta)}(h) + G_\lambda^{(\alpha,\beta)}(-h) - 2)\mathcal{H}(f)(\lambda),$$

Since

$$G_\lambda^{(\alpha,\beta)}(h) = \varphi_\lambda^{\alpha,\beta}(h) + \frac{\rho}{4(\alpha+1)} \sinh(2h) \varphi_\lambda^{\alpha+1,\beta+1}(h),$$

and $\varphi_\lambda^{\alpha,\beta}$ is even, then

$$\mathcal{H}(\tau_h^{(\alpha,\beta)} f + \tau_{-h}^{(\alpha,\beta)} f - 2f)(\lambda) = 2(\varphi_\lambda^{\alpha,\beta}(h) - 1)\mathcal{H}(f)(\lambda).$$

By Lemma 1.3, we have the result. □

Theorem 2.3. *Let* $f(x)$ *belong to* $Lip(\delta,\gamma,p)$ *. Then*

$$\int_{|\lambda|\ge r} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) = O\left(\frac{r^{-q\delta}}{(\log r)^{q\gamma}}\right), \quad as \quad r \to \infty,$$

where $p \in [1,2)$ *and* q *such that* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $f \in Lip(\delta,\gamma,p)$. Then we have

$$\|\tau_h^{(\alpha,\beta)} f(x) + \tau_{-h}^{(\alpha,\beta)} f(x) - 2f(x)\|_{p,\alpha,\beta} = O\left(\frac{h^\delta}{(\log\frac{1}{h})^\gamma}\right) \quad \text{as} \quad h \to 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}} |\varphi_\lambda^{\alpha,\beta}(h)-1|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \le \left(\frac{c_p}{2}\right)^q \|\tau_h^{(\alpha,\beta)} f(x)+\tau_{-h}^{(\alpha,\beta)} f(x)-2f(x)\|^q_{p,\alpha,\beta}.$$

If $|\lambda| \in [\frac{1}{h}, \frac{2}{h}]$, then $|\lambda h| \ge 1$ and (iii) of Lemma 1.2 implies that

$$1 \le \frac{1}{c^q} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^q.$$

Then

$$\begin{aligned}
\int_{\frac{1}{h}\le|\lambda|\le\frac{2}{h}} |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &\le \frac{1}{c^q} \int_{\frac{1}{h}\le|\lambda|\le\frac{2}{h}} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\
&\le \frac{1}{c^q} \int_{\mathbb{R}} |1 - \varphi_\lambda^{\alpha,\beta}(h)|^q |\mathcal{H}f(\lambda)|^q d\sigma(\lambda) \\
&\le \left(\frac{c_p}{2c}\right)^q \|\tau_h^{(\alpha,\beta)} f(x) + \tau_{-h}^{(\alpha,\beta)} f(x) - 2f(x)\|^q_{p,\alpha,\beta} \\
&= O\left(\frac{h^{q\delta}}{(\log\frac{1}{h})^{q\gamma}}\right).
\end{aligned}$$

We obtain

$$\int_{r\le|\lambda|\le 2r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)\le C\frac{r^{-q\delta}}{(\log r)^{q\gamma}},\quad r\to\infty.$$

where C is a positive constant. Now,

$$\begin{aligned}\int_{|\lambda|\ge r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r\le|\lambda|\le 2^{i+1}r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)\\ &\le C\left(\frac{r^{-q\delta}}{(\log r)^{q\gamma}}+\frac{(2r)^{-q\delta}}{(\log 2r)^{q\gamma}}+\frac{(4r)^{-2\delta}}{(\log 4r)^{2\gamma}}+\cdots\right)\\ &\le C\frac{r^{-q\delta}}{(\log r)^{q\gamma}}\left(1+2^{-q\delta}+(2^{-q\delta})^2+(2^{-q\delta})^3+\cdots\right)\\ &\le K_\delta\frac{r^{-q\delta}}{(\log r)^{q\gamma}},\end{aligned}$$

where $K_\delta=C(1-2^{-q\delta})^{-1}$ since $2^{-q\delta}<1$.
Consequently

$$\int_{|\lambda|\ge r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)=O\left(\frac{r^{-q\delta}}{(\log r)^{q\gamma}}\right),\quad as\quad r\to\infty.$$

□

Definition 2.4. Let $\gamma>0$. A function $f\in L^p_{\alpha,\beta}(\mathbb{R})$ is said to be in the (ψ,γ,p)-Cherednik-Opdam Lipschitz class, denoted by $Lip(\psi,\gamma,p)$, if

$$\|\tau_h^{(\alpha,\beta)}f(x)+\tau_{-h}^{(\alpha,\beta)}f(x)-2f(x)\|_{p,\alpha,\beta}=O\left(\frac{\psi(h)}{(\log\frac{1}{h})^\gamma}\right)\quad \text{as}\quad h\to 0,$$

where ψ is a continuous increasing function on $[0,\infty),\psi(0)=0$ and $\psi(ts)=\psi(t)\psi(s)$ for all $t,s\in[0,\infty)$.

Theorem 2.5. *Let $f(x)$ belong to $Lip(\psi,\gamma,p)$. Then*

$$\int_{|\lambda|\ge r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)=O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right),\quad as\quad r\to\infty,$$

where $p\in[1,2)$ and q such that $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $f\in Lip(\psi,p)$. Then we have

$$\|\tau_h^{(\alpha,\beta)}f(x)+\tau_{-h}^{(\alpha,\beta)}f(x)-2f(x)\|_{p,\alpha,\beta}=O\left(\frac{\psi(h)}{(\log\frac{1}{h})^\gamma}\right)\quad \text{as}\quad h\to 0.$$

From Lemma 2.2, we have

$$\int_{\mathbb{R}}|\varphi_\lambda^{\alpha,\beta}(h)-1|^q|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)\le\left(\frac{c_p}{2}\right)^q\|\tau_h^{(\alpha,\beta)}f(x)+\tau_{-h}^{(\alpha,\beta)}f(x)-2f(x)\|^q_{p,\alpha,\beta}.$$

If $|\lambda|\in[\frac{1}{h},\frac{2}{h}]$, then $|\lambda h|\geq 1$ and (iii) of Lemma 1.2 implies that

$$1\leq\frac{1}{c^q}|1-\varphi_\lambda^{\alpha,\beta}(h)|^q.$$

Then

$$\begin{aligned}\int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &\leq \frac{1}{c^q}\int_{\frac{1}{h}\leq|\lambda|\leq\frac{2}{h}}|1-\varphi_\lambda^{\alpha,\beta}(h)|^q|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)\\ &\leq \frac{1}{c^q}\int_{\mathbb{R}}|1-\varphi_\lambda^{\alpha,\beta}(h)|^q|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)\\ &\leq \left(\frac{c_p}{2c}\right)^q\|\tau_h^{(\alpha,\beta)}f(x)+\tau_{-h}^{(\alpha,\beta)}f(x)-2f(x)\|_{p,\alpha,\beta}^q\\ &= O\left(\frac{\psi(h)^q}{(\log\frac{1}{h})^{q\gamma}}\right)=O\left(\frac{\psi(h^q)}{(\log\frac{1}{h})^{q\gamma}}\right).\end{aligned}$$

We obtain

$$\int_{r\leq|\lambda|\leq 2r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)\leq C\frac{\psi(r^{-q})}{(\log r)^{q\gamma}},\quad r\to\infty.$$

where C is a positive constant. Now,

$$\begin{aligned}\int_{|\lambda|\geq r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda) &= \sum_{i=0}^{\infty}\int_{2^i r\leq|\lambda|\leq 2^{i+1}r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)\\ &\leq C\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}+\frac{\psi((2r)^{-q})}{(\log 2r)^{q\gamma}}+\frac{\psi((4r)^{-q})}{(\log 4r)^{q\gamma}}+\cdots\right)\\ &\leq C\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\left(1+\psi(2^{-q})+(\psi(2^{-q}))^2+(\psi(2^{-q}))^3+\cdots\right)\\ &\leq K_\delta\frac{\psi(r^{-q})}{(\log r)^{q\gamma}},\end{aligned}$$

where $K_\delta=C(1-\psi(2^{-q}))^{-1}$ since $\psi(2^{-q})<1$.
Consequently

$$\int_{|\lambda|\geq r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)=O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right),\quad as\quad r\to\infty.$$

□

3. Conclusions

In this work we have succeded to generalise the theorem in [3] for the Cherednik-Opdam transform in the space $L^p_{\alpha,\beta}(\mathbb{R})$. We proved that $f(x)$ belong to $Lip(\psi,\gamma,p)$. Then

$$\int_{|\lambda|\geq r}|\mathcal{H}f(\lambda)|^q d\sigma(\lambda)=O\left(\frac{\psi(r^{-q})}{(\log r)^{q\gamma}}\right),\quad as\quad r\to\infty,$$

where $p \in [1, 2)$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$.

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