Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **6**(2)(2017), 62-76

Existence and Uniqueness of a Certain Type of Subdirect Product

H. Khabazian, Department of Mathematical Science
Isfahan University of Technology, Isfahan, Iran [1]

Abstract. We introduce a "$l\mathbb{F}$-type subdirect product" and show that every ring is uniquely a $l\mathbb{F}$-type subdirect product of a family of $l\mathbb{C}$-simple rings. Then,, we show some applications in maximal ring of quotients and Martindale's ring of quotients.

Keywords: $l\mathbb{F}$-type subdirect product, ring of quotients, $l\mathbb{C}$-ideal, $l\mathbb{C}$-simple, $r\mathbb{A}\mathbb{I}$-semiprime.

2010 Mathematics subject classification: 16S60, 16D25

Introduction

Any ring can be written as subdirect product of a family of rings in many ways. For any set S of ideals of a ring R with zero intersection, R is isomorpic to a subdirect product of the family $\{R/I \mid I \in S\}$. But this representation is not unique, even if S is a set of certain ideals, unless certain types of subdirects is considered.

[1]Corresponding author: khabaz@cc.iut.ac.ir
Received: 26 August 2016
Revised: 13 November 2016
Accepted: 3 December 2016

In section 1 standard facts are collected and $l\mathbb{C}$-simple rings are introduced. Then in section 2, $l\mathbb{F}$-type subdirect product is introduced and we show that every left faithful $l\mathbb{A}\mathbb{I}$-Noetherian ring is isomorphic to a $l\mathbb{F}$-type subdirect product of a finite family of $l\mathbb{C}$-simple rings and this representation is unique. In section 3, applying maximal ring of quotients, we show some applications.

In this paper, for any set S of subgroups of an additive group we set $\Sigma(S) = \sum_{I\in S} I$ and S is called independent if $\sum_{I\in S} I$ is a direct sum.
For any class $\mathcal{C}$ of subgroups, a $\mathcal{C}$-subgroup means a subgroup from the class $\mathcal{C}$, the class of minimal $\mathcal{C}$-subgroups is shown by $\mathcal{C}^{\mathrm{mn}}$ and the class of maximal $\mathcal{C}$-subgroups is shown by $\mathcal{C}^{\mathrm{mx}}$.
For classes $\mathcal{C}$ and $\mathcal{F}$ of subgroups, the class of subgroups which are a $\mathcal{C}$-subgroup and a $\mathcal{F}$-subgroup is shown by $\mathcal{C} \cap \mathcal{F}$.
For an additive group M, the set of $\mathcal{C}$-subgroups of M is shown by $\langle \mathcal{C} \colon M \rangle$, M is called **$\mathcal{C}$-simple** if it has no proper $\mathcal{C}$-subgroup, M is called **$\mathcal{C}$-Artinian** if M satisfies the descending chain condition on $\mathcal{C}$-subgroups, in other words, if whose $\mathcal{C}$-subgroups satisfy DCC, M is called **$\mathcal{C}$-Noetherian** if M satisfies the ascending chain condition on $\mathcal{C}$-subgroups, in other words, if whose $\mathcal{C}$-subgroups satisfy ACC, and M is called **$\mathcal{C}$-ind.finite** if every independent set of $\mathcal{C}$-subgroups is finite. For an additive group M and $K \subseteq M$, the set of $\mathcal{C}$-subgroups of M containing K is shown by $\langle \mathcal{C} \supseteq K \rangle$ and we say that K is **completely $\mathcal{C}$-uniform** if $K \neq 0$ and for any $\mathcal{C}$-subgroups I and L, $I \cap L = 0$ implies $K \cap I = 0$ or $K \cap L = 0$. Finally, the class of completely $\mathcal{C}$-uniform subgroups is shown by $\mathcal{C}^{\mathrm{cu}}$.
For any family S of subsets of a set, we set $\mathrm{Int}(S) = \bigcap_{I\in S} I$ and $\mathrm{Un}(S) = \bigcup_{I\in S} I$.

1 Preliminaries

Definition 1.1 Let R ba a ring and $\mathcal{C}$ be a class of subgroups.

1. R is said to be **$\mathcal{C}$-semiprime** if there exists no nonzero nilpotent $\mathcal{C}$-subgroup.
2. R is said to be **$\mathcal{C}$-prime** if for every $\mathcal{C}$-subgroups I and J, $IJ = 0$ implies either $I = 0$ or $J = 0$.

Definition 1.2 Let R ba a ring.

1. $I \subseteq R$ is said to be **left inner faithful** if $I \cap \mathrm{ann_l}(I) = 0$.
2. $I \subseteq R$ is called a **left annihilator** if $\mathrm{ann_l}(\mathrm{ann_r}(I)) = I$.
3. An ideal which is a left annihilator is called a **left annihilator ideal**.
4. For the subgroups U and I, we set $(U\colon I)_\mathrm{l} = \{x \in R \mid xI \subseteq U\}$.

In this paper, in the category of rings, the class of left inner faithful subgroups is shown by $\mathrm{l}\mathbb{IF}$, the class of left inner faithful ideals I for which $\mathrm{ann_l}(\mathrm{ann_l}(I)) = I$ is shown by $\mathrm{l}\mathbb{C}$ and the class of left inner faithful subgroups I for which $\mathrm{ann_l}(I)$ is also left inner faithful is shown by $\mathrm{dl}\mathbb{IF}$. Also the class of ideals is shown by $\mathbb{I}$, the class of left ideals is shown by $\mathrm{l}\mathbb{I}$, the class of right ideals is shown by $\mathrm{r}\mathbb{I}$, the class of left annihilator ideals is shown by $\mathrm{l}\mathbb{AI}$, the class of right annihilator ideals is shown by $\mathrm{r}\mathbb{AI}$ and the class of left faithful subgroups is shown by $\mathrm{l}\mathbb{F}$. Thus, $\mathrm{dl}\mathbb{IF}$-ideals means a left inner faithful ideal I for which $\mathrm{ann_l}(I)$ is also left inner faithful.
Recall that according to the terminologies used in this paper, $\mathbb{I} \cap \mathrm{l}\mathbb{IF}$ indicates the class of left inner faithful ideals, $\mathrm{l}\mathbb{C}^{\mathrm{mn}}$ is the class of minimal $\mathrm{l}\mathbb{C}$-ideals, and $\Sigma\langle \mathrm{l}\mathbb{C}^{\mathrm{mn}}\colon R\rangle$ is the sum of the minimal $\mathrm{l}\mathbb{C}$-ideals of R.

It is good to know that since every $\mathrm{l}\mathbb{C}$-ideal is a left annihilator ideal, we have

$$\mathbb{I}-\text{Noetherian} \Rightarrow \mathrm{r}\mathbb{AI}-\text{Noetherian} \Rightarrow \mathrm{l}\mathbb{AI}-\text{Artinian} \Rightarrow \mathrm{l}\mathbb{C}-\text{Artinian}$$

$$\mathbb{I}-\text{Noetherian} \Rightarrow \mathrm{l}\mathbb{AI}-\text{Noetherian} \Rightarrow \mathrm{l}\mathbb{C}-\text{Noetherian}$$

Lemma 1.3 Let R be a ring. If S is an independent set of right ideals and I is a left annihilator, then $I \cap \Sigma(S) = \Sigma\{I \cap J \mid J \in S\}$.

proof. We may assume that S has only two elements A and B. Suppose $v \in I \cap (A \oplus B)$. There exist $a \in A$ and $b \in B$ with $v = a + b$. For every $x \in \mathrm{ann_r}(I)$ we have $ax + bx = vx = 0$, implying $ax = 0$. Thus, $a\mathrm{ann_r}(I) = 0$, implying $a \in \mathrm{ann_l}(\mathrm{ann_r}(I)) = I$. Consequently $a \in I \cap A$. Similarly, $b \in I \cap B$. ■

Lemma 1.4 Let R be a ring. For any independent set S of left inner faithful ideals, $\Sigma(S)$ is also a left inner faithful ideal.

proof. For every $J \in S$ we have $\mathrm{ann_l}(\Sigma(S)) \subseteq \mathrm{ann_l}(J)$, implying $\mathrm{ann_l}(\Sigma(S)) \cap J = 0$. Thus by Lemma 1.3,

$$\mathrm{ann_l}(\Sigma(S)) \cap \Sigma(S) = \Sigma\{\mathrm{ann_l}(\Sigma(S)) \cap J \mid J \in S\} = 0$$

Lemma 1.5 Let R be a ring. Any set S of left annihilator ideals in which for every distinct elements $I, J \in S$ we have $I \cap J = 0$, is independent.

proof. Let $I \in S$. Set $T = S - \{I\}$. We have $I \cap \Sigma(T) = \Sigma(\{I \cap J \mid J \in T\}) = 0$ by Lemma 1.3. ■

Lemma 1.6 Let R be a ring.

1. An ideal K is a dl$\mathbb{IF}$-ideal iff $\mathrm{ann_l}(K)$ is a left inner faithful ideal and $K \subseteq \mathrm{ann_l}(\mathrm{ann_l}(K))$.

2. If I is a dl$\mathbb{IF}$-ideal, then $\mathrm{ann_l}(I)$ is a l$\mathbb{C}$-ideal.

3. If I and J are left inner faithful ideals, then so are IJ and $I \cap J$. Also we have $\mathrm{ann_l}(IJ) = \mathrm{ann_l}(I \cap J)$.

4. If I and J are dl$\mathbb{IF}$-ideals, then so is $I + J$.

5. If I and J are l$\mathbb{C}$-ideals, then so is $I \cap J$.

proof. (1 and 2) Straightforward.
(3) It is easy to see that $\mathrm{ann_l}(IJ) \cap J \subseteq \mathrm{ann_l}(I)$. Thus $\mathrm{ann_l}(IJ) \cap (I \cap J) = 0$, implying $\mathrm{ann_l}(I \cap J) \cap (I \cap J) = 0$, $\mathrm{ann_l}(IJ) \cap (IJ) = 0$ and $\mathrm{ann_l}(IJ) = \mathrm{ann_l}(I \cap J)$.
(4) Since $\mathrm{ann_l}(I + J) = \mathrm{ann_l}(I) \cap \mathrm{ann_l}(J)$, $\mathrm{ann_l}(I + J)$ is a l$\mathbb{IF}$-ideal by (3). On the other hand $I \subseteq \mathrm{ann_l}(\mathrm{ann_l}(I)) \subseteq \mathrm{ann_l}(\mathrm{ann_l}(I + J))$ and similarly, $J \subseteq \mathrm{ann_l}(\mathrm{ann_l}(I + J))$, implying $I + J \subseteq \mathrm{ann_l}(\mathrm{ann_l}(I + J))$. Applying (1) completes the proof.
(5) $\mathrm{ann_l}(I) + \mathrm{ann_l}(J)$ is a dl$\mathbb{IF}$-ideal by (2) and (4). Thus $\mathrm{ann_l}(\mathrm{ann_l}(I) + \mathrm{ann_l}(J))$ is a l$\mathbb{C}$-ideal by (2). On the other hand,

$$\mathrm{ann_l}(\mathrm{ann_l}(I) + \mathrm{ann_l}(J)) = \mathrm{ann_l}(\mathrm{ann_l}(I)) \cap \mathrm{ann_l}(\mathrm{ann_l}(J)) = I \cap J$$

Lemma 1.7 Let R be a ring and U be a l$\mathbb{C}$-ideal.

1. If I is a left inner faithful ideal, then so is I/U, also we have $\mathrm{ann_l}(I/U) = \mathrm{ann_l}(I \cap \mathrm{ann_l}(U))/U$.

2. If I is a dl$\mathbb{IF}$-ideal, then so is I/U, also we have $\mathrm{ann_l}(\mathrm{ann_l}(I/U)) = \mathrm{ann_l}(\mathrm{ann_l}(I \cap \mathrm{ann_l}(U)) \cap \mathrm{ann_l}(U))/U$.

Proof. (1) Set $M = \mathrm{ann_l}(U)$. M is a left inner faithful ideal and $\mathrm{ann_l}(M) = U$. So, IM is a left inner faithful ideal and

$$\mathrm{ann_l}(I/U) = \mathrm{ann_l}(I/\mathrm{ann_l}(M)) = \mathrm{ann_l}(IM)/\mathrm{ann_l}(M) =$$

$$\mathrm{ann_l}(IM)/U = \mathrm{ann_l}(I \cap M)/U$$

by Lemma 1.6. Thus, $(I \cap \mathrm{ann_l}(IM))M \subseteq IM \cap \mathrm{ann_l}(IM) = 0$, so $(I \cap \mathrm{ann_l}(IM)) \subseteq \mathrm{ann_l}(M)$, implying $(I/U) \cap \mathrm{ann_l}(I/U) = 0$.
(2) I/U is a left inner faithful ideal and $\mathrm{ann_l}(I/U) = \mathrm{ann_l}(I \cap \mathrm{ann_l}(U))/U$ by (1). On the other hand $I \cap \mathrm{ann_l}(U)$ is a dl$\mathbb{IF}$-ideal by Lemma 1.6, so $\mathrm{ann_l}(I \cap \mathrm{ann_l}(U))$ is a left inner faithful ideal, thus $\mathrm{ann_l}(I \cap \mathrm{ann_l}(U))/U$ is a left inner faithful ideal. The rest is clear by (1). ■

Lemma 1.8 Let R be a ring, U be a l$\mathbb{C}$-ideal and I be an ideal containing U.

1. I is a left inner faithful ideal iff I/U is so. In this case $\mathrm{ann_l}(I \cap \mathrm{ann_l}(U) + U) = \mathrm{ann_l}(I)$.
2. I is a dl$\mathbb{IF}$-ideal iff I/U is so. In this case, $\mathrm{ann_l}(\mathrm{ann_l}(I/U)) = \mathrm{ann_l}(\mathrm{ann_l}(I))/U$.
3. I is a l$\mathbb{C}$-ideal iff I/U is so.

Proof. (1$\Rightarrow$) Follows from Lemma 1.7.
(1$\Leftarrow$) We have $(U : I)_l \cap I \subseteq U$, thus $\mathrm{ann_l}(I) \cap I \subseteq \mathrm{ann_l}(U) \cap U = 0$.
Set $J = \mathrm{ann_l}(I \cap \mathrm{ann_l}(U) + U)$. $I \cap \mathrm{ann_l}(U) + U$ is a left inner faithful ideal by Lemma 1.6 and Lemma 1.5, so $(I \cap \mathrm{ann_l}(U) + U) \cap J = 0$, implying $I \cap \mathrm{ann_l}(U) \cap J = 0$ and $U \cap J = 0$. Thus $J \subseteq \mathrm{ann_l}(U)$, so $I \cap J = 0$, implying $J \subseteq \mathrm{ann_l}(I)$. On the other hand, it is clear tat $\mathrm{ann_l}(I) \subseteq J$.
(2$\Rightarrow$) Follows from Lemma 1.7.
(2$\Leftarrow$) I is a left inner faithful ideal, so $\mathrm{ann_l}(I/U) = \mathrm{ann_l}(I \cap \mathrm{ann_l}(U))/U$ by Lemma 1.7, thus $\mathrm{ann_l}(I \cap \mathrm{ann_l}(U))$ is a left inner faithful ideal by (1). Consequently, $I \cap \mathrm{ann_l}(U)$ is a dl$\mathbb{IF}$-ideal, implying that $A = U + I \cap \mathrm{ann_l}(U)$ is a dl$\mathbb{IF}$-ideal by Lemma 1.6. On the other hand $\mathrm{ann_l}(A) = \mathrm{ann_l}(I)$ by (1), so $\mathrm{ann_l}(I)$ is a left inner faithful ideal. Therefore I is a dl$\mathbb{IF}$-ideal. The rest can be proven easily by applying (1) and Lemma 1.7.
(3) Follows from (2). ■

Corollary 1.9 Let R be a ring and U be a $\mathrm{l}\mathbb{C}$-ideal. U is a maximal $\mathrm{l}\mathbb{C}$-ideal iff R/U is a $\mathrm{l}\mathbb{C}-$simple ring.

Lemma 1.10 Let R be a ring.

1. If S is a set of maximal $\mathrm{l}\mathbb{C}$-ideals with $\mathrm{Int}(S) = 0$, then $S = \langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle$.

2. $\langle \mathrm{l}\mathbb{C}^{\mathrm{mn}} \colon R \rangle$ is an independent set.

Proof. (1) Let U be a maximal $\mathrm{l}\mathbb{C}$-ideal. There exists $V \in S$ with $\mathrm{ann}_{\mathrm{l}}(U) \cap \mathrm{ann}_{\mathrm{l}}(V) \neq 0$, then $\mathrm{ann}_{\mathrm{l}}(U) = \mathrm{ann}_{\mathrm{l}}(V)$, implying $U = V$. Thus $U \in S$.
(2) Follows from Lemma 1.5. ∎

Lemma 1.11 Let R be a ring with $\mathrm{Int}(\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle) = 0$.

1. Every nonzero $\mathrm{l}\mathbb{C}$-ideal contains a minimal $\mathrm{l}\mathbb{C}$-ideal.

2. Every proper $\mathrm{l}\mathbb{C}$-ideal is contained in a maximal $\mathrm{l}\mathbb{C}$-ideal.

3. For every maximal $\mathrm{l}\mathbb{C}$-ideal K, $\mathrm{ann}_{\mathrm{l}}(K) = \mathrm{Int}(\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle - \{K\})$.

Proof. (1) Let K be a $\mathrm{l}\mathbb{C}$-ideal containing no minimal $\mathrm{l}\mathbb{C}$-ideal. For every maximal $\mathrm{l}\mathbb{C}$-ideal J we have $K \cap \mathrm{ann}_{\mathrm{l}}(J) = 0$ by Lemma 1.6, implying $K \subseteq \mathrm{ann}_{\mathrm{l}}(\mathrm{ann}_{\mathrm{l}}(J)) = J$. Thus $K \subseteq \mathrm{Int}(\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle) = 0$.
(2) Let J be a proper $\mathrm{l}\mathbb{C}$-ideal. $\mathrm{ann}_{\mathrm{l}}(J)$ is a nonzero $\mathrm{l}\mathbb{C}$-ideal, so contains a minimal $\mathrm{l}\mathbb{C}$-ideal I by (1). Then $J = \mathrm{ann}_{\mathrm{l}}(\mathrm{ann}_{\mathrm{l}}(J)) \subseteq \mathrm{ann}_{\mathrm{l}}(I)$, on the other hand $\mathrm{ann}_{\mathrm{l}}(I)$ is a maximal $\mathrm{l}\mathbb{C}$-ideal.
(3) $\mathrm{ann}_{\mathrm{l}}(K)$ is a minimal $\mathrm{l}\mathbb{C}$-ideal, so for every $K \neq L \in \langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle$, $\mathrm{ann}_{\mathrm{l}}(K) \cap L \neq 0$, implying $\mathrm{ann}_{\mathrm{l}}(K) \subseteq L$. So, $\mathrm{ann}_{\mathrm{l}}(K) \subseteq \mathrm{Int}(\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle - \{K\})$. On the other hand $K \cap \mathrm{Int}(\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle - \{K\}) = 0$, implying $\mathrm{Int}(\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon R \rangle - \{K\}) \subseteq \mathrm{ann}_{\mathrm{l}}(K)$. ∎

Proposition 1.12 Let R be a left faithful ring. The following are equivalent.

1. R is $\mathrm{l}\mathbb{C}-$Artinian.

2. R is $\mathrm{l}\mathbb{C}-$Noetherian.

3. R is $\mathrm{l}\mathbb{C}-$ind.finite.

4. $\langle \mathrm{l}\mathbb{C}^{\mathrm{mn}} \colon R \rangle$ is finite and every nonzero $\mathrm{l}\mathbb{C}$-ideal contains a minimal $\mathrm{l}\mathbb{C}$-ideal.

5. $\langle \mathrm{l}\mathbb{C} \colon R \rangle$ is finite.

6. $\langle \mathrm{l}\mathbb{C}^{\mathrm{mn}}\colon R\rangle$ is finite and $\Sigma\langle \mathrm{l}\mathbb{C}^{\mathrm{mn}}\colon R\rangle$ is left faithful.

7. $\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}}\colon R\rangle$ is finite and $\mathrm{Int}\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}}\colon R\rangle = 0$.

In this case for any l$\mathbb{C}$-ideal K we have $K = \mathrm{Int}\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \supseteq K\rangle$.

Proof. (1$\Leftrightarrow$2) Follows from Lemma 1.6.
(2$\Rightarrow$3) Temporarily suppose that there exists an infinite independent set of l$\mathbb{C}$-ideals. Then there exists an infinite independent set $\{J_i \mid i \geq 1\}$ of l$\mathbb{C}$-ideals. Set $N_n = \mathrm{ann_l}(\mathrm{ann_l}(\sum_{i=1}^n J_i))$. $\sum_{i=1}^n J_i$ is a dlI$\mathbb{F}$-ideal by Lemma 1.6, so N_n is a l$\mathbb{C}$-ideal by Lemma 1.6. Also, $\sum_{i=1}^n J_i \subseteq N_n$ by Lemma 1.6. On the other hand we have $J_{n+1} \cap \sum_{i=1}^n J_i = 0$, then $J_{n+1} \subseteq \mathrm{ann_l}(\sum_{i=1}^n J_i)$, so $N_n \subseteq \mathrm{ann_l}(J_{n+1})$, implying $N_n \cap J_{n+1} = 0$. Thus $N_n \subset N_{n+1}$ for all $n \geq 1$ which is a contradiction.
(3$\Rightarrow$2) Temporarily suppose that there exists a set $\{N_n \mid n \geq 1\}$ of l$\mathbb{C}$-ideals with $N_n \subset N_{n+1}$ for all $n \geq 1$. Set $J_n = N_{n+1} \cap \mathrm{ann_l}(N_n)$. J_n is a l$\mathbb{C}$-ideal by Lemma 1.6. On the other hand $J_i \subseteq N_{i+1} \subseteq N_{n+1}$ for all $1 \leq i \leq n$, implying $\sum_{i=1}^n J_i \subseteq N_{n+1}$. Furthermore $J_{n+1} \subseteq \mathrm{ann_l}(N_{n+1})$, implying $J_{n+1} \cap \sum_{i=1}^n J_i = 0$. Thus $\{J_n \mid n \geq 1\}$ is an infinite independent set of l$\mathbb{C}$-ideals which is a contradiction.
(1 and 3$\Rightarrow$4) Follows from Lemma 1.11.
(4$\Rightarrow$5 and 6) We show that for any l$\mathbb{C}$-ideal K we have $K = \mathrm{Int}\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \supseteq K\rangle$. Set $J = \mathrm{Int}\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \supseteq K\rangle$. J is a l$\mathbb{C}$-ideal by Lemma 1.6, so it is enough to show that $J \cap \mathrm{ann_l}(K) = 0$. Temporarily suppose that $J \cap \mathrm{ann_l}(K) \neq 0$. $J \cap \mathrm{ann_l}(K)$ contains a minimal l$\mathbb{C}$-ideal I by Lemma 1.11, then $K \subseteq \mathrm{ann_l}(I)$, so $\mathrm{ann_l}(I) \in \langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \supseteq K\rangle$, thus $J \subseteq \mathrm{ann_l}(I)$, implying $I \subseteq \mathrm{ann_l}(J)$ which is a contradiction.
(5$\Rightarrow$1 and 2) It is obvious.
(6$\Rightarrow$4) Straightforward.
(6$\Leftrightarrow$7) Follows from Lemma 1.6. ■

Lemma 1.13 Let R ba a ring, U be a left annihilator ideal and $I \subseteq R$.

1. $(U\colon I)_{\mathrm{l}} = \mathrm{ann_l}(\mathrm{ann_l}(I\mathrm{ann_r}(U)))$.

2. If $U \subseteq I$ and I/U is a left annihilator, then so is I.

Proof. (1) We have $(U : I)_{\mathrm{l}} I \subseteq U$, so $(U : I)_{\mathrm{l}} I\mathrm{ann_r}(U) = 0$, implying $(U : I)_{\mathrm{l}} \subseteq \mathrm{ann_l}(I\mathrm{ann_r}(U))$. On the other hand $\mathrm{ann_l}(I\mathrm{ann_r}(U))I\mathrm{ann_r}(U) = 0$, so

$\mathrm{ann_l}(I\mathrm{ann_r}(U))I \subseteq \mathrm{ann_l}(\mathrm{ann_r}(U)) = U$, implying $\mathrm{ann_l}(I\mathrm{ann_r}(U)) \subseteq (U\!:\!I)_\mathrm{l}$.
(2) Setting $N = (U\!:\!I)_\mathrm{r}$ we have, $\mathrm{ann_r}(I/U) = N/U$ so,

$$I/U = \mathrm{ann_l}(\mathrm{ann_r}(I/U)) = \mathrm{ann_l}(N/U) = (U\!:\!N)_\mathrm{l}/U$$

implying $I = (U\!:\!N)_\mathrm{l} = \mathrm{ann_l}(N\mathrm{ann_r}(U))$ by (1). ■

Proposition 1.14 Let R be a l$\mathbb{AI}$–semiprime ring. For every l$\mathbb{C}$-ideal U, R/U is also a l$\mathbb{AI}-$semiprime ring.

Proof. Let I be an ideal containing U such that I/U is a nilpotent left annihilator ideal. There exists $n \geq 1$ with $(I/U)^n = 0$, implying $I^n \subseteq U$. I is a left annihilator ideal by Proposition 1.12, so is $I \cap \mathrm{ann_l}(U)$. On the other hand, $(I \cap \mathrm{ann_l}(U))^n \subseteq I^n \subseteq U$ and $(I \cap \mathrm{ann_l}(U))^n \subseteq \mathrm{ann_l}(U)^n \subseteq \mathrm{ann_l}(U)$, so $(I \cap \mathrm{ann_l}(U))^n \subseteq U \cap \mathrm{ann_l}(U) = 0$, thus $I \cap \mathrm{ann_l}(U) = 0$, implying $I \subseteq \mathrm{ann_l}(\mathrm{ann_l}(U)) = U$. Thus $I/U = 0$. ■

Proposition 1.15 Let R be a semiprime ring. For every l$\mathbb{C}$-ideal U, R/U is also a semiprime ring.

Proof. Let I be an ideal containing U such that I/U is nilpotent. There exists $n \geq 1$ with $(I/U)^n = 0$, implying $I^n \subseteq U$. Thus $(I \cap \mathrm{ann_l}(U))^n \subseteq I^n \subseteq U$ and $(I \cap \mathrm{ann_l}(U))^n \subseteq \mathrm{ann_l}(U)^n \subseteq \mathrm{ann_l}(U)$, so $(I \cap \mathrm{ann_l}(U))^n \subseteq U \cap \mathrm{ann_l}(U) = 0$, consequently $I \cap \mathrm{ann_l}(U) = 0$, implying $I \subseteq \mathrm{ann_l}(\mathrm{ann_l}(U)) = U$. Thus $I/U = 0$.

Lemma 1.16 Any left faithful l$\mathbb{C}$-simple ring is indecomposable (as a ring).

Proof. Let R be l$\mathbb{C}$-simple ring and A and B be ideals with $R = A \oplus B$. We have $(B \cap \mathrm{ann_l}(B))R = (B \cap \mathrm{ann_l}(B))(A \oplus B) = 0$, so $B \cap \mathrm{ann_l}(B) = 0$. On the othe hand, $A \subseteq \mathrm{ann_l}(B)$, implying $\mathrm{ann_l}(B) = A$. Similarly, $\mathrm{ann_l}(A) = B$. Therefore, $A = 0$ or $B = 0$.

Lemma 1.17 Let R be a left faithful l$\mathbb{C}$-simple ring. . For every l$\mathbb{C}$-ideal U, $U + \mathrm{ann_l}(U)$ is a left faithful ideal and contains $\Sigma\langle \mathrm{r}\mathbb{I} \cap \mathrm{l}\mathbb{AI}^{\mathrm{cu}}\!: R\rangle$.

Proof. Let J be a completely l$\mathbb{AI}$-uniform right ideal. Then, $J \cap U = 0$ or $J \cap \mathrm{ann_l}(U) = 0$, so $JU = 0$ or $J\mathrm{ann_l}(U) = 0$, implying $J \subseteq \mathrm{ann_l}(U)$ or $J \subseteq \mathrm{ann_l}(\mathrm{ann_l}(U)) = U$. Thus, $J \subseteq U + \mathrm{ann_l}(U)$.

It worth to mention that every minimal right ideal, minimal ideal and minimal left annihilator ideal is a completely l$\mathbb{AI}$-uniform right ideal.

2 $\mathbb{lF}$-type subdirect product

Definition 2.1 We use the notation $S \subseteq_{\text{sd}} \prod_{i \in I} R_i$ to indicate that the ring S is a subdirect product of the family $\{R_i \mid i \in I\}$ of rings. In this case:

1. For every $J \subseteq I$ we set $S_J = \{u \in S \mid \forall i \notin J, \pi_i(u) = 0\}$
2. For every $j \in I$ we set $S_{*j} = \pi_j(S_j)$
3. If for every $j \in I$, S_{*j} is a $\mathcal{C}$-subgroup of R_j, then we say that S is a **$\mathcal{C}$-type** subdirect product of the family $\{R_i \mid i \in I\}$ of rings and we write $S \subseteq^{\mathcal{C}}_{\text{sd}} \prod_{i \in I} R_i$.

It is clear that $\iota_j(S_{*j}) = S_j$ for every $j \in I$ and $\sum_{i \in I} S_i = \bigoplus_{i \in I} S_{*i}$.

Lemma 2.2 Let $\{R_i \mid i \in I\}$ be a family of rings and $S \subseteq^{\mathrm{l}\mathbb{F}}_{\text{sd}} \prod_{i \in I} R_i$.

1. For every $J \subseteq I$, $\mathrm{ann_l}(S_J) = S_{I-J}$ and S_J is a $\mathrm{l}\mathbb{C}$-ideal.
2. If A is a left faithful left ideal of S, then $\pi_j(A)$ is left faithful for every $j \in I$.

Proof. (1) Straightforward.
(2) Set $I = \mathrm{ann_l}(\pi_j(A))$. We have $IS_{*j} \subseteq S_{*j}$, so $\iota_j(IS_{*j}) \subseteq S_j \subseteq S$. On the other hand, $\pi_j(\iota_j(IS_{*j})A) = IS_{*j}\pi_j(A) \subseteq I\pi_j(A) = 0$, also for every $j \neq i \in I$, $\pi_i(\iota_j(IS_{*j})A) = \pi_i(\iota_j(IS_{*j}))\pi_j(A) = 0$. So, $\iota_j(IS_{*j})A = 0$, thus $\iota_j(IS_{*j}) = 0$, hence $IS_{*j} = 0$, implying $I = 0$. ■

Lemma 2.3 Let $\{R_i \mid i \in I\}$ be a family of $\mathrm{l}\mathbb{C}-$simple rings and $S \subseteq^{\mathrm{l}\mathbb{F}}_{\text{sd}} \prod_{i \in I} R_i$. $\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon S \rangle = \{S_{I-\{j\}} \mid j \in I\}$ and $\langle \mathrm{l}\mathbb{C}^{\mathrm{mn}} \colon S \rangle = \{S_j \mid j \in I\}$.

Proof. Let $j \in I$. $S_{I-\{j\}}$ is a $\mathrm{l}\mathbb{C}$-ideal by Lemma 2.2. On the other hand $S/S_{I-\{j\}} \cong R_j$, thus $S/S_{I-\{j\}}$ is a $\mathrm{l}\mathbb{C}-$simple ring, consequently $S_{I-\{j\}}$ is a maximal $\mathrm{l}\mathbb{C}$-ideal by Corollary 1.9. Furthermore, $\mathrm{Int}(\{S_{I-\{j\}} \mid j \in I\}) = 0$. Therefore $\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}} \colon S \rangle = \{S_{I-\{j\}} \mid j \in I\}$ by Lemma 1.10. ■

Proposition 2.4 Let $\{R_i \mid i \in I\}$ be a family of rings and $S \subseteq^{\mathrm{l}\mathbb{F}}_{\text{sd}} \prod_{i \in I} R_i$.

1. If S is $\mathrm{l}\mathbb{A}\mathbb{I}-$semiprime, then R_j is $\mathrm{l}\mathbb{A}\mathbb{I}-$semiprime for each $j \in I$.

2. S is semiprime iff for each $j \in I$, R_j is semiprime.

Proof. (1) $S_{I-\{j\}}$ is a $\mathrm{l}\mathbb{C}$-ideal by Lemma 2.2, so $S/S_{I-\{j\}}$ is a $\mathrm{l}\mathbb{AI}-$semiprime ring by Proposition 1.14. On the other hand we have $R_j \cong S/S_{I-j}$.
(2$\Rightarrow$) $S_{I-\{j\}}$ is a $\mathrm{l}\mathbb{C}$-ideal by (2-2), so $S/S_{I-\{j\}}$ is a semiprime ring by Proposition 1.15. On the other hand we have $R_j \cong S/S_{I-\{j\}}$.
(2$\Leftarrow$). Let $K \subseteq S$ be a an ideal with zero square. Then for every $j \in I$, $\pi_j(K)^2 = \pi_j(K^2) = 0$ and $\pi_j(K)$ is an ideal, implying $\pi_j(K) = 0$. Thus $K = 0$.

Theorem 2.5 Let $\{R_i \mid i \in I\}$ and $\{Q_u \mid u \in A\}$ be a families of $\mathrm{l}\mathbb{C}-$simple left faithful rings, $S \subseteq^{\mathrm{l}\mathbb{F}}_{\mathrm{sd}} \prod\limits_{i\in I} R_i$ and $T \subseteq^{\mathrm{l}\mathbb{F}}_{\mathrm{sd}} \prod\limits_{u\in A} Q_u$. If $\phi : S \longrightarrow T$ is an isomorphism, then there exists a bijection $f : I \longrightarrow A$ and for each $i \in I$, an isomorphism $\phi_i : R_i \longrightarrow Q_{fi)}$ such that on S we have $\phi = \prod_{i\in I} \phi_i$.

Proof. Follows from Lemma 2.3 we have

$$\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}}\colon S\rangle = \{S_{I-\{j\}} \mid j \in I\} \text{ and } \langle \mathrm{l}\mathbb{C}^{\mathrm{mx}}\colon T\rangle = \{T_{A-\{u\}} \mid u \in A\}$$

So there is a bijection $f : I \longrightarrow A$ such that for each $i \in I$, $\phi(S_{I-\{i\}}) = T_{A-\{f(i)\}}$. The map $\phi_i : R_i \longrightarrow Q_{f(i)}$ defined by $\phi_i(\pi_i(x)) = \pi_{f(i)}(\phi(x))$ is well defined and an isomorphism. ■

Theorem 2.6 Let R be a nonzero ring with $\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}}\colon R\rangle = 0$. R is a $\mathrm{l}\mathbb{F}-$type subdirect product of a family of $\mathrm{l}\mathbb{C}-$simple rings and this representation is unique.

Proof. We may have $\langle \mathrm{l}\mathbb{C}^{\mathrm{mx}}\colon R\rangle = \{U_i \mid i \in I\}$ for some set I. Set $R_i = R/U_i$ and consider the homomorphism $\Phi : R \longrightarrow \prod_{i\in I} R_i$ given by $\Phi(r) = \{r + U_i \mid i \in I\}$. Now set $S = \Phi(R)$. Clearly Φ is a monomorphism. For each $j \in I$ we set $V_j = \cap_{j\neq i\in I} U_i$. We have

$$S_j = \{\iota_j(r + U_j) \mid r \in V_j\}$$

On the other hand we have $V_j = \mathrm{ann_l}(U_j)$ by Lemma 1.11. Thus $S_{*j} = \mathrm{ann_l}(U_j)/U_j$, implying $\mathrm{ann_l}(S_{*j}) = \mathrm{ann_l}(\mathrm{ann_l}(U_j))/U_j = 0$. Finally R_j is a $\mathrm{l}\mathbb{C}-$simple ring by Corollary 1.9. ■

Theorem 2.7 Any left faithful $\mathrm{l}\mathbb{C}$–Noetherian ring is isomorphic to a $\mathrm{l}\mathbb{F}$–type subdirect product of a finite family of $\mathrm{l}\mathbb{C}-$simple rings and this representation is unique.

Proof. Follows from Theorem 2.6 and Proposition 1.12. ■

Theorem 2.8 Any $l\mathbb{AI}-$semiprime $l\mathbb{C}$-Noetherian ring is isomorphic to a $l\mathbb{F}-$type subdirect product of a finite family of $l\mathbb{AI}-$prime rings and this representation is unique.

Proof. It is easy to see that a ring is $l\mathbb{AI}-$prime iff it is $l\mathbb{AI}-$semiprime and $l\mathbb{C}$-simple. Applying Theorem 2.7 and Lemma 2.4 completes the proof. ■

Theorem 2.9 Any semiprime $l\mathbb{C}$-Noetherian ring is isomorphic to a $l\mathbb{F}-$type subdirect product of a finite family of prime rings and this representation is unique.

Proof. It is easy to see that a ring is prime iff it is semiprime and $l\mathbb{C}$-simple. Applying Theorem 2.7 and Lemma 2.4 completes the proof. ■

3 Applications

In [3, (13.21)], it is shown that the maximal right ring of quotients of a left faithful ring R exists and is the unique ring extension Q of R such that

1. For every dense right ideal L of R and any R-homomorphism $f : L \longrightarrow R$ there exists $q \in Q$ such that $f(x) = qx$ for all $x \in L$.
2. For every $q \in Q$ there exists dense right ideal L of R with $qL \subseteq R$.
3. Every dense right ideal I of R is left Q-faithful ($q \in Q$, $qI = 0$ implies $q = 0$).

Also, this extension is shown by $\mathrm{Q}^{\mathrm{r}}_{\max}(R)$. Therefore, by this description, we have Proposition 3.5. But before that we need some lemmas.

For a ring R, $I \subseteq R$ and $l \in R$ we set $(I : l)_{\mathrm{r}} = \{r \in R \mid lr \in I\}$. Pay attention that for any $I, J \subseteq R$, $r(J : r)_{\mathrm{r}} \subseteq J$, so $r(J : r)_{\mathrm{r}}I \subseteq JI$, implying $(J : r)_{\mathrm{r}}I \subseteq (JI : r)_{\mathrm{r}}$. Thus, for any dense right ideal J and any left faithful right ideal I, JI is a dense right ideal. ■

Lemma 3.1 Let Q be a ring and R be a subring of Q such that every dense right ideal of R is left Q-faithful. For any right ideal $L \subseteq R$ and any R-homomorphism $f : L \longrightarrow Q$, if $f(I \cap L) = 0$ for a dense right ideal I of R, then $f = 0$.

Proof. For every $l \in L$, $f(l)(I\!:\!l)_{\mathrm{r}} = f(l(I\!:\!l)_{\mathrm{r}}) \subseteq f(I \cap L) = 0$, so $f(l) = 0$. ■

Lemma 3.2 Let R be a ring and S be a subring of R. If $N \subseteq S$ is a left R-faithful ideal of R, then for any dense right ideal $J$of S, JN is a dense right ideal of R.

Proof. Let $r \in R$. First we show that $(J:r)_{\mathrm{r}}$ is left R-faithful. Suppose $t(J\!:\!r)_{\mathrm{r}} = 0$ for a $t \in R$. Then, for every $u \in N$, $tu(J\!:\!ru)_{\mathrm{r}} = 0$, so $tu(S \cap (J\!: ru)_{\mathrm{r}}) = 0$, implying $tu = 0$, Thus, $tN = 0$, so $t = 0$. On the other hand $(J\!:\!r)_{\mathrm{r}}N \subseteq (JN\!:\!r)_{\mathrm{r}}$, so $(JN\!:\!r)_{\mathrm{r}}$ is left faithful. ■

Lemma 3.3 Let Q be a ring and R be a subring of Q such that for every $q \in Q$, there exists a left Q-faithful $L \subseteq R$ with $qL \subseteq R$. If R is l$\mathbb{C}$-simple, then so is Q.

Proof. Firs we show that for every right ideal A of Q, $\mathrm{ann}_{\mathrm{l}}(R \cap A) = \mathrm{ann}_{\mathrm{l}}(A)$ in Q. Let $q \in A$. There exists a left Q-faithful $L \subseteq R$ such that $qL \subseteq R$. Then, $qL \subseteq R \cap A$, so $\mathrm{ann}_{\mathrm{l}}(R \cap A)qL = 0$, implying $\mathrm{ann}_{\mathrm{l}}(R \cap A)q = 0$. Thus, $\mathrm{ann}_{\mathrm{l}}(R \cap A)A = 0$. It is easy to see that for every right ideal A of Q, $R \cap A = 0$ implies $A = 0$. Now let A and B be disjoit ideals of Q with $\mathrm{ann}_{\mathrm{l}}(A) = B$ and $\mathrm{ann}_{\mathrm{l}}(B) = A$. Set $I = R \cap A$ and $J = R \cap B$. I and J are disjoit ideals of R. Also in R, $\mathrm{ann}_{\mathrm{l}}(I) = R \cap B = J$ and $\mathrm{ann}_{\mathrm{l}}(J) = R \cap A = I$. Thus, $I = 0$ or $J = 0$, implying $A = 0$ or $B = 0$. ■

Lemma 3.4 Let $\{R_i \mid i \in I\}$ be a family of left faithful rings. Set $S = \prod_{i \in I} R_i$.

1. If K_j is a dense right ideal of R_j for each $j \in I$, then $\bigoplus_{i \in I} K_i$ is a dense right ideal of S.
2. If L is a dense right ideal of S, then L_{*j} and so $\pi_j(L)$ is a dense right ideal of R_j for each $j \in I$.

Proof. (1) Let $x, y \in S$ and $y(\bigoplus_{i \in I} K_i : x)_{\mathrm{r}} = 0$. For each $j \in I$ we have, $\iota_j((K_j\!:\!\pi_j(x))_{\mathrm{r}}) \subseteq (\bigoplus_{i \in I} K_i\!:\!x)_{\mathrm{r}}$, so $\pi_j(y)(K_j\!:\!\pi_j(x))_{\mathrm{r}} = 0$, implying $\pi_j(y) = 0$. Thus, $y = 0$.
(2) Let $x, y \in R_j$ and $y(L_{*j} : x)_{\mathrm{r}} = 0$. We have $(L : \iota_j(x))_{\mathrm{r}} = \prod_{i \in I} N_i$, where $N_j = (L_{*j} : x)_{\mathrm{r}}$ and $N_i = R_i$ for $j \neq i \in I$. Thus, $\iota_j(y)(L : \iota_j(x))_{\mathrm{r}} = 0$, so $\iota_j(y) = 0$, implying $y = 0$. ■

Proposition 3.5 Let R be a left faithful ring and S be a subring of R, If S contains a left R-faithful ideal of R, then $\mathrm{Q}^{\mathrm{r}}_{\max}(S) = \mathrm{Q}^{\mathrm{r}}_{\max}(R)$.

Proof. S contains a left R-faithful ideal N of R. Set $Q = \mathrm{Q}^{\mathrm{r}}_{\max}(R)$. First let J be a dense right ideal of S and $qJ = 0$ for a $q \in Q$. JN is a dense right ideal of S by Lemma 3.2 and $qJN = 0$, implying $q = 0$. Thus, every dense right ideal of S is left Q-faithful.
Now let J be a dense right ideal of S and $f : J \longrightarrow S$ be a S-homomorphism. Set $L = JN$, L is a dense right ideal of R contained in J by Lemma 3.2. Let $x \in L$ and $r \in R$. For every $u \in N$, $f(x)ru = f(xru) = f(xr)u$, Hence, $(f(x)r - f(xr))N = 0$, implying $f(x)r = f(xr)$. Thus, $f : L \longrightarrow S$ is an R-homomorphism, so there exists $q \in Q$ such that $f(x) = qx$ for all $x \in L$. The map $g : J \longrightarrow Q$ given by $g(x) = f(x) - qx$ is a S-homomorphism and $g(L) = 0$. On the other hand, L is dense right ideal of S, so $g = 0$ by Lemma 3.1. Thus, $f(x) = qx$ for all $x \in J$.
Finally, let $q \in Q$. There exists dense right ideal L of R with $qL \subseteq R$. Thus, $qLN \subseteq RN \subseteq S$. On the other hand, LN is a dense right ideal of R, so is a dense right ideal of S. ■

Lemma 3.6 Let $\{R_i \mid i \in I\}$ be a family of left faithful rings. Then, $\mathrm{Q}^{\mathrm{r}}_{\max}(\prod\limits_{i\in I} R_i) = \prod\limits_{i\in I}\mathrm{Q}^{\mathrm{r}}_{\max}(R_j)$.

Proof. Set $S = \prod\limits_{i\in I} R_i$ and $Q = \prod\limits_{i\in I}\mathrm{Q}^{\mathrm{r}}_{\max}(R_j)$. First, let L be a dense right ideal of S and $f : L \longrightarrow S$ be a S-homomorphism. Set $N = L(\sum_{i\in I} S_i)$. N is a dense right ideal of S and $N = \sum_{i\in I} LS_i$. Also, for every $j \in I$, $f(LS_j) = f(L)S_j \subseteq S_j$ and $\pi_j(LS_j)$ is a dense right ideal of R_j by Lemma 3.4. Consider the well defined R_j-homomorphism $f_j : \pi_j(LS_j) \longrightarrow R_j$ given by $f_j(\pi_j(x)) = \pi_j(f(x))$. There exists $q_j \in \mathrm{Q}^{\mathrm{r}}_{\max}(R_j)$ such that $f_j(\pi_j(x)) = q_j\pi_j(x)$ for all $x \in LS_j$. Set $q = \{q_i \mid i \in I\}$. Then, $f(x) = qx$ for all $x \in N$. Thus, $f(x) = qx$ for all $x \in L$ by Lemma 3.1.
Now let $q \in Q$. For each $j \in I$, there exists a dense right ideal $K_j \subseteq R_j$ such that $\pi_j(q)K_j \subseteq R_j$. Thus, $q\bigoplus\limits_{i\in I} K_i \subseteq S$. On the other hand, $\bigoplus\limits_{i\in I} K_i$ is a dense right ideal of S by Lemma 3.4.
Finally, let L be a dense right ideal of S, $q \in Q$ and $qL = 0$. Then, for each $j \in I$, $\pi_j(q)\pi_j(L) = 0$, implying $\pi_j(q) = 0$ because $\pi_j(L)$ is a dense right ideal by Lemma 3.4. Thus, $q = 0$. ■

Proposition 3.7 Let $\{R_i \mid i \in I\}$ be a family of rings and $S \subseteq^{\mathrm{l}\mathbb{F}}_{\mathrm{sd}} \prod_{i\in I} R_i$. Then, $\mathrm{Q}^{\mathrm{r}}_{\max}(S) = \prod_{i\in I} \mathrm{Q}^{\mathrm{r}}_{\max}(R_j)$.

Proof. We have $\bigoplus_{i\in I} S_{*j} = \sum_{j\in I} S_j$, so $\bigoplus_{i\in I} S_{*j}$ is a left faithful ideal of S and a left faithful ideal of $\prod_{i\in I} S_{*j}$. Thus,

$$\mathrm{Q}^{\mathrm{r}}_{\max}(S) = \mathrm{Q}^{\mathrm{r}}_{\max}(\bigoplus_{i\in I} S_{*j}) = \mathrm{Q}^{\mathrm{r}}_{\max}(\prod_{i\in I} S_{*j}) = \prod_{i\in I} \mathrm{Q}^{\mathrm{r}}_{\max}(S_{*j}) = \prod_{i\in I} \mathrm{Q}^{\mathrm{r}}_{\max}(R_j)$$

by Proposition 3.5 and Lemma 3.6. ■

In [3, (14.7)], Martindale's right rings of quotients is introduced for semiprime rings. We can extend this definition to any left faithful ring R as follow.

$$\mathrm{Q}^{\mathrm{r}}(R) = \{q \in \mathrm{Q}^{\mathrm{r}}_{\max}(R) \mid qA \subseteq R \text{ for some left faithful ideal } A\}$$

Proposition 3.8 Let $\{R_i \mid i \in I\}$ be a family of rings and $S \subseteq^{\mathrm{l}\mathbb{F}}_{\mathrm{sd}} \prod_{i\in I} R_i$. Then, $\mathrm{Q}^{\mathrm{r}}(S) = \prod_{i\in I} \mathrm{Q}^{\mathrm{r}}(R_i)$.

Proof. Let $q \in \mathrm{Q}^{\mathrm{r}}(S)$. Then $q \in \mathrm{Q}^{\mathrm{r}}_{\max}(S)$ and $qA \subseteq S$ for some left faithful ideal A of S. Let $j \in I$. $\pi_j(A)$ is a left faithful ideal of R_j by Lemma 2.2, on the other hand, $\pi_j(q)\pi_j(A) \subseteq R_j$, so $\pi_j(q) \in \mathrm{Q}^{\mathrm{r}}(R_j)$. Thus, $q \in \prod_{i\in I} \mathrm{Q}^{\mathrm{r}}(R_i)$. Now let $q \in \prod_{i\in I} \mathrm{Q}^{\mathrm{r}}(R_i)$. Let $j \in I$ and set $q_j = \pi_j(q)$. Then, $q_j \in \mathrm{Q}^{\mathrm{r}}_{\max}(R_j)$ and $q_j K_j \subseteq R_j$ for a left faithful ideal K_j of R_j. $K_i S_{*j}$ is also a left faithful ideal of R_j. and $q_j K_i S_{*j} \subseteq S_{*j}$. Thus, $A = \sum_{j\in I} \iota_j(K_j S_{*j})$ is a left faithful ideal of S and $qA \subseteq S$. Therefore, $q \in \mathrm{Q}^{\mathrm{r}}(S)$. ■

Corollary 3.9 For any $\mathrm{l}\mathbb{C}$-simple ring R, $\mathrm{Q}^{\mathrm{r}}_{\max}(R)$ and $\mathrm{Q}^{\mathrm{r}}(R)$ are also $\mathrm{l}\mathbb{C}$-simple.

Proof. Follows from Lemma 3.3. ■

In this order, applying Theorem 2.7, Proposition 3.7, Proposition 3.8, Corollary 3.9 and Lemma 1.16, provide us with the decomposition of the maximal right ring of quotients and Martindale's right ring of quotients of any left left faithful $\mathrm{l}\mathbb{C}$-Noetherian ring.

Theorem 3.10 Any left faithful $\mathrm{l}\mathbb{C}$-Noetherian ring R having no proper left faithful ideal which contains $\Sigma\langle \mathrm{r}\mathbb{I} \cap \mathrm{l}\mathbb{A}\mathbb{I}^{\mathrm{cu}} \colon R\rangle$ is isomorphic to a direct product of a finite family of $\mathrm{l}\mathbb{C}$-simple rings R_i with no proper left faithful ideal which contains $\Sigma\langle \mathrm{r}\mathbb{I} \cap \mathrm{l}\mathbb{A}\mathbb{I}^{\mathrm{cu}} \colon R_i\rangle$ and this representation is unique.

Proof. Let R be a left faithful $\mathrm{l}\mathbb{C}$-Noetherian ring with no proper left faithful ideal which contains $\Sigma\langle \mathrm{r}\mathbb{I} \cap \mathrm{l}\mathbb{A}\mathbb{I}^{\mathrm{cu}} \colon R\rangle$. We have $R \cong S \subseteq^{\mathrm{l}\mathbb{F}}_{\mathrm{sd}} \prod_{i=1}^{n} R_i$, where each R_i is a $\mathrm{l}\mathbb{C}$-simple ring by Theorem 2.7. On the other hand, for each $1 \leq j \leq n$, $S_{I-\{j\}} + S_j$ is a left faithful ideal and contains $\Sigma\langle \mathrm{r}\mathbb{I} \cap \mathrm{l}\mathbb{A}\mathbb{I}^{\mathrm{cu}} \colon S\rangle$ by Lemma 1.17, so $S = S_{I-\{j\}} + S_j$, implying $\pi_j(S_j) = R_j$. Thus, $S = \prod_{i=1}^{n} R_i$. ■

Theorem 2.9 is obtained in [1, Theorem 17 and Theorem 19] in a long process. $\mathrm{r}\mathbb{A}\mathbb{I}-$semiprime property has been taken under consideration in [2, Proposition 2.7 and Theorem 2.8] and can be characterized as follow:

Proposition 3.11 Let R be a $\mathrm{l}\mathbb{I}-$Artinian ring.

1. $\mathcal{Z}(_R R)$ is the nilpotent right annihilator ideal containing all nilpotent right annihilator ideals.
2. R is left nonsingular iff R is $\mathrm{r}\mathbb{A}\mathbb{I}-$semiprime.
3. $R/\mathcal{Z}(_R R)$ is left nonsingular and $\mathrm{r}\mathbb{A}\mathbb{I}-$semiprime.

References

[1] H. Khabazian, THE STRUCTURE OF RINGS IN WHICH THE INTERSECTION OF INESSENTIAL PRIME IDEALS IS ZERO, Bulletin of the Iranian Mathematical Society, Vol. 25, No. 1 (1999), 1-23.

[2] H. Khabazian, SOME CHARACTERIZATIONS OF ARTINIAN RINGS, International Electronic Journal of Algebra, Vol. 9 (2011), 1-9.

[3] T. Y. Lam, Lectures on Modules and Rings, Spring-Verlag, New York, Inc., (1998).