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Generalized Helices and Singular Points

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Abstract. In this paper, we define $X-$slant helix in Euclidean 3-space and we obtain helix, slant helix, clad and g-clad helix as special case of the $X-$slant helix. Then we study Darboux and tangential darboux developable surfaces, and their singular points. Especially we show that singular locus of the surface is coincide with the striction line of the surface.

Keywords: clad helices, g-clad helices, Darboux surface, striction line.

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1. Introduction

Let $\alpha = \alpha(s)$ be a curve in Euclidean $3-$space with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$, where κ and τ represent curvature and torsion of the curve α. A general helix in Euclidean $3-$space is a curve whose tangent vector makes a constant angle with a fixed direction (axis of the helix). By using curvature functions we have a necessary and sufficient condition for a helix that the ratio of curvature to torsion be constant. Then Izumiya and Takeuchi defined a new special curve called slant helix its normal vector makes a constant angle with a fixed straight line [6]. Also they proved that α is a slant helix if and only if $\dfrac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\dfrac{\tau}{\kappa}\right)'$

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is a constant function. In [2], authors show that Salkowski and anti-Salkowski curves make a constant angle with a fixed direction. That is, these types of curves are examples of slant helices. Also in [3] Kula and Yayli have studied the relationship between the slant helix and spherical helix. They showed that the tangent indicatrix and binormal indicatrix of a slant helix are spherical helices. Then Takahashi and Takeuchi [7] introduced new kinds of curves which are called clad helix and g-clad helix. A curve γ with non-zero curvature is called clad helix and g-clad helix if the spherical image of the unit principal normal is a part of a cylindrical helix and slant helix in $\mathbb{S}^2$, respectively. Then they gave characterizations of clad helices and g-clad helices in terms of the curvatures.

Motivated these papers, we define $X-$slant helix in terms of the orthonormal frame $\left\{X, Y=\frac{X'}{\|X'\|}, Z=X\wedge Y\right\}$ and give the characterization of the $X-$slant helix. Since we can obtain the helix, slant helix, clad helix and g-clad helix in special cases, we called the curve as generalized helix. Then we consider Darboux surface

$$F_{(Z,X)}(s,u)=Z(s)+uX(s)$$

and tangential darboux developable surface

$$F_{(\overline{D},Y)}(s,u)=\overline{D}(s)+uY(s)$$

of the $X-$slant helix where
$\overline{D}(s)=\left(\dfrac{1}{\sqrt{k_1^2+k_2^2}}\right)(s)(k_2(s)X(s)+k_1(s)Z(s))$. Their special cases with Frenet frame $\{T,N,B\}$ were studied in [6] and some other cases were studied in [7]. In this paper we give an important geometric interpretation of the singular locus of the Darboux and tangential darboux developable surfaces, then we show that singular locus of these surfaces are striction lines of the surfaces. Also we give examples and draw their pictures by using Mathematica.

2. Preliminaries

In this section, we review some basic concepts of slant, clad and g-clad helices in Euclidean $3-$space. A curve $\alpha: I\subset\mathbb{R}\to E^3$ is called unit speed curve (parametrized by an arc-length parameter), if $\|\alpha'(s)\|=1$. Let $\{T,N,B\}$ denotes the Frenet frame of the curve α. Then the Frenet equations are given as the following

$$\begin{aligned}
T'(s) &= \kappa(s)N(s)\\
N'(s) &= -\kappa(s)T(s)+\tau(s)B(s)\\
B'(s) &= -\tau(s)N(s)
\end{aligned}$$

One knows that a curve is a helix if and only if $\dfrac{\tau(s)}{\kappa(s)}$ =const.
In [1], Uzunoğlu et al. define an alternative moving frame

$$\left\{N, C = \frac{N'}{\| N' \|}, W = N \times C\right\}$$

with

$$\begin{aligned} N' &= fC \\ C' &= -fN + gW \\ W' &= -gC \end{aligned}$$

where $f = \sqrt{\kappa^2 + \tau^2}$ and $g = \dfrac{\kappa^2}{\kappa^2 + \tau^2}\left(\dfrac{\tau}{\kappa}\right)'$. By using the alternative moving frame, we can say that a curve is called slant helix if and only if $\sigma = \frac{g(s)}{f(s)}$ is constant.
In [7], Takahashi and Takeuchi introduced new types of curves such as clad helix and g-clad helix. Then they give characterizations of these curves as the following

Proposition 2.1. *Let γ be a unit speed space curve with $\kappa(s) \neq 0$. Then γ is a clad helix if and only if*

$$\varphi(s) = \left(\frac{\sigma'}{\sqrt{\kappa^2 + \tau^2}(1 + \sigma^2)^{\frac{3}{2}}}\right)(s)$$

is a constant function.

Proposition 2.2. *Let γ be a unit speed space curve with $\kappa(s) \neq 0$. Then γ is a g-clad helix if and only if*

$$\psi(s) = \left(\frac{\varphi'}{\sqrt{\kappa^2 + \tau^2}(1 + \sigma^2)^{\frac{1}{2}}(1 + \varphi^2)^{\frac{3}{2}}}\right)(s)$$

is a constant function.

Now we define general orthonormal frame of the curve. Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed curve and X be a unit vector field along the curve α such that $X \mid_{\alpha(s)} = X(s)$. In terms of $\left\{X, Y = \frac{X'}{\|X'\|}, Z = X \times Y\right\}$ the orthonormal frame of the curve is as the following

$$\begin{aligned} X' &= k_1 Y \\ Y' &= -k_1 X + k_2 Z \\ Z' &= -k_2 Y \end{aligned}$$

(1) Case $X = T$, we get Frenet frame $\{T, N, B\}$ with curvatures κ and τ.

(2) Case $X = N$, we get an alternative moving frame $\{N, C, W\}$ with curvatures f and g.

Let arc-length parameter of the vector field $X(s)$ be s_x and we obtain

$$\frac{ds_x}{ds} = k_1 \tag{2.1}$$

If we derivative with respect to arc-length parameter of the vector field $X(s)$ we get

$$\begin{aligned} \frac{d}{ds_x}X &= Y \\ \frac{d}{ds_x}Y &= -X + \frac{k_2}{k_1}Z \\ \frac{d}{ds_x}Z &= -\frac{k_2}{k_1}Y \end{aligned}$$

Darboux vector of this frame is

$$\overline{W}(s) = \frac{k_2(s)}{k_1(s)}X(s) + Z(s) \tag{2.2}$$

and $k_g(s) = \frac{k_2(s)}{k_1(s)}$ is the geodesic curvature of the vector field $X(s)$.

3. Basic Notions and Properties

In this section we define $X-$slant helix and we get the helix, slant helix, clad and g-clad helix as a special case of the $X-$slant helix. Then we obtain the singular locus of the Darboux surface and tangential darboux developable surface.

Definition 3.1. Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed curve in Euclidean 3-space. A curve α is called $X-$slant helix if the vector field X makes a constant angle θ with a fixed direction U, that is

$$\langle X, U \rangle = \cos\theta = \text{constant} \tag{3.1}$$

along the curve α.

Theorem 3.2. *Let α be a unit speed space curve in Euclidean 3-space. Then the curve α is a $X-$slant helix if and only if the geodesic curvature of the vector field X is a constant.*

Proof. The curve α is called $X-$slant helix if the vector field X makes a constant angle θ with a fixed direction U. Then,

$$\begin{aligned} \langle X(s), U \rangle &= \cos\theta \\ \langle X'(s), U \rangle &= 0 \\ k_1(s)\langle Y(s), U \rangle &= 0 \end{aligned}$$

Hence the axis of the $X-$slant helix is obtained as

$$U = \cos\theta X(s) + \sin\theta Z(s).$$

Since the direction U is a fixed, then we get

$$(k_1(s)\cos\theta - k_2(s)\sin\theta)Y(s) = 0$$

and

$$\frac{k_2(s)}{k_1(s)} = k_g(s) = \cot\theta$$

which completes the proof. □

Corollary 3.3. *Modified Darboux vector field $\overline{W}$ has the same direction with the axis U of the $X-$slant helix.*

Proof. The axis of the $X-$slant helix is obtained as

$$U = \cos\theta X(s) + \sin\theta Z(s)$$

Then

$$\begin{aligned} U &= \sin\theta\left(\frac{\cos\theta}{\sin\theta}X(s) + Z(s)\right) \\ &= \sin\theta\overline{W}(s) \end{aligned}$$

which completes the proof. □

Corollary 3.4. *From the orthonormal frame $\left\{X, Y = \frac{X'}{\|X'\|}, Z\right\}$ we have the following states*
(i) If $X(s) = T(s)$, we get $\{T, N, B\}$ and $k_g = \dfrac{\tau(s)}{\kappa(s)}$
(ii) If $X(s) = N(s)$ we get $\left\{N, C = \frac{N'}{\|N'\|}, W\right\}$ and $k_g = \dfrac{g}{f}$
(iii) If $X(s) = C(s)$ we get $\left\{C, M = \frac{C'}{\|C'\|}, D\right\}$ and $k_g = \varphi(s)$
(iv) If $X(s) = M(s)$ we get $\left\{M, L = \frac{M'}{\|M'\|}, K\right\}$ and $k_g = \psi(s)$
Hence we get helix, slant helix, clad helix and g-clad helix as special cases of $X-$slant helix.

4. Developable Surfaces and Singular Points

A ruled surface in $\mathbb{R}^3$ is locally the map

$$F(\gamma, \delta) : I \times \mathbb{R} \to \mathbb{R}^3$$

defined by $F(\gamma,\delta)(t,u) = \gamma(t) + u\delta(t)$, where $\gamma : I \to \mathbb{R}^3$ is base curve and $\delta : I \to \mathbb{R}^3\backslash\{0\}$ is director curve and I is an open interval or a unit circle $\mathbb{S}^1$. A ruled surface $F(\gamma,\delta)(t,u) = \gamma(t)+u\delta(t)$ is called developable if

$$\det(\gamma', \delta, \delta') = 0$$

and striction line of the surface is defined as

$$\beta(s) = \gamma(s) - \frac{< \gamma', \alpha' >}{< \alpha', \alpha' >}\alpha(s).$$

In the paper [6], Izumiya and Takeuchi defined Darboux developable of the curve γ as

$$F_{(B,T)}(s,u) = B(s) + uT(s)$$

where $T(s)$ and $B(s)$ are tangent and binormal vectors of the curve γ. Also

$$F_{(\overline{D},N)}(s,u) = \overline{D}(s) + uN(s)$$

is called tangential Darboux developable of γ where $\overline{D}(s) = (\frac{1}{\sqrt{\tau^2+\kappa^2}})(s)(\tau(s)T(s)+\kappa(s)B(s))$ is a unit Darboux vector field.

In this section, we consider Darboux surface $F_{(Z,X)}(s,u) = Z(s) + uX(s)$ of the unit speed curve α. Then we show that the Darboux surface is a developable surface which has zero Gaussian curvature. There is no any geometric interpretation of singular points in previous papers. Hence we investigate the singular locus of the surface and we show that it coincide with striction line of the surface.

Proposition 4.1. *Let* $\alpha : I \subset \mathbb{R} \to E^3$ *be a unit speed curve in Euclidean 3-space and* $X(s)$ *be a unit vector field along the curve* α. *Darboux surface* $F_{(Z,X)}(s,u) = Z(s) + uX(s)$ *is a developable surface and its striction line is* $\overline{W}(s)$.

Proof. From the orthonormal moving frame $\{X, Y, X \wedge Y = Z\}$ we easily get

$$\det(Z', X, X') = 0. \tag{4.1}$$

Hence, the Darboux surface is a developable surface.
Let $\overline{\gamma}$ be a striction line of the surface, then

$$\begin{aligned} \overline{\gamma} &= Z - \frac{\langle Z', X' \rangle}{\langle X', X' \rangle}X \\ &= Z + \frac{k_2}{k_1}X \\ &= \overline{W} \end{aligned}$$

which completes the proof. □

Corollary 4.2. *Singular locus of the Darboux surface* $F_{(Z,X)}(s,u) = Z(s) + uX(s)$ *is striction line of the surface.*

Proof. The singular locus of the surface $F_{(Z,X)}$ is given by

$$\sigma(s) = Z(s) + k_g(s)X(s)$$

For the Darboux surface $F_{(Z,X)}(s,u) = Z(s) + uX(s)$ we calculate that

$$F_s \wedge F_u = (k_2 - uk_1)Z = 0.$$

Hence (s_0, u_0) is a singular point of $F_{(Z,X)}$ if and only if $u = \frac{k_2}{k_1} = k_g$. Then

$$\sigma(s) = \overline{W}(s)$$

This completes the proof. $\square$

Proposition 4.3. *Let $\alpha : I \subset \mathbb{R} \to E^3$ be a unit speed curve in Euclidean 3-space with $\kappa(s) \neq 0$, the followings are equivalent.*
(1) The Darboux surface $F_{(Z,X)}(s,u) = Z(s) + uX(s)$ of α is a conical surface and its peak point is $\overline{W}$.
(2) The curve α is $X-$slant helix and its axis has the same direction with $\overline{W}$.

Proof. Using the Eq. (2.2) we get

$$F_{(Z,X)}(s,u) = \overline{W}(s) + (u - k_g)X(s)$$

If we derivative of $\overline{W}(s)$ with respect to parameter s, we get

$$\begin{aligned} \frac{d\overline{W}}{ds} &= \frac{d}{ds}\left(Z(s) + \frac{k_2(s)}{k_1(s)}X(s)\right) \\ &= \frac{d}{ds}\left(\frac{k_2(s)}{k_1(s)}\right)X(s) \end{aligned}$$

If F is a conical surface then $\overline{W}$ is a constant. That is, $\dfrac{k_2}{k_1}$ =const. From Theorem 3.2, α is a $X-$slant helix.
Conversely, if α is a $X-$slant helix, then $\overline{W}$ is a constant. So the ruled developable Darboux surface $F_{(Z,X)}$ is a conical surface. $\square$

By using the Proposition4.3 we obtain Proposition (3.4) in [6] , Proposition (4.5) and (4.7) in [7].

Corollary 4.4. *(1) $F(s,u) = W(s) + uN(s)$ is a conical surface iff α is a slant helix in terms of alternative moving frame $\left\{N, C = \frac{N'}{\|N'\|}, W\right\}$*
(2) $F(s,u) = D(s) + uC(s)$ is a conical surface iff α is a clad helix in terms of the moving frame $\left\{C, M = \frac{C'}{\|C'\|}, D\right\}$
(3) $F(s,u) = K(s) + uM(s)$ is a conical surface iff α is a g-clad helix in terms of the moving frame $\left\{M, L = \frac{M'}{\|M'\|}, K\right\}$

The singular points of a Darboux developable surface is investigated in [4, 5].

Theorem 4.5. *Let* $\alpha : I \subset \mathbb{R} \to E^3$ *be a unit speed curve in Euclidean 3-space with* $\kappa(s) \neq 0$. *Then we have the following states*
(1) The Darboux surface of α *is locally diffeomorphic to a cuspidaledge* $C \times \mathbb{R}$ *at* $F_{(Z,X)}(s_0, u_0)$ *if and only if* $u_0 = k_g(s_0) \neq 0$, $k_g'(s_0) = 0$.
(2) The Darboux surface of α *is locally diffeomorphic to a swallowtail* SW *at* $F_{(Z,X)}(s_0, u_0)$ *if and only if* $u_0 = k_g(s_0) \neq 0$, $k_g'(s_0) = 0$, $k_g''(s_0) \neq 0$
(3) The Darboux surface of α *is locally diffeomorphic to a cuspidalcross cap* CCR *at* $F_{(Z,X)}(s_0, u_0)$ *if and only if* $u_0 = k_g(s_0) = 0$, $k_g'(s_0) \neq 0$.

By using this theorem we obtain Theorem (3.3) in [6] , Theorem (4.4) and (4.6) in [7].

Now we study tangential Darboux developable surface and its singular points.

$$F_{(\overline{D},Y)}(s,u) = \overline{D}(s) + uY(s)$$

is called tangential Darboux developable of γ where
$\overline{D}(s) = \left(\dfrac{1}{\sqrt{k_1^2 + k_2^2}}\right)(s)(k_2(s)X(s) + k_1(s)Z(s))$ is a unit Darboux vector field with respect to orthormal frame $\{X, Y, Z\}$.

Proposition 4.6. *Let* $\alpha : I \subset \mathbb{R} \to E^3$ *be a unit speed curve in Euclidean 3-space with* $\kappa(s) \neq 0$. *Singular locus of the tangential Darboux developable surface* $F_{(\overline{D},Y)}(s,u) = \overline{D}(s) + uY(s)$ *is coincide with striction line of the surface and it is given by*

$$\sigma(s) = \overline{D}(s) + \lambda(s)Y(s)$$

where $\lambda(s) = \left(\dfrac{k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}}\left(\dfrac{k_2}{k_1}\right)'\right)(s)$.

Proof. Striction line of the tangential darboux surface is given by

$$\begin{aligned}
\overline{\gamma}(s) &= \overline{D}(s) - \frac{\langle \overline{D}', Y' \rangle}{\langle Y', Y' \rangle} Y \\
&= \overline{D}(s) + \left(\frac{k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}}(\frac{k_2}{k_1})'\right)(s)Y(s) \\
&= \overline{D}(s) + \lambda(s)Y(s)
\end{aligned}$$

where $\lambda(s) = \left(\dfrac{k_1^2}{(k_1^2 + k_2^2)^{\frac{3}{2}}}(\dfrac{k_2}{k_1})'\right)(s)$. Also for the surface $F_{(\overline{D},Y)}(s,u) = \overline{D}(s) + uY(s)$, we get

$$F_s = (-\lambda(s) + u)(-k_1(s)X(s) + k_2(s)Z(s)) \quad \text{and} \quad F_u = Y(s)$$

Hence,

$$F_s \times F_u = (\lambda(s) - u)(k_1(s)Z(s) + k_2(s)X(s))$$

(s_0, u_0) is a singular point of the surface if and only if $\lambda(s) = u$. So, singular locus of the surface is obtained

$$\sigma(s) = \overline{D}(s) + \lambda(s)Y(s)$$

which completes the proof. $\square$

Proposition 4.7. *Let* $\alpha : I \subset \mathbb{R} \to E^3$ *be a unit speed curve in Euclidean 3-space with* $\kappa(s) \neq 0$. *Tangential Darboux developable surface* $F_{(\overline{D},Y)}(s,u) = \overline{D}(s) + uY(s)$ *is a conical surface if and only if* $\lambda(s) = \left(\dfrac{k_1^2}{(k_1^2+k_2^2)^{\frac{3}{2}}}\left(\dfrac{k_2}{k_1}\right)'\right)(s)$ *is constant.*

Proof. Singular locus of the tangential darboux surface $F_{(\overline{D},Y)}$ is given by

$$\sigma(s) = \overline{D}(s) + \lambda(s)Y(s).$$

Since $F_{(\overline{D},Y)}(s,u) = \overline{D}(s) + uY(s)$ is a conical surface if and only if $\sigma'(s) = 0$, by using the orthonormal frame $\{X, Y, Z\}$ we obtain

$$\sigma'(s) = \lambda'(s)Y(s)$$

Therefore, $\sigma'(s) = 0$ if and only if $\lambda(s) = const.$ $\square$

Remark 4.8. If we take the Frenet frame $\{T, N, B\}$ we get $\lambda(s) = \left(\dfrac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\dfrac{\tau}{\kappa})'\right)(s)$. Then we can say that the tangential Darboux developable surface $F_{(\overline{D},N)}(s,u) = \overline{D}(s) + N(s)$ is a conical surface if and only if $\dfrac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\dfrac{\tau}{\kappa})'$ is constant where $\overline{D}(s) = \left(\dfrac{1}{\sqrt{\kappa^2+\tau^2}}\right)(s)(\tau(s)T(s) + \kappa(s)B(s))$. So the curve α is a slant helix. This case is obtained by Izumiya and Takeuchi in [6].

Corollary 4.9. *Let* $\alpha : I \subset \mathbb{R} \to E^3$ *be a unit speed curve in Euclidean 3-space. The followings are equivalent*
(i) Tangential Darboux developable surface $F_{(\overline{D},C)}(s,u) = \overline{D}(s) + uC(s)$, *where* $\overline{D}(s) = \left(\dfrac{1}{\sqrt{f^2+g^2}}\right)(s)(g(s)N(s) + f(s)W(s))$ *is a conical surface.*
(ii) α *is a clad helix.*

Proof. From the Proposition 4.7, tangential darboux developable surface $F_{(\overline{D},C)}(s,u) = \overline{D}(s) + uC(s)$ is a conical surface if and only if

$$\lambda(s) = \left(\frac{f^2}{(f^2+g^2)^{\frac{3}{2}}} (\frac{g}{f})' \right)(s)$$

is a constant. By using the equations $f = \sqrt{\kappa^2+\tau^2}$ and $\dfrac{g}{f} = \sigma$, we get

$$\begin{aligned} \lambda(s) &= \left(\frac{f^2}{(f^2+g^2)^{\frac{3}{2}}} (\frac{g}{f})' \right)(s) \\ &= \frac{\sigma'(s)}{\sqrt{\kappa^2+\tau^2}(1+\sigma^2(s))^{\frac{3}{2}}} \\ &= \varphi(s) \end{aligned}$$

which completes the proof. □

This corollary is coincide with proposition (4.5) in [7]. Also when we take the frame $\left\{C, M = \frac{C'}{\|C'\|}, D\right\}$ we obtain proposition (4.7) in [7].

5. Examples

In this section we give examples of Darboux surface of a helix and a slant helix. For a curve helix the Darboux surface is a conical surface and $\overline{W}$ is a constant, for a slant helix singular locus of the Darboux surface is a striction line of the surface. Then we draw their pictures with Mathematica.

Example 5.1. $\alpha(s) = (\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}})$ is a unit speed cylindrical helix with $\kappa(s) = \frac{1}{2}$ and $\tau(s) = \frac{1}{2}$. Darboux developable surface of the curve α is

$$F_{(B,T)}(s,u) = \begin{pmatrix} \frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}} - u\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} + u\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} + u\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\overline{W}(s) = \frac{\tau(s)}{\kappa(s)}T(s) + B(s) = (0,0,\sqrt{2})$$

Since $\overline{W}$ is a constant, Darboux surface $F_{(B,T)}(s,u) = B(s) + uT(s)$ is a conical surface.

Example 5.2. $\alpha(s) = (\frac{2}{5}\sin 2s - \frac{1}{40}\sin 8s, -\frac{2}{5}\cos 2s + \frac{1}{40}\cos 8s, \frac{4}{15}\sin 3s)$ is a unit speed slant helix with $\kappa(s) = -4\sin 3s$ and $\tau(s) = 4\cos 3s$.

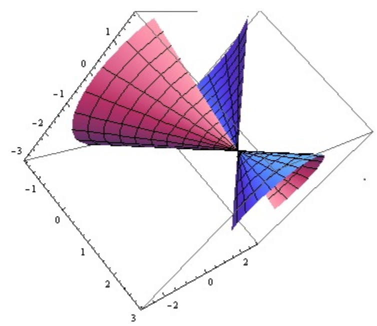

FIGURE 1. Darboux surface of the curve α

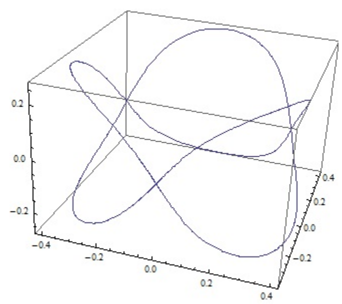

FIGURE 2. The curve α

$$
\begin{aligned}
T(s) &= \left(\frac{4}{5}\cos 2s - \frac{1}{5}\cos 8s, \frac{4}{5}\sin 2s - \frac{1}{5}\sin 8s, \frac{4}{5}\cos 3s\right)\\
N(s) &= \frac{1}{\sin 3s}\left(-\frac{2}{5}\sin 2s + \frac{2}{5}\sin 8s, \frac{2}{5}\cos 2s - \frac{2}{5}\cos 8s, -\frac{3}{5}\sin 3s\right)\\
B(s) &= \left(\begin{array}{c} -\frac{12}{25}\sin 2s + \frac{3}{25}\sin 8s - \frac{8}{25}\cot 3s(\cos 2s - \cos 8s),\\ \frac{12}{25}\cos 2s - \frac{3}{25}\cos 8s + \frac{8}{25}\cot 3s(-\sin 2s + \sin 8s),\\ \frac{2}{5\sin 3s}(1-\cos 6s) \end{array}\right)
\end{aligned}
$$

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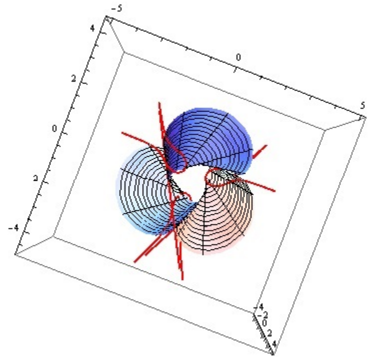

FIGURE 3. Darboux surface and its striction line

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