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Inverse problem for Sturm–Liouville operators with a transmission and parameter dependent boundary conditions

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Abstract. In this manuscript, we consider the inverse problem for non self-adjoint Sturm–Liouville operator $-D^2+q$ with eigenparameter dependent boundary and discontinuity conditions inside a finite closed interval. We prove by defining a new Hilbert space and using spectral data of a kind, the potential function can be uniquely determined by a set of value of eigenfunctions at an interior point and part of two sets of eigenvalues.

Keywords: Inverse Sturm–Liouville equation, Non self–adjoint operator, Jump condition, Hilbert space.

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1. Introduction

We consider the boundary value problem $L = L(q(x), h, H, a)$

$$\ell y := -y'' + qy = \lambda y, \quad x \in [0,\, a) \cup (a,\, \pi] \tag{1.1}$$

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$$U(y) := y'(0) - hy(0) = 0, \tag{1.2}$$
$$V(y) := Hy'(\pi) + \lambda y(\pi) = 0, \tag{1.3}$$

with the jump conditions

$$U_1(y) := y(a+0) - a_1 y(a-0) = 0,$$
$$U_2(y) := y'(a+0) - a_2 y'(a-0) - a_3 y(a-0) = 0, \tag{1.4}$$

where $q \in L^2[0,\pi]$, $h, H \in \mathbb{R}$, $0 < H < \infty$. $a \in (0,\pi)$, a_1, a_2, a_3 are real, and a_1, a_2 have the same sign.

The method of separation of variables for solving PDEs with discontinuous boundary conditions naturally led to ODE with discontinuities inside of the interval which often appear in mathematics. Inverse spectral problem consists in recovering operators from their spectral characteristics. For example the mathematical formulation of a large variety of technical and physical problem led to inverse problems such as identifying the density of the vibrating string from data collected from the sets of frequencies of oscillations of the vibrating string with barrier.

The inverse spectral Sturm–Liouville problem can be regarded as three aspects, e.g., existence, uniqueness and reconstruction of the coefficients given specific properties of eigenvalues and eigenfunctions. Here we want to look at the question of uniqueness for the above problem using two sets of spectra, or one spectrum plus part of a set of value of eigenfunctions at an interior point.

The applications of boundary value problems with discontinuity conditions inside the interval are connected with discontinuous material properties. Inverse problems with a discontinuity condition inside the interval play an important role in mathematics, mechanics, radio electronics, geophysics, and other fields of science and technology. As a rule, such problems are related to discontinuous and non–smooth properties of a medium (e.g., see [12]–[14] and [18]).

There are various formulations of the inverse problems and the corresponding uniqueness theorems. Ambarzumian [1] considered the questions of uniqueness in a special case. He considered equation (1.1) with the boundary conditions

$$y'(0) = 0 = y'(\pi) \tag{1.5}$$

and the equation

$$y'' + \lambda y = 0$$

with the same end conditions. He showed that if the two systems have the same spectrum, $\{\lambda_n\}_0^\infty = \{n^2\}_0^\infty$, then $q(x)$ is identically zero. Note that Ambarzumian's result [1] is an exception from the rule. In general, the specification of the spectrum does not uniquely determine the potential function. The idea of using two sequences of eigenvalues seems to have originated in Borg's paper [2] on inverse spectral theory. He showed that if $q(x)$ is symmetric, i.e.,

$$q(x) = q(\pi - x),$$

then the spectrum of equation (1.1) corresponding to the end condition (1.5) or to the end condition

$$y(0) = 0 = y(\pi)$$

determines $q(x)$ uniquely. We refer to the somewhat complementary surveys in [6]–[11], [17] and [19] for further aspects of this field. For general background on inverse Sturm–Liouville problems we refer (e.g.) to the monographs [4], [17], [19], and [26]. More recently, Yong and Guo [3] have investigated inverse spectra problems for a differential pencil and have determined a differential pencil from interior spectral data. In fact in this work, we formulate a new inverse spectral problem for discontinuous and parameter dependent boundary conditions. One can find the similar works for continuous and discontinuous conditions in [14], [15] and [22]–[25].

Remark 1.1. The same result can be obtained by the same method in the more general case of the parameter dependent boundary conditions

$$\begin{aligned} &y'(0) - hy(0) = 0, \\ &\lambda(y'(\pi) + H_1 y(\pi)) - H_2 y(\pi) - H_3 y'(\pi) = 0. \end{aligned}$$

2. Asymptotic form of solutions and eigenvalues

In this section, we introduce the special inner product in the Hilbert space $(L_2(0,a) \oplus L_2(a,\pi)) \oplus \mathbb{C}$ and define a linear operator A in it such that the considered problem (1.1)–(1.4) can be interpreted as the eigenvalue problem of A. So, we define a new Hilbert space inner product on $\mathbf{H} := (L_2(0,a) \oplus L_2(a,\pi)) \oplus \mathbb{C}$ by

$$\langle F, G \rangle_{\mathbf{H}} = |a_1| \int_0^a f\bar{g} + \frac{1}{|a_2|} \int_a^\pi f\bar{g} + \frac{1}{H|a_2|} f_1 \bar{g_1},$$

where $F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}$, $G = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in \mathbf{H}$ and $\bar{g}$, $\bar{g_1}$ are the conjugate of g, g_1 respectively. We define $R_1(u) := u(\pi)$ and $R_1'(u) := Hu'(\pi)$. We now define the operator

$$A : \mathbf{H} \to \mathbf{H}$$

by

$$AF = \begin{pmatrix} \ell f \\ -R_1'(f) \end{pmatrix} \quad \text{where } F = \begin{pmatrix} f(x) \\ R_1(f) \end{pmatrix} \in D(A),$$

with domain

$$D(A)=\left\{F=\begin{pmatrix} f(x)\\ f_1\end{pmatrix}\;\middle|\; \begin{array}{l} f(x),f'(x)\in AC[0,a)\cup(a,\pi],\ f(a\pm 0),\\ f'(a\pm 0)\text{ is defined, } \ell f\in L^2[(0,a)\cup(a,\pi)],\\ U(f)=U_1(f)=U_2(f)=0, f_1=R_1(f).\end{array}\right\}$$

Thus, we can change the boundary value problem (1.1)–(1.4) of the form

$$AU=\lambda U \qquad U:=\begin{pmatrix} u(x)\\ R_1(u)\end{pmatrix}\in D(A),$$

in the Hilbert space $\mathbf{H}$. It is easy that verified the eigenvalues of the operator A coincide with those of the problem (1.1)–(1.4).

Remark 2.1. We note that the operator L is non self–adjoint in the $L^2[0,\pi]$, where the operator A defined in the new Hilbert space $\mathbf{H}$ is self–adjoint. The proof of this result is the same as [16].

Suppose that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1.1) under the initial conditions:

$$\varphi(0,\lambda)=1, \qquad \varphi'(0,\lambda)=h, \tag{2.1}$$

and

$$\psi(\pi,\lambda)=H, \qquad \psi'(\pi,\lambda)=-\lambda. \tag{2.2}$$

By attaching a subscript 1 or 2 to the functions φ and ψ, we mean to refer to the first subinterval $[0,a)$ or to the second subinterval $(a,\pi]$. By $\varphi_{1n}(x)$ we mean $\varphi(x,\lambda_n)$ for $x\in[0,a)$ and by $\varphi_{2n}(x)$ we mean $\varphi(x,\lambda_n)$ for $x\in(a,\pi)$. Therefore we see that

$$\varphi_n(x):=\varphi(x,\lambda_n)=\begin{cases}\varphi_{1n}(x), & x<a,\\ \varphi_{2n}(x), & x>a.\end{cases}$$

It is easy to see that equation (1.1) under the initial conditions (2.1) or (2.2) has a unique solution $\varphi_1(x,\lambda)$ or $\psi_2(x,\lambda)$, which is an entire function of $\lambda\in\mathbb{C}$ for each fixed point $x\in[0,a)$ or $x\in(a,\pi]$. From the linear differential equations we obtain that the Wronskians

$$\Delta_1(\lambda):=W(\varphi_1(x,\lambda),\psi_1(x,\lambda))$$

and

$$\Delta_2(\lambda):=W(\varphi_2(x,\lambda),\psi_2(x,\lambda)),$$

are independent on $x\in[0,a)\cup(a,\pi]$ respectively.

Lemma 2.2. *The equality $\Delta_2(\lambda)=a_1a_2\Delta_1(\lambda)$ holds for each $\lambda\in\mathbb{C}$.*

Corollary 2.3. *The zeros of $\Delta(\lambda)=\Delta_2(\lambda)=a_1a_2\Delta_1(\lambda)$ are coincide, and eigenvalues of the problem* (1.1)–(1.4) *coincide with the zeros of the function $\Delta(\lambda)$.*

Corollary 2.4. *By self–adjointness of A and Corollary 2.3., all eigenvalues of the problem* (1.1)–(1.4) *are real and simple.*

Theorem 2.5. *Let $\lambda = \rho^2$, $\tau := \mathrm{Im}\rho$. For equation* (1.1) *with boundary conditions* (1.3) *and jump conditions* (1.4) *as $|\lambda| \to \infty$, the following asymptotic formulas hold.*

$$\varphi(x;\lambda) = \begin{cases} \cos\rho x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\frac{\sin\rho x}{\rho} + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & x < a, \\ (b_1\cos\rho x + b_2\cos\rho(2a-x)) + f_1(x)\frac{\sin\rho x}{\rho} & x > a, \\ \quad + f_2(x)\frac{\sin\rho(2a-x)}{\rho} + O\left(\frac{\exp(|\tau|x)}{\rho^2}\right), & \end{cases}$$

$$\varphi'(x;\lambda) = \begin{cases} -\rho\sin\rho x + \left(h + \frac{1}{2}\int_0^x q(t)dt\right)\cos\rho x + O\left(\frac{\exp(|\tau|x)}{\rho}\right) & x < a, \\ \rho[(-b_1\sin\rho x + b_2\sin\rho(2a-x))] + f_1(x)\cos\rho x & x > a, \\ \quad - f_2(x)\cos\rho(2a-x) + O\left(\frac{\exp(|\tau|x)}{\rho}\right), & \end{cases}$$

where

$$b_1 := \frac{a_1 + a_2}{2}, \qquad b_2 := \frac{a_1 - a_2}{2}$$

and

$$f_1(x) = b_1\left(h + \frac{1}{2}\int_0^x q(t)dt\right) + \frac{a_3}{2},$$
$$f_2(x) = b_2\left(h - \frac{1}{2}\int_0^x q(t)dt + \int_0^a q(t)dt\right) + \frac{a_3}{2}.$$

Then the characteristic function is

$$\begin{aligned}\Delta(\lambda) &= \rho^2(b_1\cos\rho\pi + b_2\cos\rho(2a-\pi)) + \rho[(f_1(\pi) - Hb_1)\sin\rho\pi \\ &\quad + (f_2(\pi) + Hb_2)\sin\rho(2a-\pi)] + O(\exp(|\tau|\pi)). \qquad (2.3)\end{aligned}$$

Proof. The arguments for obtaining the asymptotic formulas is similar to that of [18]. Note that by changing x to $\pi - x$ one can obtain the asymptotic form of $\psi(x,\lambda)$ and $\psi'(x,\lambda)$. □

We consider the boundary value problem $L_\circ = L(0;0;H;a)$ such that

$$\ell_\circ y := -y'' = \rho^2 y, \qquad (2.4)$$

with the boundary conditions

$$U_\circ(y) := y'(0) = 0,$$
$$V(y) = Hy'(\pi) + \lambda y(\pi) = 0,$$

and the jump conditions

$$y(a+0) = a_1 y(a-0), \qquad y'(a+0) = a_2 y'(a-0). \qquad (2.5)$$

Let $\varphi_\circ(x;\rho)$ be the following form

$$\varphi_\circ(x;\rho) = \begin{cases} \cos\rho x, & x < a, \\ b_1\cos\rho x + b_2\cos\rho(2a-x), & x > a. \end{cases} \qquad (2.6)$$

One can see that $\varphi_\circ(x,\rho)$ is the solution of (2.4) under the initial conditions $\varphi_\circ(0,\rho)=1$ and $\varphi'_\circ(0,\rho)=0$ and the jump conditions (2.5). Let $\Delta_\circ(\rho)$ be a characteristic function of problem $L_\circ$. By using (2.6) it is easy to see that the characteristic function related to $L_\circ$ is

$$\Delta_\circ(\rho)=\rho[\rho(b_1\cos\rho\pi+b_2\cos\rho(2a-\pi))+H(-b_1\sin\rho\pi+b_2\sin\rho(2a-\pi))]. \tag{2.7}$$

The roots ρ_n° of this equation are eigenvalues of problem $L_\circ$. We now prove the main results of this section.

Lemma 2.6. *The eigenvalues of problem* $L_\circ$ *are*

$$\rho_n^\circ=n-1+\eta_n,$$

where $\sup_n\ \eta_n<M$ *and for sufficiently large* ρ, $\eta_n\in(0,1)$ *for* $n\in\mathbb{N}$.

Proof. For the operator $L_\circ$ we can define the operator $A_\circ$ similar than the operator A but different boundary and jump conditions. From Remark 2.1 we find that the operator $A_\circ$ is self-adjoint. Indeed the zeros of the entire function $\Delta_\circ(\rho)$ are simple and coincide with the eigenvalues of $L_\circ$ for which its eigenvalues are real. We restrict the domain of $\Delta_\circ(\rho)$ to real line. From the fact that a_1 and a_2 have the same sign, we see that $b_1>b_2$ and so for sufficiently large ρ the sign of $\Delta_\circ(\rho)$ depend on the first term of $\Delta_\circ(\rho)$. By substituting the points $n-1$ and n instead of ρ we conclude that $\Delta_\circ(n-1)\Delta_\circ(n)<0$ for $n\in\mathbb{N}$. According to continuity and differentiability of $\Delta_\circ(\rho)$ there is a point, say η_n, in the interval $(0,1)$ such that $\Delta_\circ(n-1+\eta_n)=0$. Now we show that there exists exactly one zero in $(n-1,n)$. Suppose that $a=\frac{p}{q}\pi$, where $\frac{p}{q}$ is a rational number in the interval $(0,1)$. By rewriting (2.7) in the following form

$$\begin{aligned}\frac{\Delta_\circ(\rho)}{\rho^2}=&\frac{b_1}{2i}\left(\exp(i\rho\pi)+\exp(-i\rho\pi)\right)\\&+\frac{b_2}{2i}\left[\exp\left(i\rho\pi\left(\frac{2p}{q}-1\right)\right)+\exp\left(-i\rho\pi\left(\frac{2p}{q}-1\right)\right)\right]\\&+\frac{H}{\rho}\left[-\frac{b_1}{2i}(\exp(i\rho\pi)-\exp(-i\rho\pi))\right.\\&\left.+\frac{b_2}{2i}\left[\exp\left(i\rho\pi\left(\frac{2p}{q}-1\right)\right)-\exp\left(-i\rho\pi\left(\frac{2p}{q}-1\right)\right)\right]\right]\end{aligned} \tag{2.8}$$

and substituting $w:=\exp(i\frac{\rho\pi}{q})$ in (2.8), we see that for sufficiently large ρ, $\frac{\Delta_\circ(\rho)}{\rho^2}$ is a polynomial of degree $2q$ in term of w. Since $\frac{\Delta_\circ(\rho)}{\rho^2}$ is periodic function with period $T=2q$, there are $2q$ zeros on the interval $(0,2q)$ and so for each interval $(n-1,n)$ there is exactly one zero. By applying the same method of [21] or as a consequence of Valiron's theorem ([5, Thm. 13.4]) we then get that $\rho_n^\circ=n-1+\eta_n$ where $\sup\eta_n\le M$. □

Theorem 2.7. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n \to \infty$:*

$$\sqrt{\lambda_n} = n - 1 + \eta_n + O\left(\frac{1}{n}\right),$$

where η_n is defined in Lemma 2.6.

Proof. From (2.3) and (2.7), we see that

$$\Delta(\rho) = \Delta_\circ(\rho) + O\left(\rho \exp(|\tau|\pi)\right).$$

For sufficiently large value of ρ, the functions $\Delta(\rho)$ and $\Delta_\circ(\rho)$ have the same number of zeros counting multiplicities according to Rouche's theorem. So if ρ_n and ρ_n° are eigenvalues of L and $L_\circ$ respectively, then we have $\rho_n = \rho_n^\circ + \varepsilon_n$ where $\lim_{n\to\infty} \varepsilon_n = 0$. Since numbers ρ_n are zeros of the characteristic function $\Delta(\rho)$, therefore we have

$$\begin{aligned} 0 = \Delta(\rho_n) =& \Delta_\circ(\rho_n) + O(\rho_n) \\ =& \Delta_\circ(\rho_n^\circ + \varepsilon_n) + O(\rho_n^\circ + \varepsilon_n) \\ =& \Delta_\circ(\rho_n^\circ) + \varepsilon_n \dot{\Delta}_\circ(\rho_n^\circ) + O(\rho_n^\circ). \end{aligned}$$

From (2.7), by a simple calculation, we have $\dot{\Delta}_\circ(\rho_n^\circ) = O({\rho_n^\circ}^2)$ and therefore $\varepsilon_n = O(\frac{1}{n})$. □

3. Main results

In this section, together with L we consider the boundary value problem $\tilde{L} = L(\tilde{q}(x); h; H; a)$ of the same form but a different coefficient $\tilde{q}$ and $a = \frac{p}{q}\pi$ where $\frac{p}{q}$ is a rational number in the interval $(0, 1)$. One can find the same results in [22], [24] and [25].

Theorem 3.1. *If*

$$\lambda_n = \tilde{\lambda}_n, \qquad W(y_n, \tilde{y}_n)_{(a-0)} = 0$$

for any $n \in \mathbb{N}$ and $a \leq \frac{\pi}{2}$, then $q(x) = \tilde{q}(x)$ a.e. on $[0, a)$.

Remark 3.2. Let $\varphi(x, \lambda)$ and $\tilde{\varphi}(x, \lambda)$ are solutions of L and $\tilde{L}$ with the initial conditions $\varphi(0) = 1$, $\varphi'(0) = h$ and $\tilde{\varphi}(0) = 1$, $\tilde{\varphi}'(0) = h$ respectively. For $x \leq a$ the following representation holds (see [4], [19])

$$\varphi(x, \lambda) = \cos(\rho x) + \int_0^x k(x, t) \cos(\rho t) dt,$$

and

$$\tilde{\varphi}(x, \lambda) = \cos(\rho x) + \int_0^x \tilde{k}(x, t) \cos(\rho t) dt,$$

where $k(x,t)$ and $\tilde{k}(x,t)$ are continuous function not depending on λ. Hence

$$\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) = \frac{1}{2}\Big[1+\cos(2\rho x)+\int_0^x v(x,t)\cos(2\rho t)dt\Big] \tag{3.1}$$

where $v(x,t)$ is a continuous function which does not depend on λ. Fix $\delta > 0$ and define $G_\delta = \{\rho : \ |\rho - \rho_n| \geq \delta\}$. Then (see[18])

$$|\Delta(\lambda)| \geq C_1|\rho|^2 \exp(|\tau|\pi), \quad \rho \in G_\delta, \tag{3.2}$$

for some constant $C_1 > 0$.

Proof of Theorem 3.1. Let

$$-\varphi'' + q\varphi = \lambda\varphi, \tag{3.3}$$

and

$$-\tilde{\varphi}'' + \tilde{q}\tilde{\varphi} = \lambda\tilde{\varphi}. \tag{3.4}$$

If we multiply (3.3) by $\tilde{\varphi}$ and (3.4) by φ and subtract them, after integrating on $[0, a-0]$ we obtain

$$\begin{aligned} G(\lambda) &= \int_0^{a-0} [\tilde{q}(x) - q(x)]\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx \\ &= [\tilde{\varphi}'(x,\lambda)\varphi(x,\lambda) - \tilde{\varphi}(x,\lambda)\varphi'(x,\lambda)]|_0^{a-0}. \end{aligned} \tag{3.5}$$

The functions φ and $\tilde{\varphi}$ are satisfying in the initial condition (1.2); from this fact and $W(\tilde{\varphi}_n, \varphi_n)_{a-0} = 0$ it follows that

$$G(\lambda_n) = 0, \qquad n \in \mathbb{N}.$$

Next we shall show that $G(\lambda) = 0$ on the whole λ-plane. We know that

$$|\cos(2\rho x)| \leq \exp(2x|\tau|).$$

From (3.1) we obtain

$$|\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)| \leq C_2 \exp(2x|\tau|), \tag{3.6}$$

for some constant C_2. From (3.5) and (3.6) we see that the entire function $G(\lambda)$ satisfies

$$|G(\lambda)| \leq C \exp(2a|\tau|) \tag{3.7}$$

for some positive constant C and $|\lambda|$ large enough. Define

$$\phi(\lambda) := \frac{\big(G(\lambda)\big)^q}{\big(\Delta(\lambda)\big)^{2p}}.$$

The definition of $\Delta(\lambda)$, $G(\lambda)$, and $a \leq \frac{\pi}{2}$ implies $\phi(\lambda)$ is an entire function. It follows from (3.2) and (3.7) that

$$\phi(\lambda) = O\left(\frac{1}{|\lambda|^{4p}}\right)$$

for sufficiently large $|\lambda|$. Thus, $\phi(\lambda)$ is bounded and so Liouville's theorem implies

$$\phi(\lambda) = 0,$$

which is equivalent to

$$G(\lambda) = 0 \qquad \text{for all } \lambda. \tag{3.8}$$

Let

$$Q(x) = \tilde{q}(x) - q(x)$$

From (3.1), (3.5) and (3.8), we obtain, on the whole λ-plane,

$$\int_0^{a-0} Q(x)[1+\cos(2\rho x)]dx + \int_0^{a-0} Q(x)\Big[\int_0^x v(x,s)\cos(2\rho s)ds\Big]dx = 0,$$

which can be rewritten as

$$\int_0^{a-0} Q(x)dx + \int_0^{a-0} \cos(2\rho s)\Big[Q(s) + \int_s^{a-0} Q(x)v(x,s)dx\Big]ds = 0. \tag{3.9}$$

Letting $\lambda \to \infty$ for real λ in (3.9), by the Riemann–Lebesque lemma we see that

$$\int_0^{a-0} Q(x)dx = 0$$

and

$$\int_0^{a-0} \cos(2\rho s)\Big[Q(s) + \int_s^{a-0} Q(x)v(x,s)dx\Big]ds = 0$$

From the completeness of the function $\cos(2\rho s)$ on the interval $[0,a]$, we have

$$Q(s) + \int_s^{a-0} Q(x)v(x,s)dx = 0, \quad 0 < s < a$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus $Q(x) = 0$ on $0 < x < a$, that is, $q(x) = \tilde{q}(x)$ almost everywhere on $[0,a)$. □

Remark 3.3. If y and z satisfy the jump conditions (1.4) and $W(y,z)_{(a-0)} = 0$ then it is easy to verify that

$$W(y,z)_{(a+0)} = 0.$$

Theorem 3.4. *Let $a \in \left(\frac{\pi}{2}, \pi\right)$ be a jump point. Let $\lambda_n = \tilde{\lambda}_n$, and $W(y_n, \tilde{y}_n)_{(a-0)} = 0$, for each $n \in \mathbb{N}$. Then $q(x) = \tilde{q}(x)$ almost everywhere on $(a, \pi]$.*

Proof. We consider the supplementary problem $\widehat{L}$ by changing x by $\pi - x$. Let $t = \pi - x$ then from Eq. (1.1) we get

$$-y'' + q(\pi - t)y = \lambda y.$$

Define $q_1(t) = q(\pi - t)$ then the above equation has the following form

$$\widehat{\ell}y := -y'' + q_1(t)y = \lambda y, \qquad 0 < t < \pi$$
$$U(y) := Hy'(0) - \lambda y(0) = 0, \qquad V(y) := y'(\pi) + hy(\pi) = 0,$$

by discontinuous conditions

$$y(\pi - a + 0) = a_1^{-1} y(\pi - a - 0),$$
$$y'(\pi - a + 0) = {a_2}^{-1} y'(\pi - a - 0) + \frac{a_3}{a_1 a_2} y(\pi - a - 0).$$

For this part we obtain $\widehat{\Delta}(\lambda)$ by the similar form of (2.3). Let $\psi(t,\lambda)$ and $\tilde{\psi}(t,\lambda)$ are solutions of $\widehat{L}$ and $\widehat{\tilde{L}}$ with the initial conditions $\psi(0) = H,\ \psi'(0) = \lambda$ and $\tilde{\psi}(0) = H,\ \tilde{\psi}'(0) = \lambda$ respectively. For $t \leq a$ the following representation holds (see [4], [19])

$$\psi(t,\lambda) = \rho\Big[\sin(\rho t) + \int_0^t \widehat{k}(t,s)\sin(\rho s)ds\Big],$$

and

$$\tilde{\psi}(t,\lambda) = \rho\Big[\sin(\rho t) + \int_0^t \widehat{\tilde{k}}(t,s)\sin(\rho s)ds\Big],$$

where $\widehat{k}(t,s)$ and $\widehat{\tilde{k}}(t,s)$ are continuous function not depending on λ. Hence

$$\psi(t,\lambda)\tilde{\psi}(t,\lambda) = \frac{\rho^2}{2}\Big[1 - \cos(2\rho t) + \int_0^t \widehat{v}(t,s)\cos(2\rho s)ds\Big]$$

where $\widehat{v}(t,s)$ is a continuous function which does not depend on λ. From (3.2) we have

$$\Big|\widehat{\Delta}(\lambda)\Big| \geq C_1|\rho|^2\exp(|\tau|\pi), \quad \rho \in G_\delta, \tag{3.10}$$

for some constant $C_1 > 0$. Define

$$\widehat{G}(\lambda) = \frac{1}{\rho^2}\int_0^{\pi-a-0}[\tilde{q}(t) - q(t)]\psi(t,\lambda)\tilde{\psi}(t,\lambda)dt$$
$$= \frac{1}{\rho^2}[\tilde{\psi}'(t,\lambda)\psi(t,\lambda) - \tilde{\psi}(t,\lambda)\psi'(t,\lambda)]|_0^{\pi-a-0}.$$

Thus the initial condition (2.2) and $W(\tilde{\psi}_n, \psi_n)_{\pi-a-0} = 0$ implies

$$\widehat{G}(\lambda_n) = 0, \quad n \in \mathbb{N}.$$

Using asymptotic form of $\psi(t,\lambda)$ and $\tilde{\psi}(t,\lambda)$ there is some constant C such that

$$\Big|\widehat{G}(\lambda)\Big| \leq C\exp(2(\pi - a)|\tau|). \tag{3.11}$$

Define

$$\widehat{\phi}(\lambda) := \frac{\big(\widehat{G}(\lambda)\big)^q}{\big(\widehat{\Delta}(\lambda)\big)^{2(q-p)}}.$$

By applying the same method of proof of Theorem 3.1 and using (3.10) and (3.11) we deduce that $|\widehat{\phi}(\lambda)|$ is entire and bounded and so $\widehat{G}(\lambda) = 0$. Therefore, $q(x) = \tilde{q}(x)$ on $(a,\pi]$. $\square$

Let $l(n)$ be a subsequence of natural numbers such that

$$l(n) = \frac{n}{\sigma}(1+\epsilon_n), \quad 0<\sigma\le 1, \quad \epsilon_n \to 0$$

and let μ_n and $\tilde{\mu}_n$ be the eigenvalues of problems (3.3), (3) and (3.4), (3) with the jump conditions (1.4), respectively, and

$$H_1 y'(\pi) + \lambda y(\pi) = 0$$

where $H \ne H_1$.

Theorem 3.5. *Let $a \in (\frac{\pi}{2}, \pi]$ be a jump point and $\sigma > \frac{2a}{\pi} - 1$. Let $\lambda_n = \tilde{\lambda}_n$, $\mu_{l(n)} = \tilde{\mu}_{l(n)}$ and $W(y_n, \tilde{y}_n)_{(a-0)} = 0$, for each $n \in \mathbb{N}$. Then $q(x) = \tilde{q}(x)$ almost everywhere on $[0,a) \cup (a,\pi]$.*

Proof. Let $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ be the eigenfunctions of $L(q(x);h;H;a)$ and $L(\tilde{q}(x);h;H;a)$ corresponding to the eigenvalues λ_n, respectively. From the fact that the eigenfunctions $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ have the same boundary condition at point π and Theorem 3.4 we conclude that $q(x) = \tilde{q}(x)$ almost everywhere on $(a,\pi]$, we obtain

$$\tilde{y}_n(x,\lambda_n) = a_n y_n(x,\lambda_n), \quad x \in (a,\pi], \quad n \in \mathbb{N}, \tag{3.12}$$

where a_n is constant. From (3.5), (3.12), (1.2)–(1.4) and assumptions we get

$$G(\lambda_n) = 0, \qquad G(\mu_{l(n)}) = 0.$$

Next, we prove $G(\lambda) = 0$, for all $\lambda \in \mathbb{C}$. From (3.1) and (3.5) we see that the entire function $G(\lambda)$ is a function of exponential type and

$$G(\lambda) \le M \exp(2ar|\sin\theta|) \tag{3.13}$$

where M is a positive number and $\mathrm{Im}\lambda = r\sin\theta$. Define the indicator of function $G(\lambda)$ by

$$h(\theta) = \limsup_{\lambda\to+\infty} \frac{\ln|G(r\exp(i\theta))|}{r}. \tag{3.14}$$

By applying (3.13) and (3.14) and using $|\mathrm{Im}\lambda| = r|\sin\theta|$ for which $\theta = \arg\lambda$, we obtain

$$h(\theta) = 2a|\sin\theta|. \tag{3.15}$$

Let $n(r)$ be the number of zeros of $G(\lambda)$ in the disk $|\lambda| \le r$. From Lemma 2.6 and Theorem 2.7 we see that there are $1 + 2r[1+o(1)]$ of λ_n and $1 + 2r\sigma[1+o(1)]$ of $\mu_{l(n)}$ located inside the disc of radius r (for sufficiently large r). Therefore

$$n(r) = 2 + 2r[1+\sigma+o(1)].$$

Hence

$$\lim_{n\to\infty} \frac{n(r)}{r} = 2(\sigma+1).$$

Using the condition $\sigma > \frac{2a}{\pi} - 1$ and from (3.15), we get

$$\lim_{n\to\infty} \frac{n(r)}{r} \geq 2(\sigma+1) > \frac{4a}{\pi} \geq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.16}$$

According to [20], for any entire function $G(\lambda)$ of exponential type, not identically zero, we see that the following inequality holds

$$\liminf_{n\to\infty} \frac{n(r)}{r} \leq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.17}$$

Also from the inequalities (3.16) and (3.17) the relation (3.8) holds. By applying the same method of the proof of Theorem 3.1, we obtain $q(x) = \tilde{q}(x)$ almost everywhere on $[0, a)$. □

Let $m(n)$ and $r(n)$ be subsequences of natural numbers such that

$$m(n) = \frac{n}{\sigma_1}(1+\epsilon_{1n}), 0 < \sigma_1 \leq 1, \epsilon_{1n} \to 0$$

and

$$r(n) = \frac{n}{\sigma_2}(1+\epsilon_{2n}), 0 < \sigma_2 \leq 1, \epsilon_{2n} \to 0.$$

Corollary 3.6. *Let $a \in (0, \frac{\pi}{2}]$ be a jump point and fix $\sigma_1 > \frac{2a}{\pi}$. Let $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ for each $n \in \mathbb{N}$ and $W(y_{m(n)}, \tilde{y}_{m(n)})_{(a-0)} = 0$. Then $q(x) = \tilde{q}(x)$ almost everywhere on $[0, a)$.*

Proof. Using the similar proof of Theorems 3.1 and 3.5 we obtained easily the result of this corollary. □

Corollary 3.7. *Let $a \in (\frac{\pi}{2}, \pi)$ be a jump point, fix $\sigma > \frac{2a}{\pi} - 1$ and $\sigma_2 > 2 - \frac{2a}{\pi}$. If for each $n \in \mathbb{N}$*

$$\lambda_n = \tilde{\lambda}_n, \qquad \mu_{l(n)} = \tilde{\mu}_{l(n)}, \qquad W(y_{r(n)}, \tilde{y}_{r(n)})_{(a-0)} = 0,$$

then $q(x) = \tilde{q}(x)$ almost everywhere on $[0, \pi]$.

Proof. Using the similar proof of Theorems 3.1, 3.5 and Corollary 3.6 we obtained easily the result of this corollary. □

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